DATA MINING 2
Linear Regression

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Contains edited slides from StatQuest

UNIVERSITÀ DI PISA

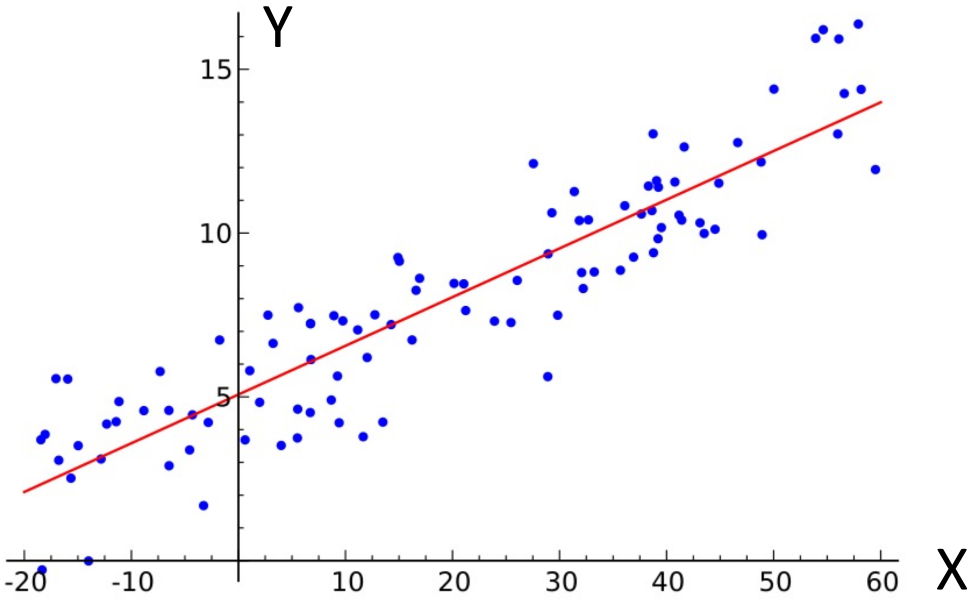
Regression

- Given a dataset containing N observations X_i, Y_i $i = 1, 2, \ldots, N$
- **Regression** is the task of learning a target function f that maps each input attribute set X into an output Y.
- The goal is to find the target function that can fit the input data with minimum error.
- The error function can be expressed as
  - Absolute Error = $\sum_i |y_i - f(x_i)|$
  - Squared Error = $\sum_i (y_i - f(x_i))^2$

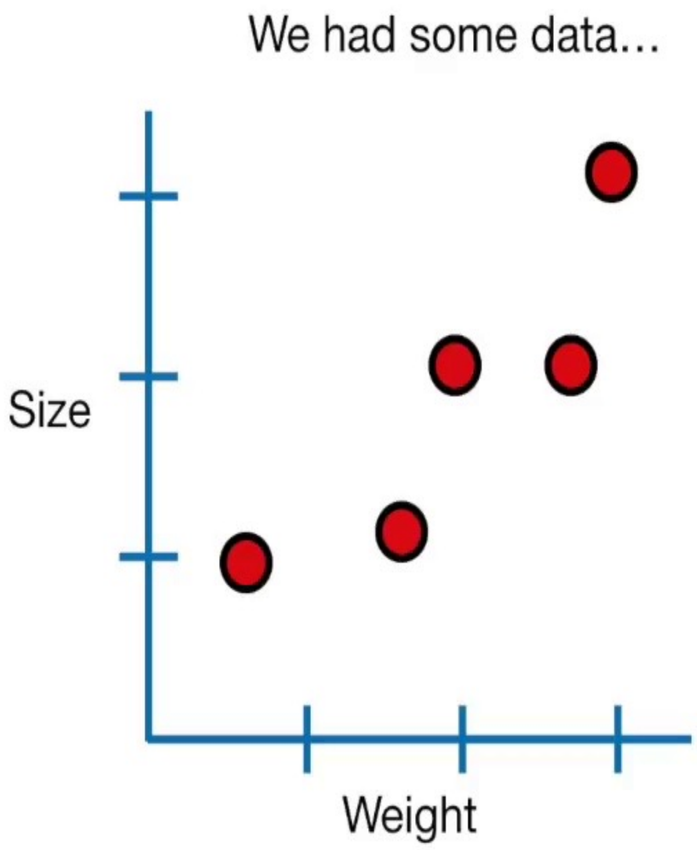
residuals

Linear Regression

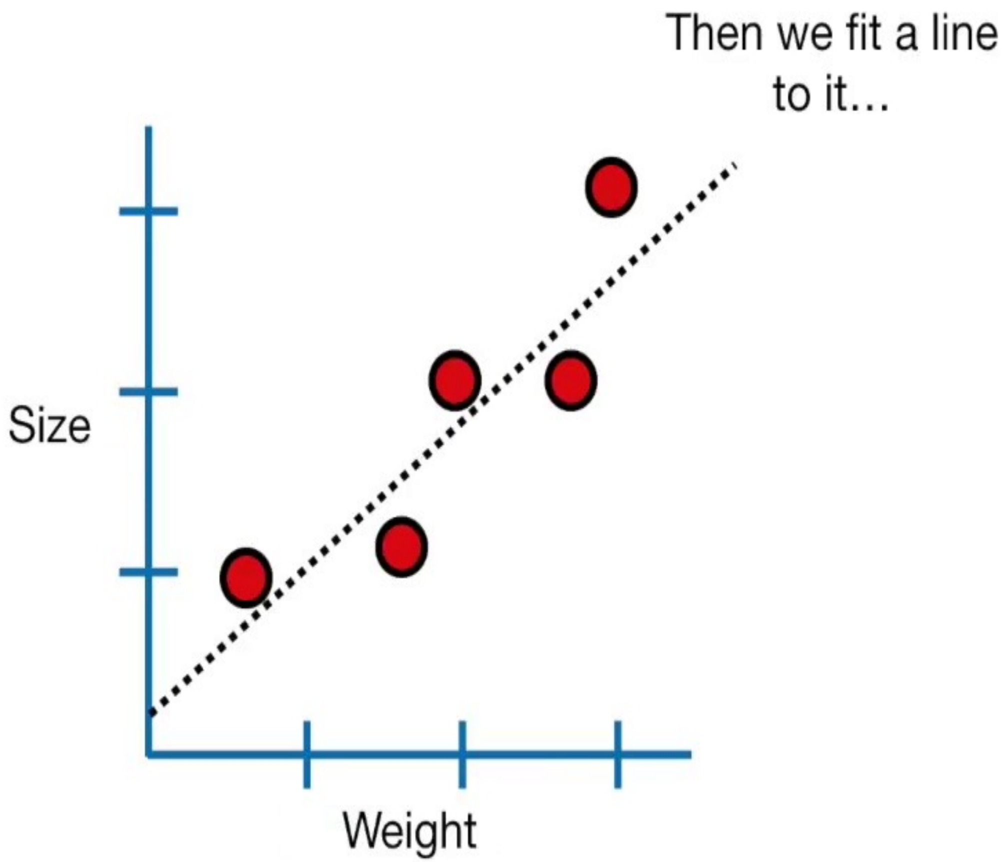
- **Linear regression** is a linear approach to modeling the relationship between a *dependent variable Y* and one or more *independent* (explanatory) variables *X*.
- The case of *one* explanatory variable is called **simple linear regression**.
- For *more than one explanatory variable*, the process is called **multiple linear regression**.
- For *multiple correlated dependent variables*, the process is called **multivariate linear regression**.



What does it mean to predict Y?



What does it mean to predict Y?



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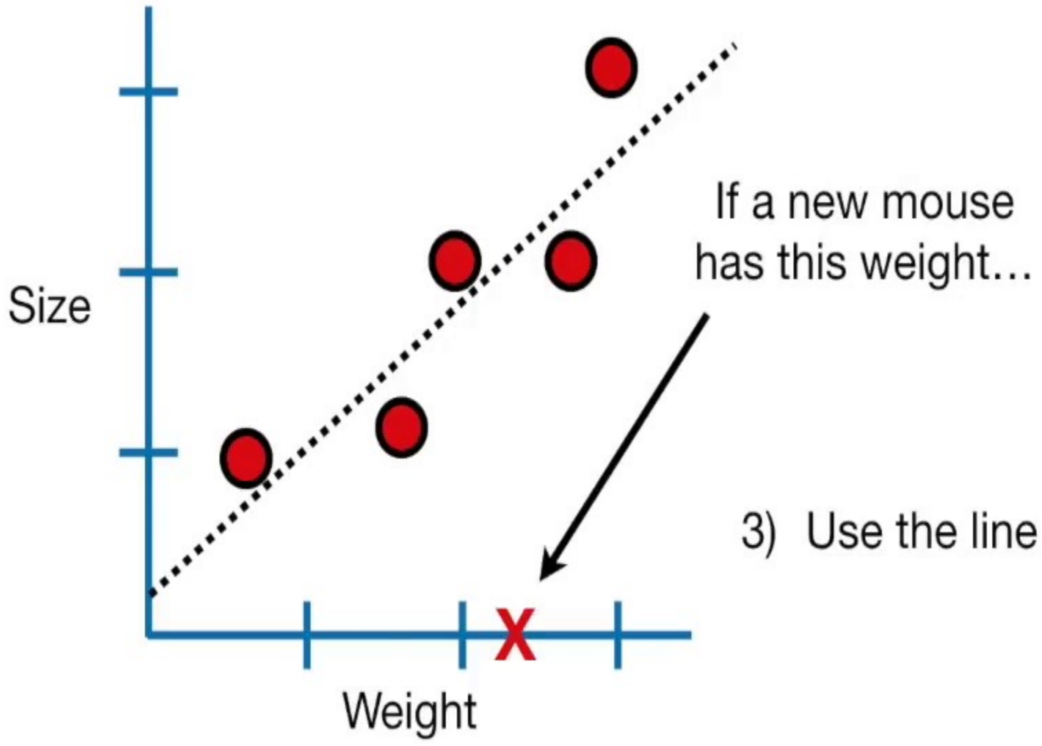
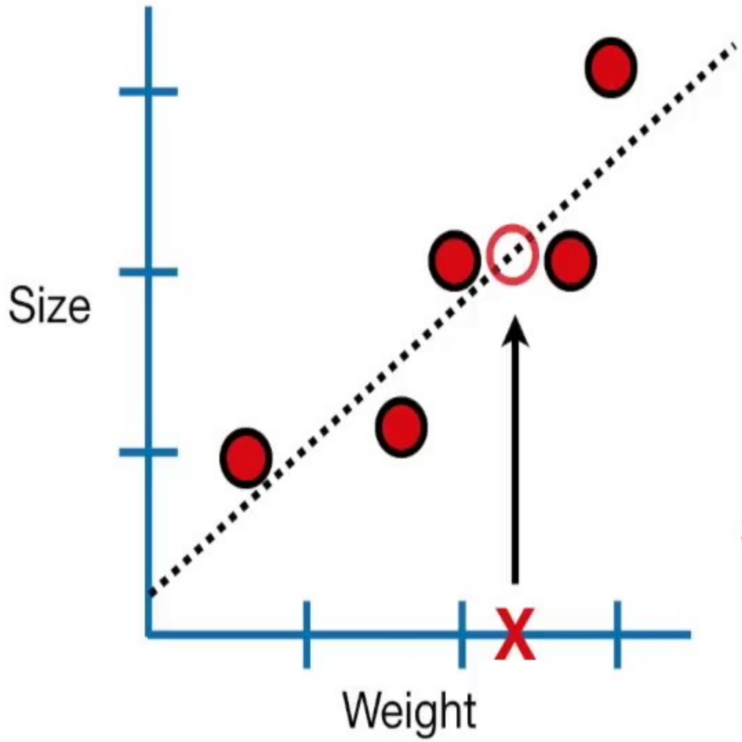


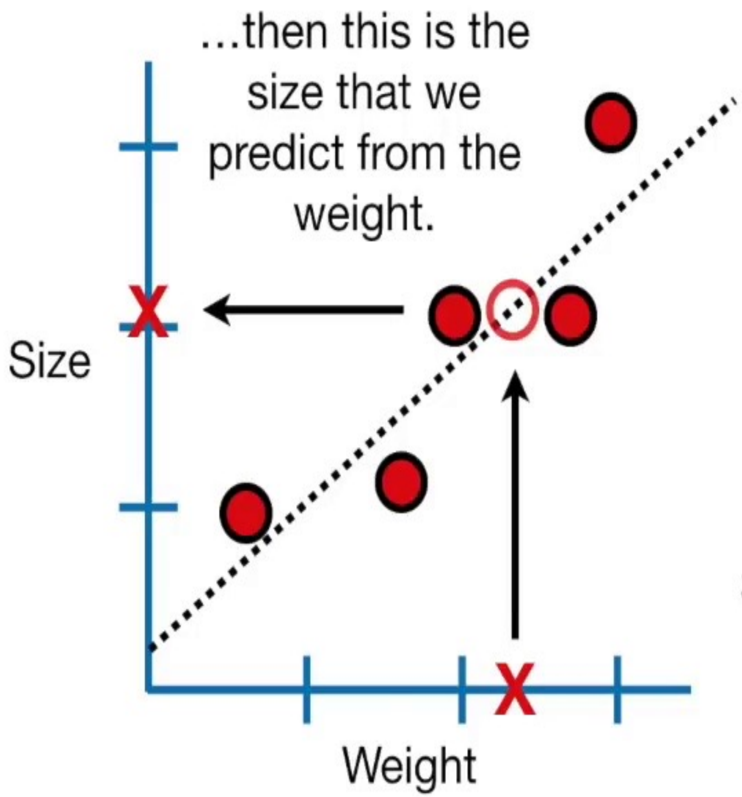
3) Use the line to predict ***size*** given ***weight***.

What does it mean to predict Y?



3) Use the line to predict ***size*** given ***weight***.

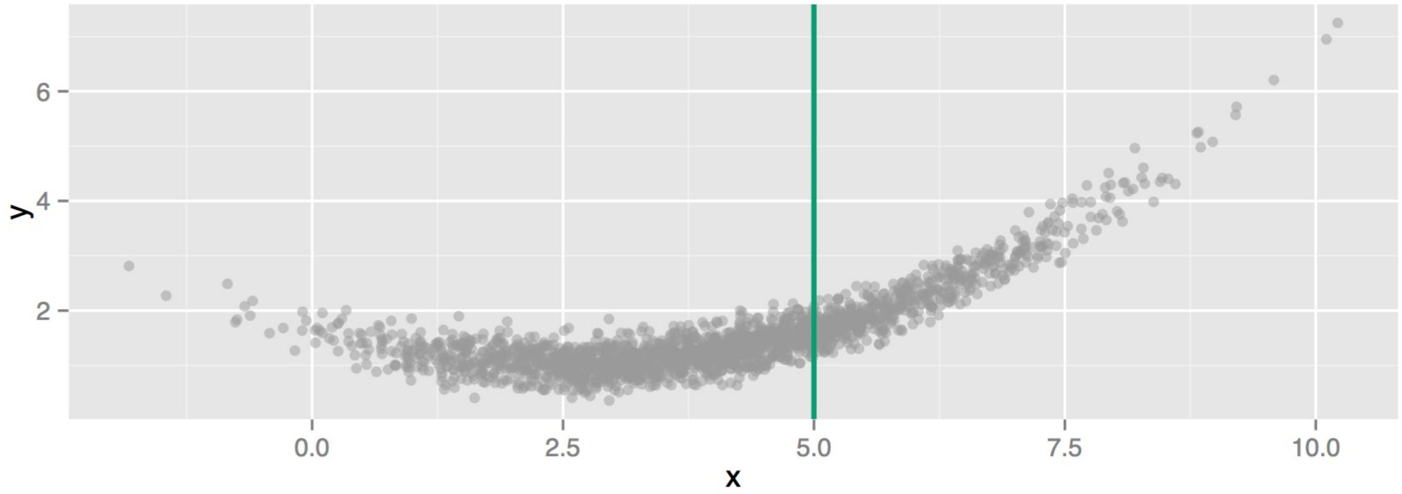
What does it mean to predict Y?



3) Use the line to predict ***size*** given ***weight***.

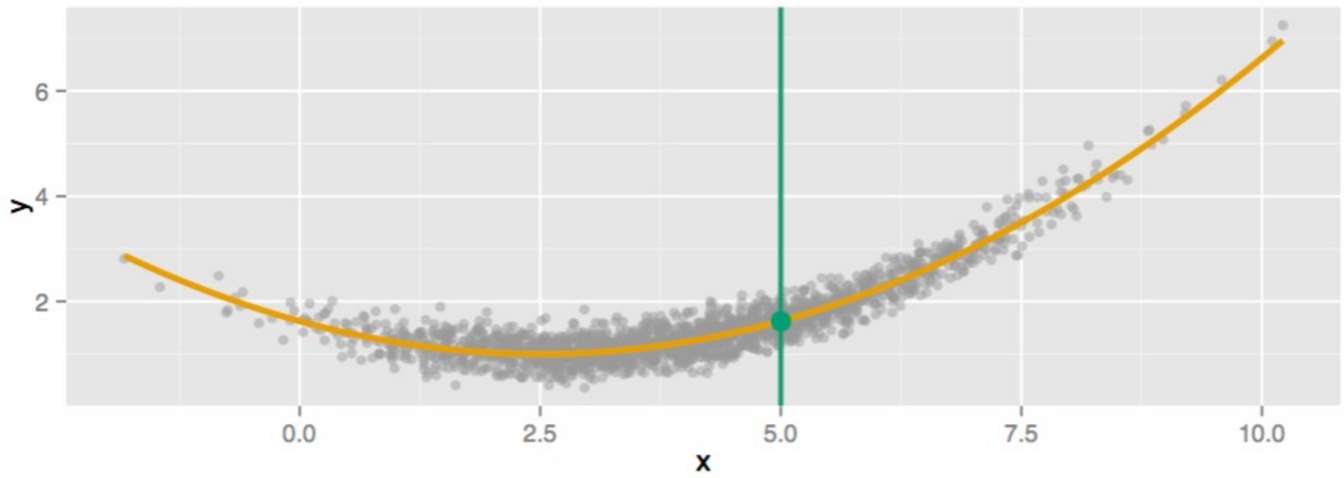
What does it mean to predict Y?

- Look at *X = 5*. There are many different *Y* values at *X=5*.
- When we say predict *Y* at *X =5*, we are really asking:
- What is the expected value (average) of *Y* at *X = 5*?

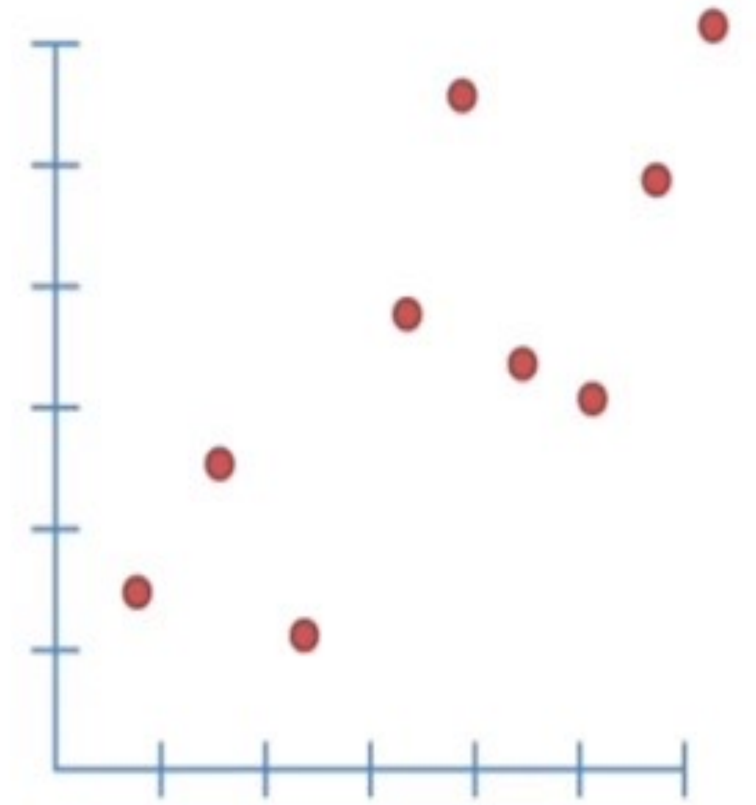


What does it mean to predict Y?

- Formally, the ***regression function*** is given by *E(Y|X=x)*. This is the expected value of Y at X=x.
- The ideal or optimal predictor of *Y* based on *X* is thus
  - f(X) = E(Y | X=x)



Fitting a Line Example



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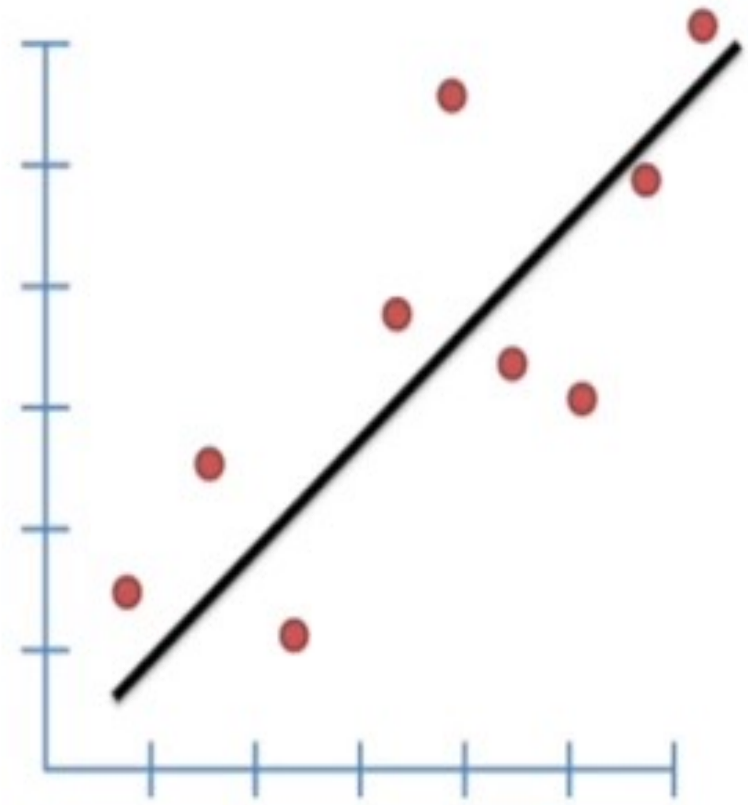

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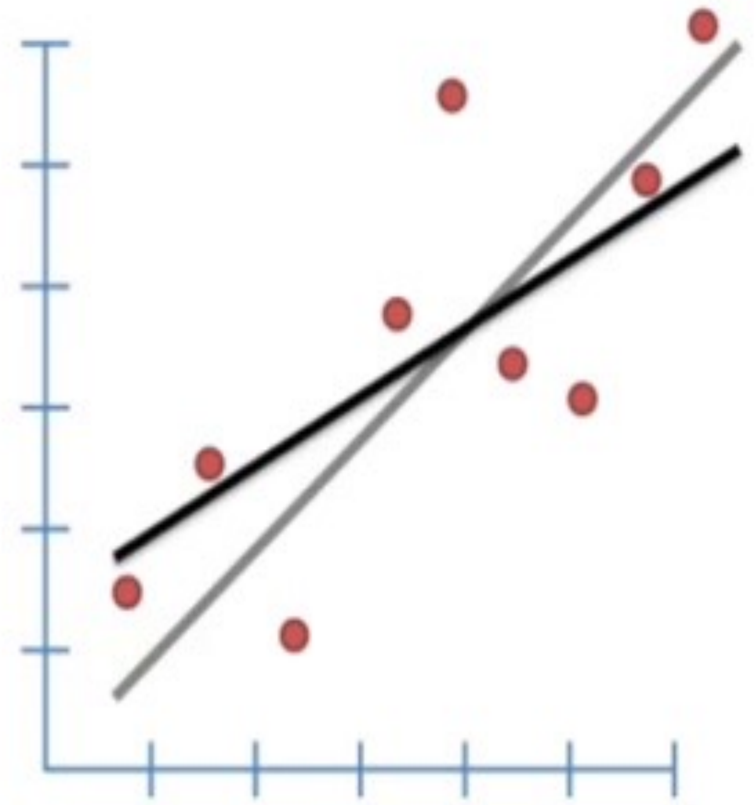

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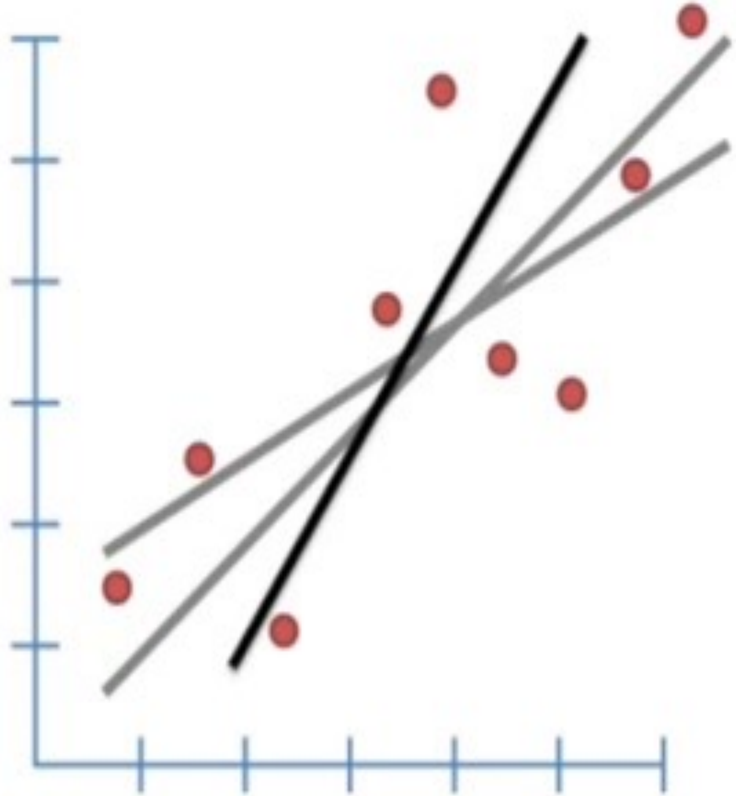
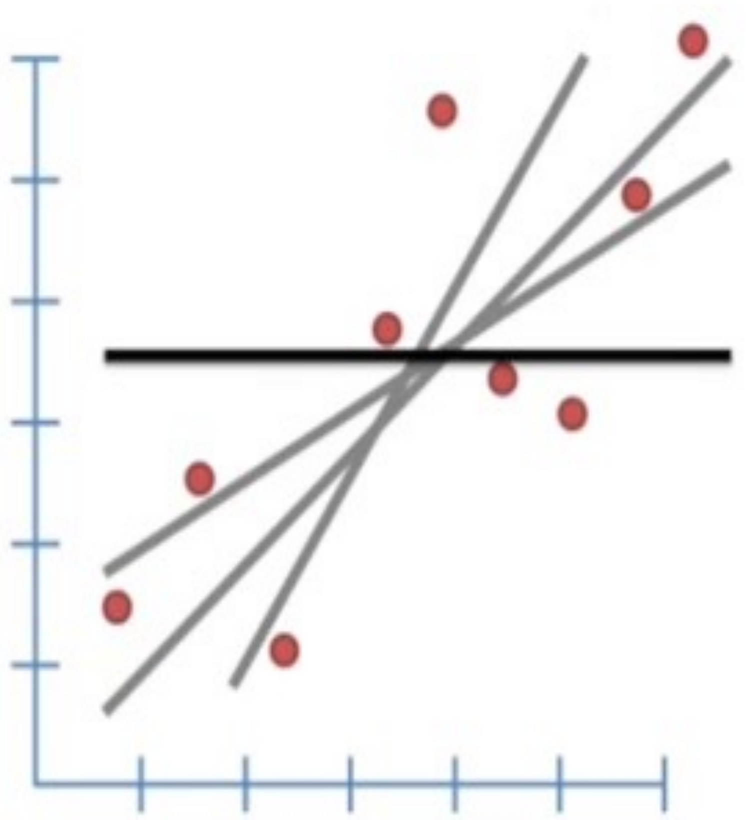
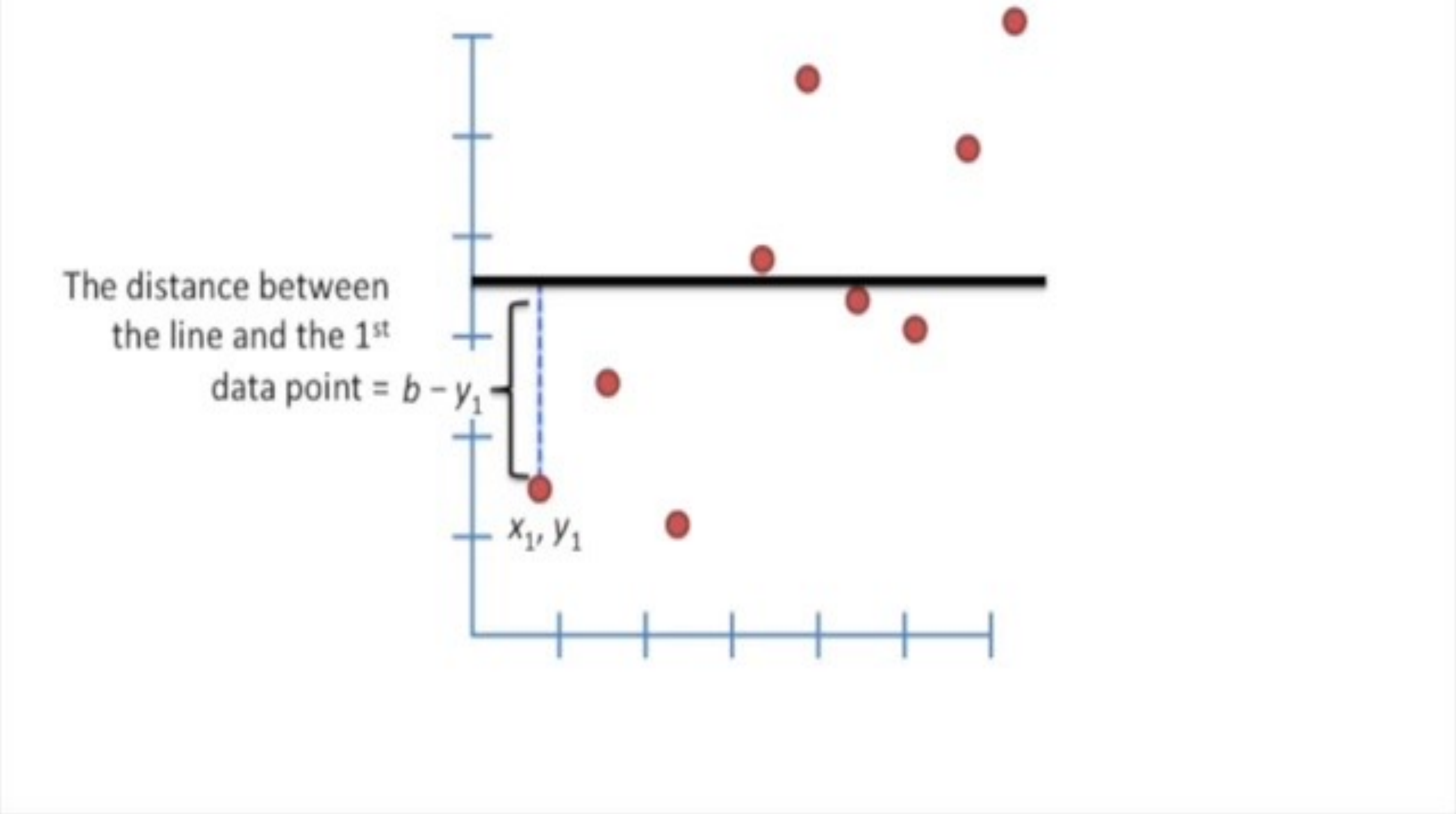


Fitting a Line Example



Measuring Line Error



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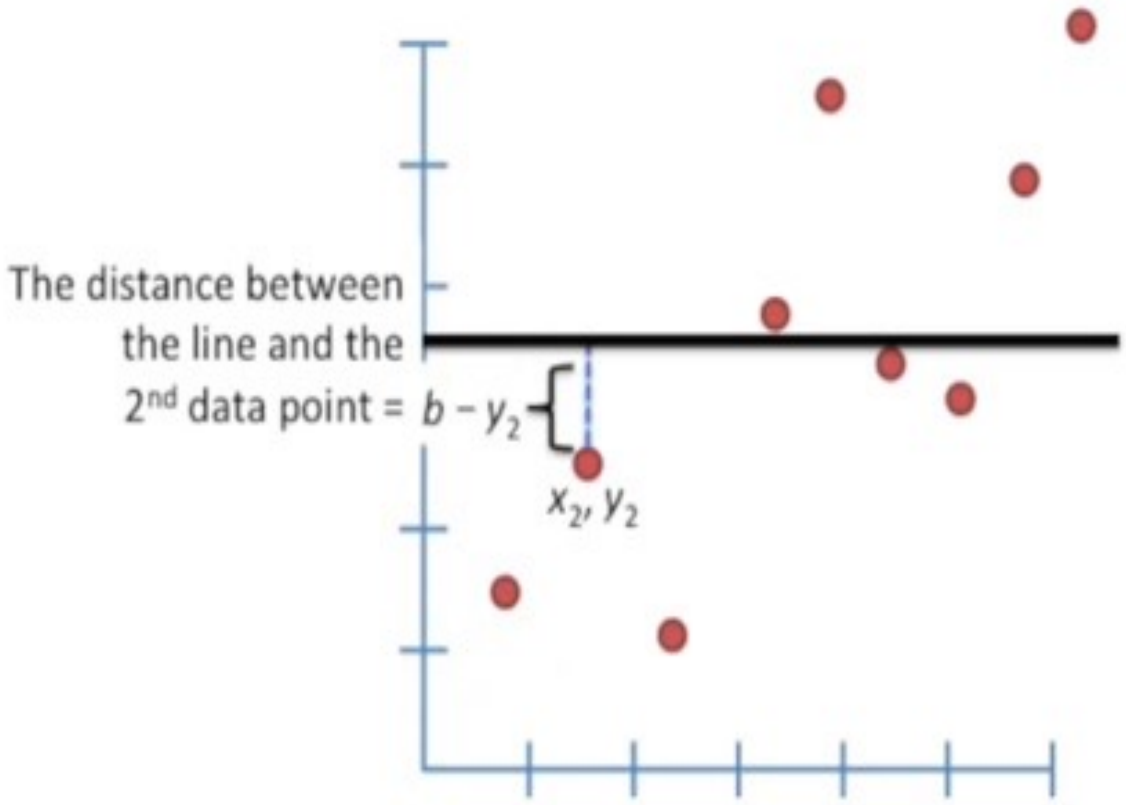
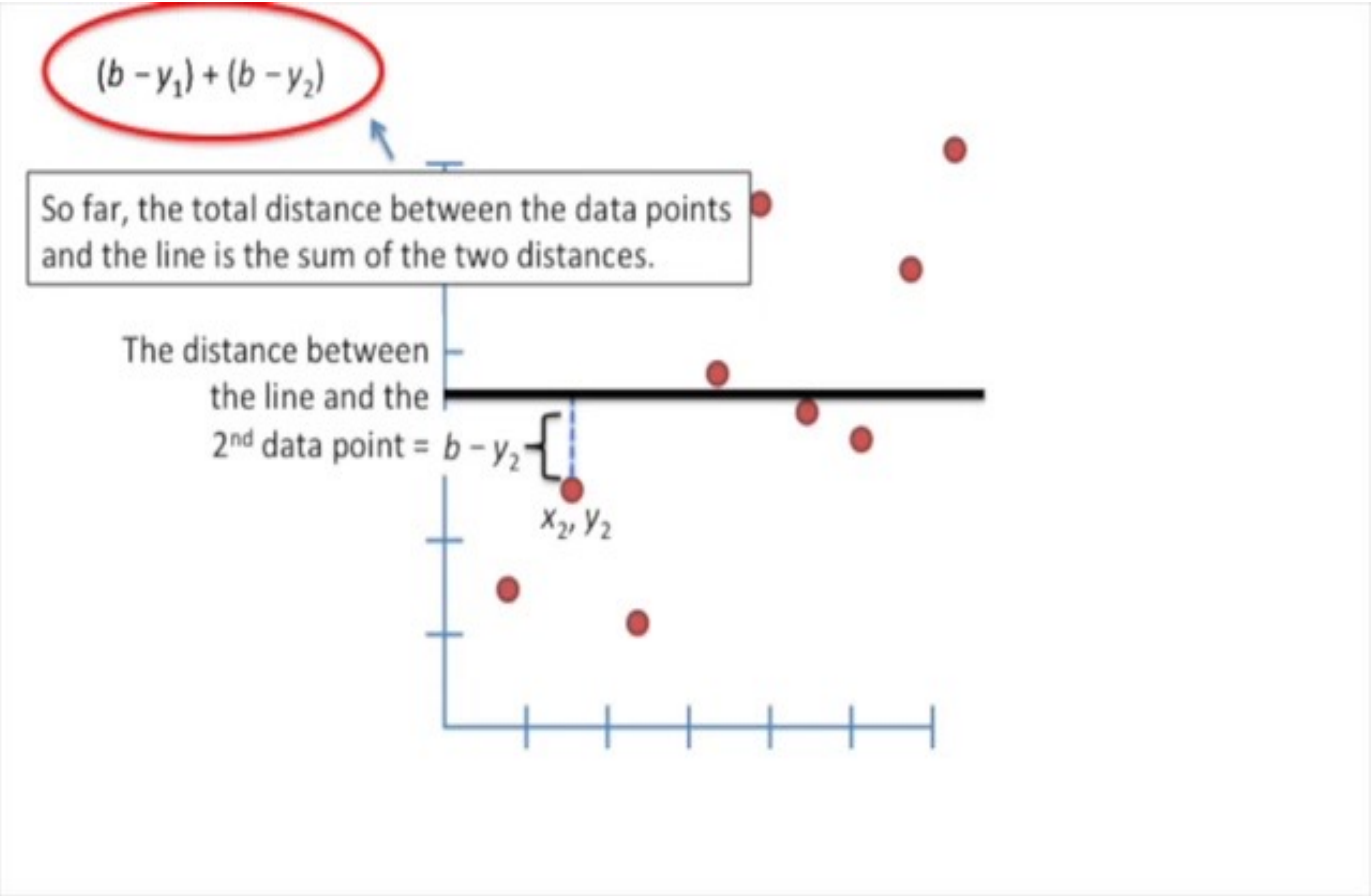
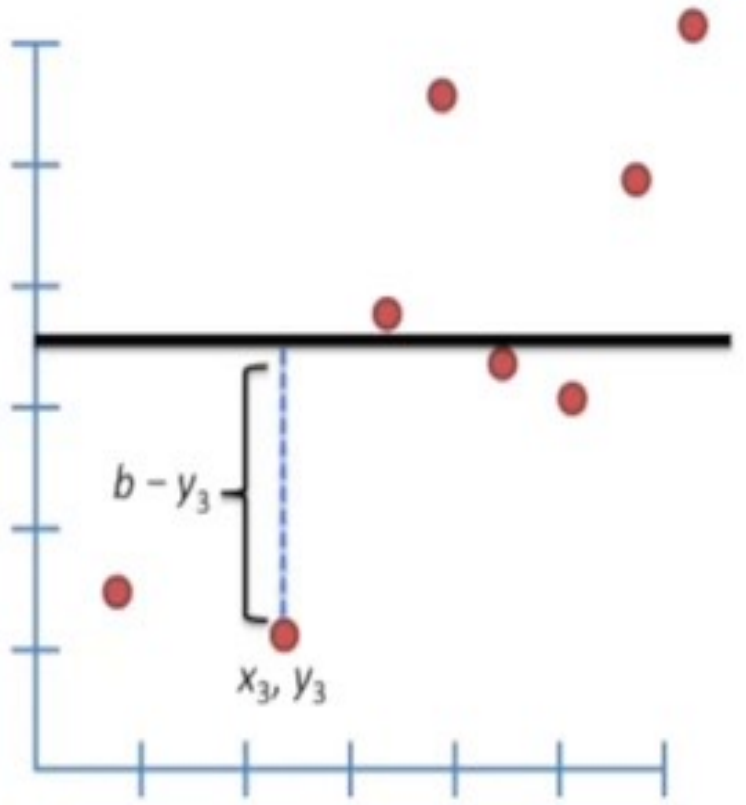
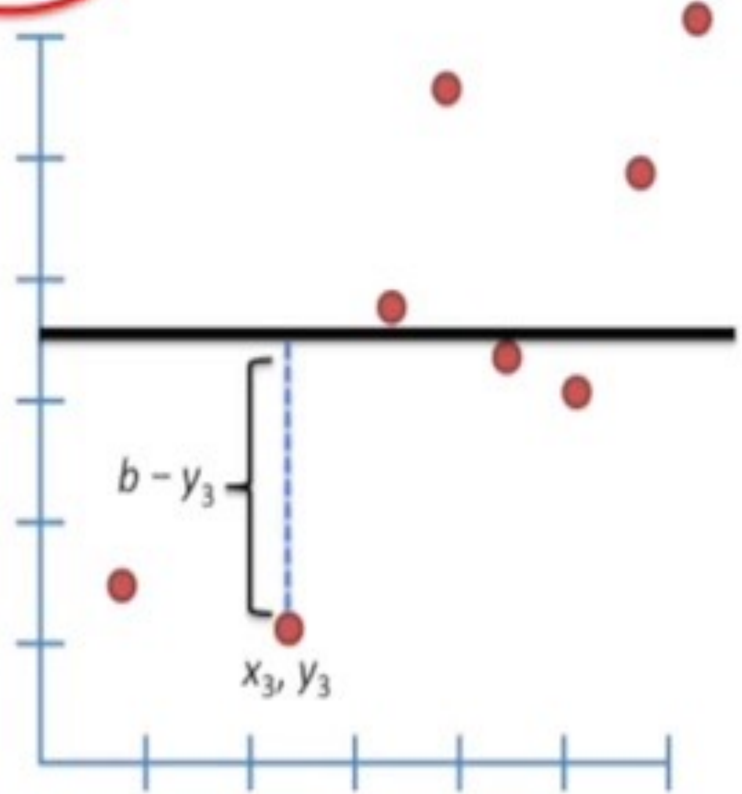



$(b - y_1) + (b - y_2)$
So far, the total distance between the data points and the line is the sum of the two distances.
The distance between the line and the 2nd data point = $b - y_2$
x_2, y_2

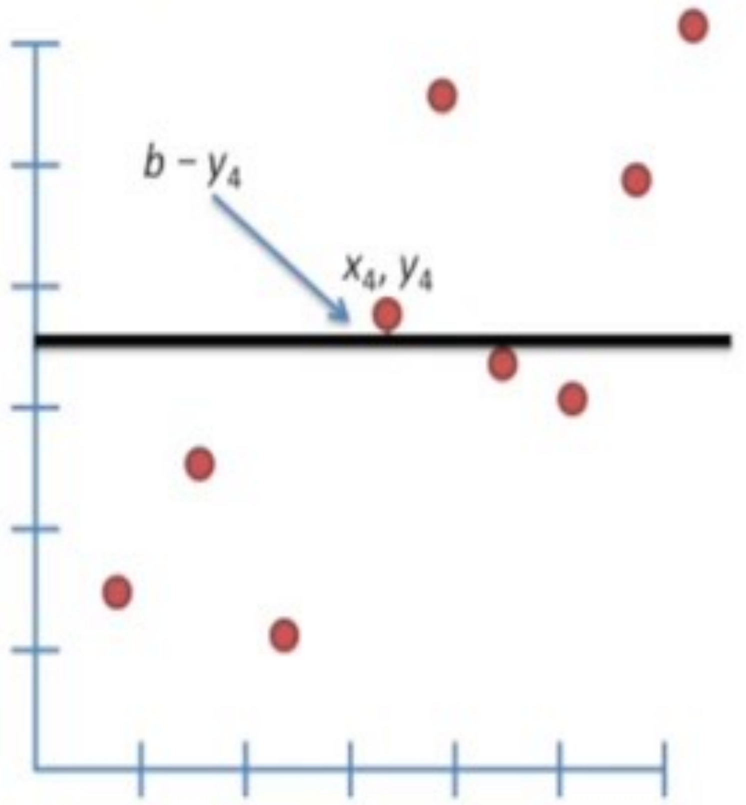
$(b - y_1) + (b - y_2)$



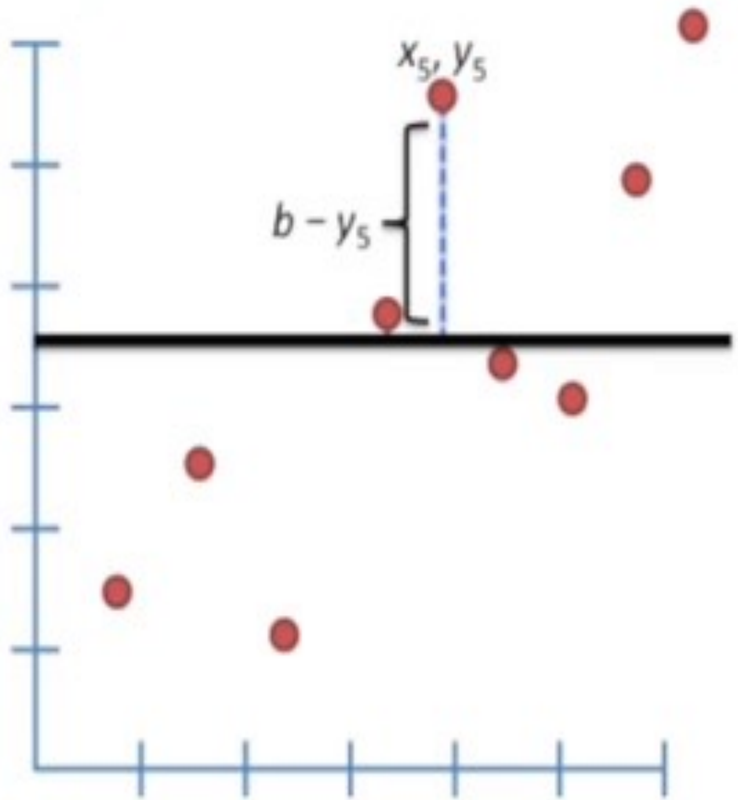
$(b - y_1) + (b - y_2) + (b - y_3)$ Now we've add the 3rd distance to our total sum.



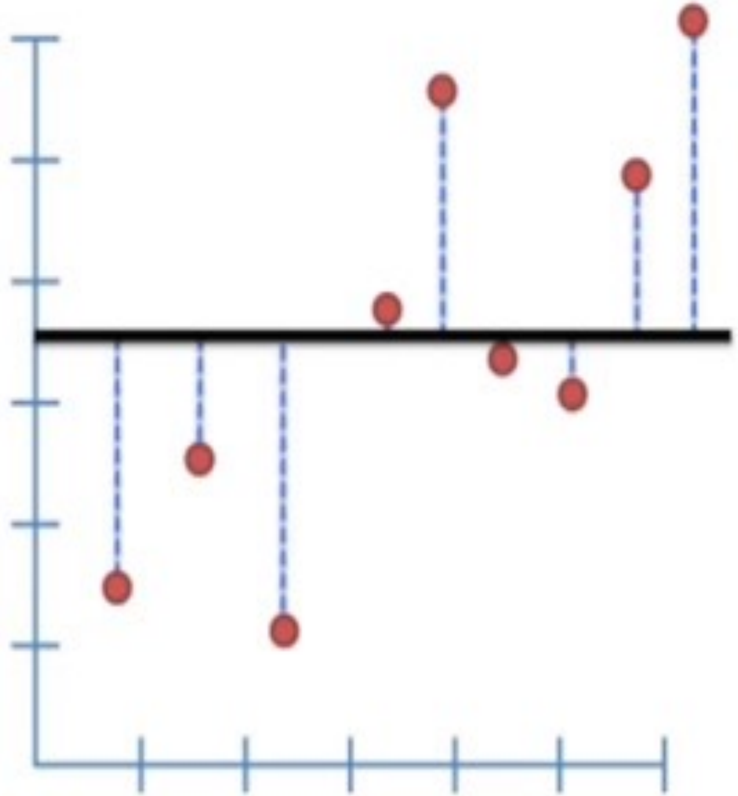
$(b - y_1) + (b - y_2) + (b - y_3)$



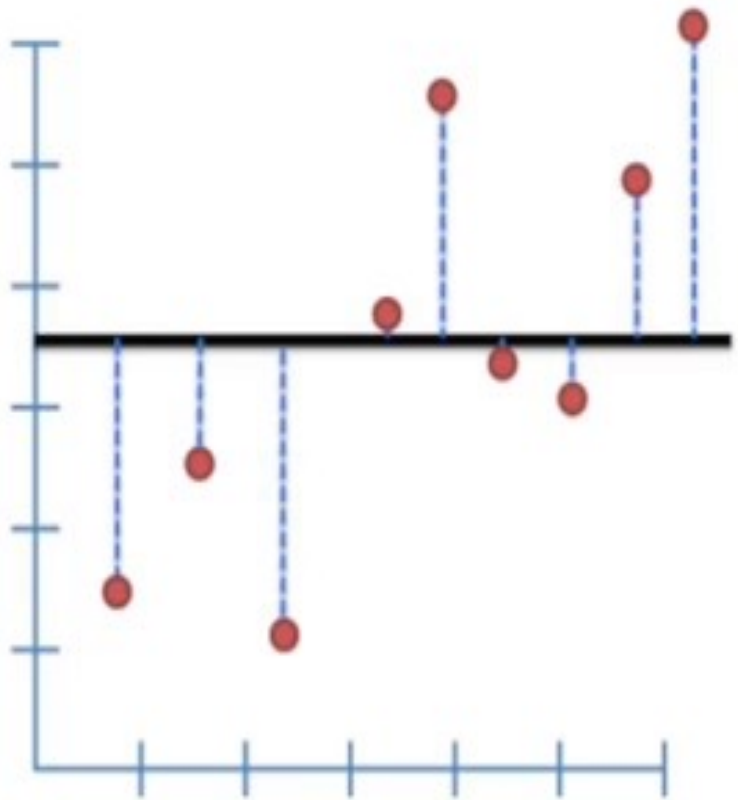
$(b - y_1) + (b - y_2) + (b - y_3) + (b - y_4)$



$$(b - y_1)^2 + (b - y_2)^2 + (b - y_3)^2 + (b - y_4)^2 + (b - y_5)^2 + (b - y_6)^2 + (b - y_7)^2 + (b - y_8)^2 + (b - y_9)^2$$



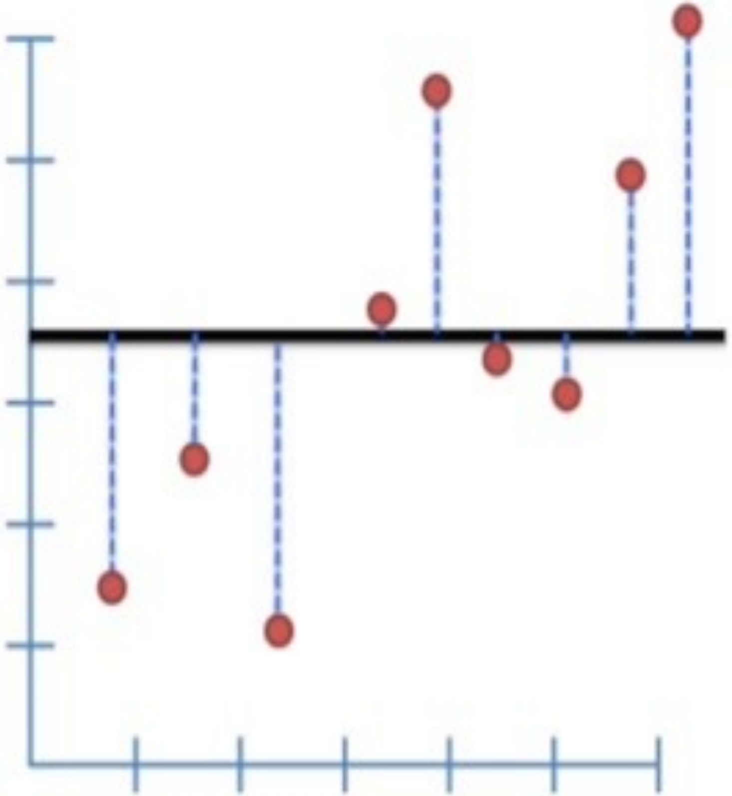
$$(b-y_1)^2+(b-y_2)^2+(b-y_3)^2+(b-y_4)^2+(b-y_5)^2+(b-y_6)^2+(b-y_7)^2+(b-y_8)^2+(b-y_9)^2$$



= 24.62

This is our measure of how well this line fits the data.

$$(b-y_1)^2+(b-y_2)^2+(b-y_3)^2+(b-y_4)^2+(b-y_5)^2+(b-y_6)^2+(b-y_7)^2+(b-y_8)^2+(b-y_9)^2$$

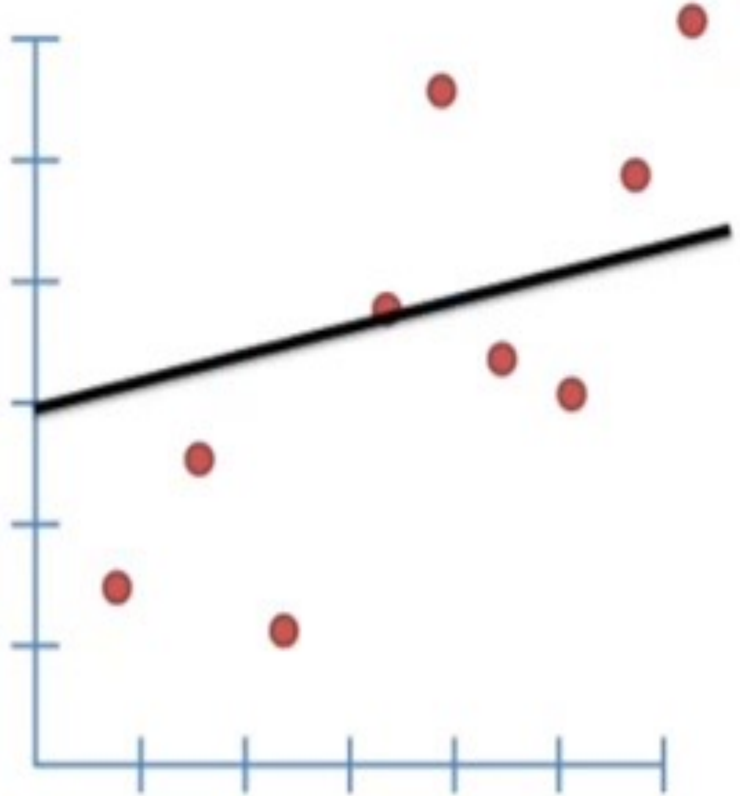


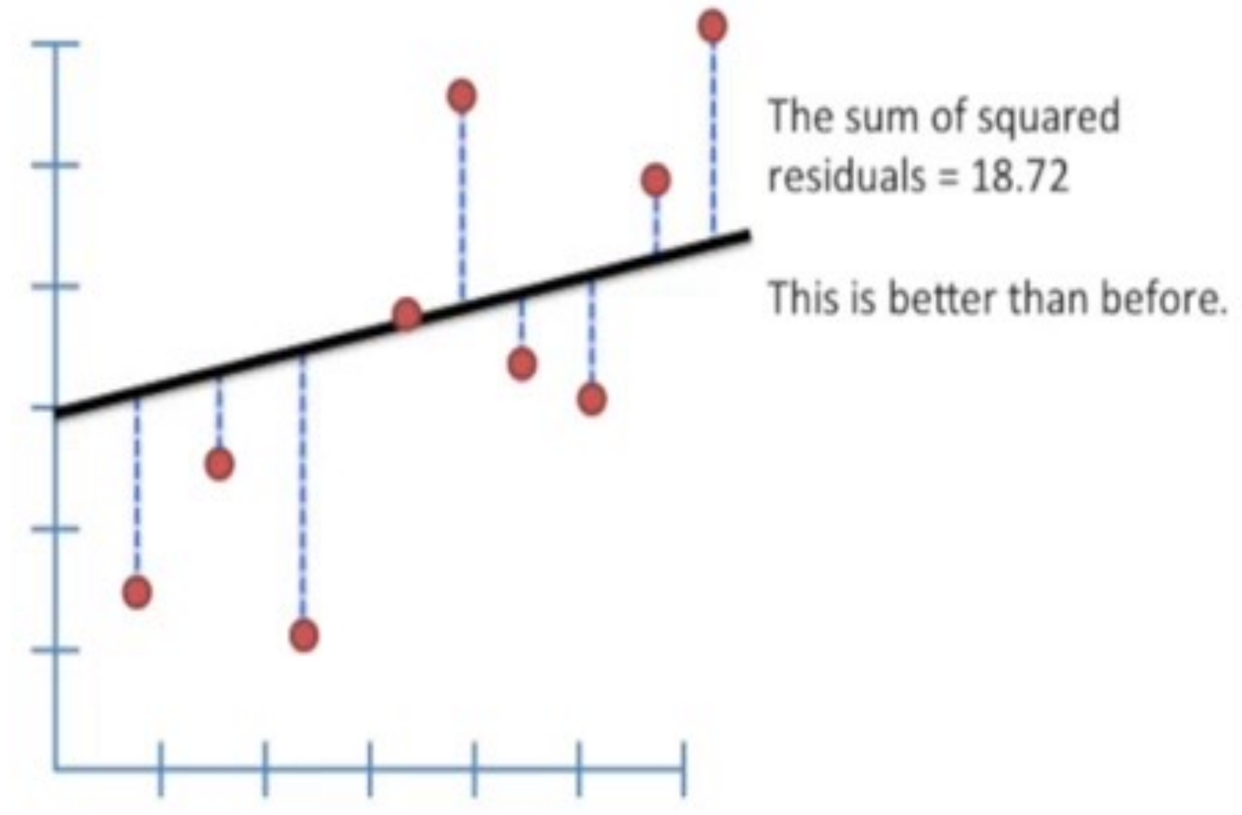
$= 24.62$

This is our measure of how well this line fits the data.

It's called the "sum of squared residuals, because the residuals are the differences between the real data and the line, and we are summing the square of these values.

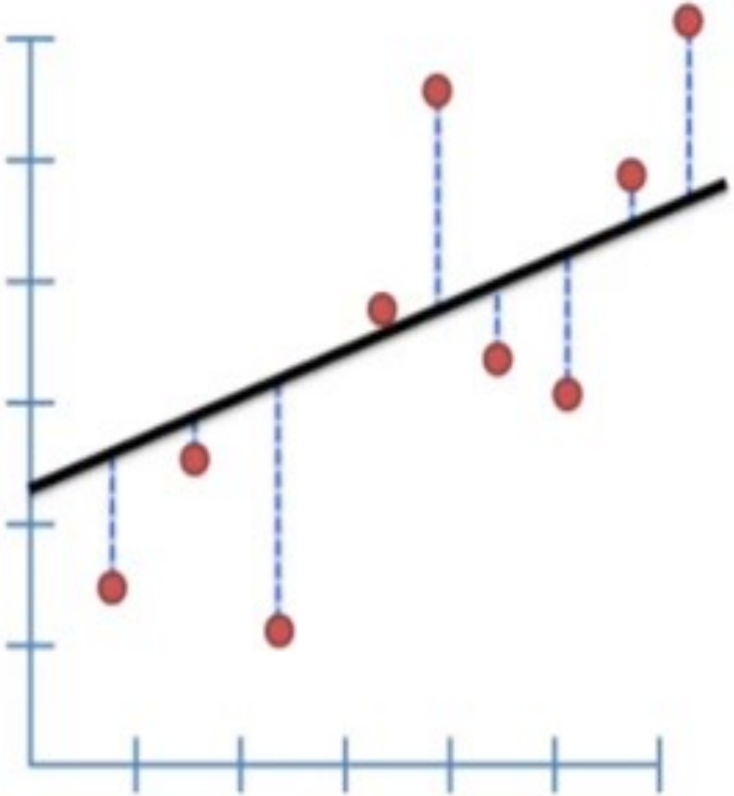
Now let's see how good the fit is if we rotate the line a little bit.



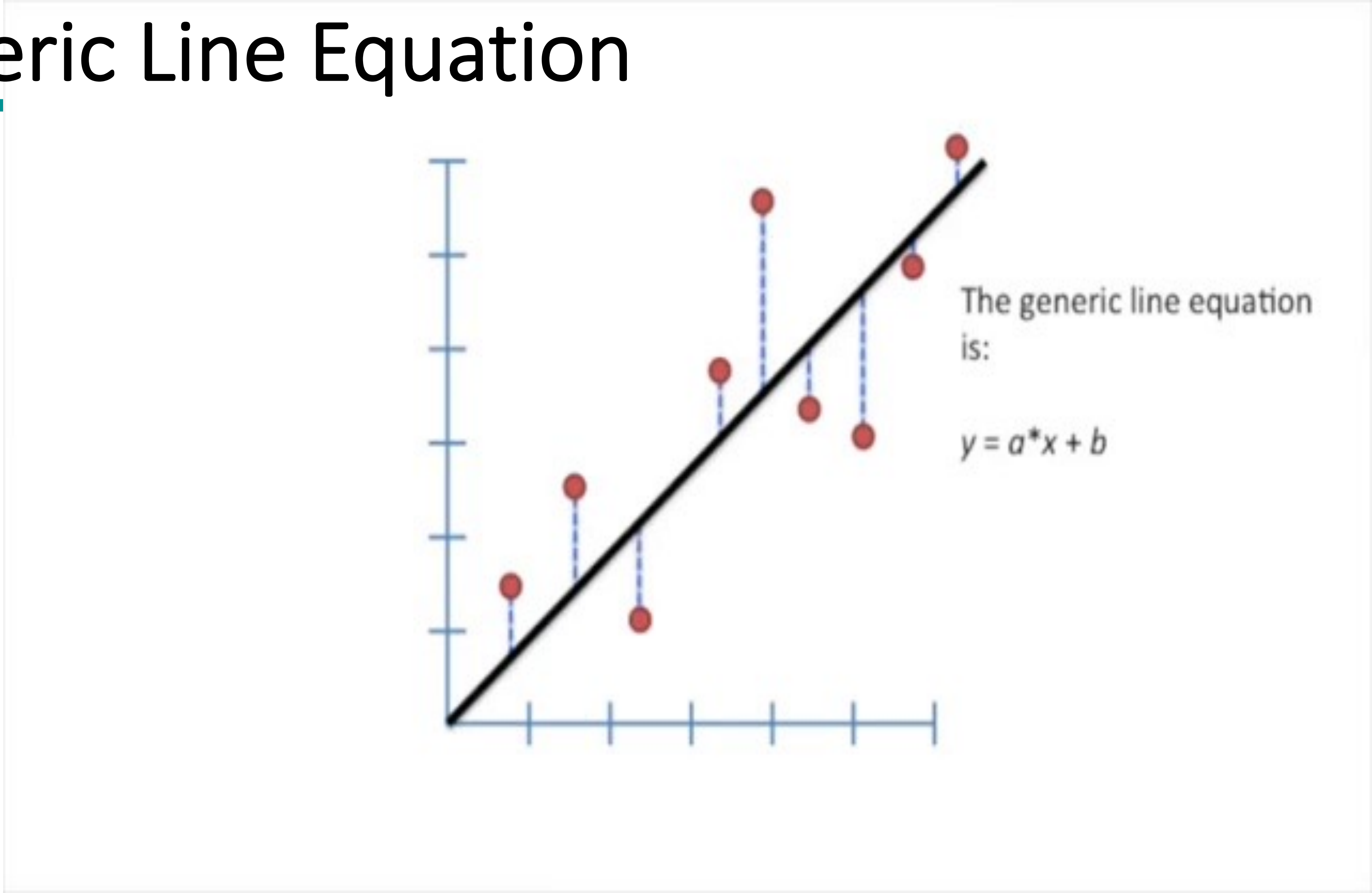

The sum of squared residuals = 18.72

This is better than before.

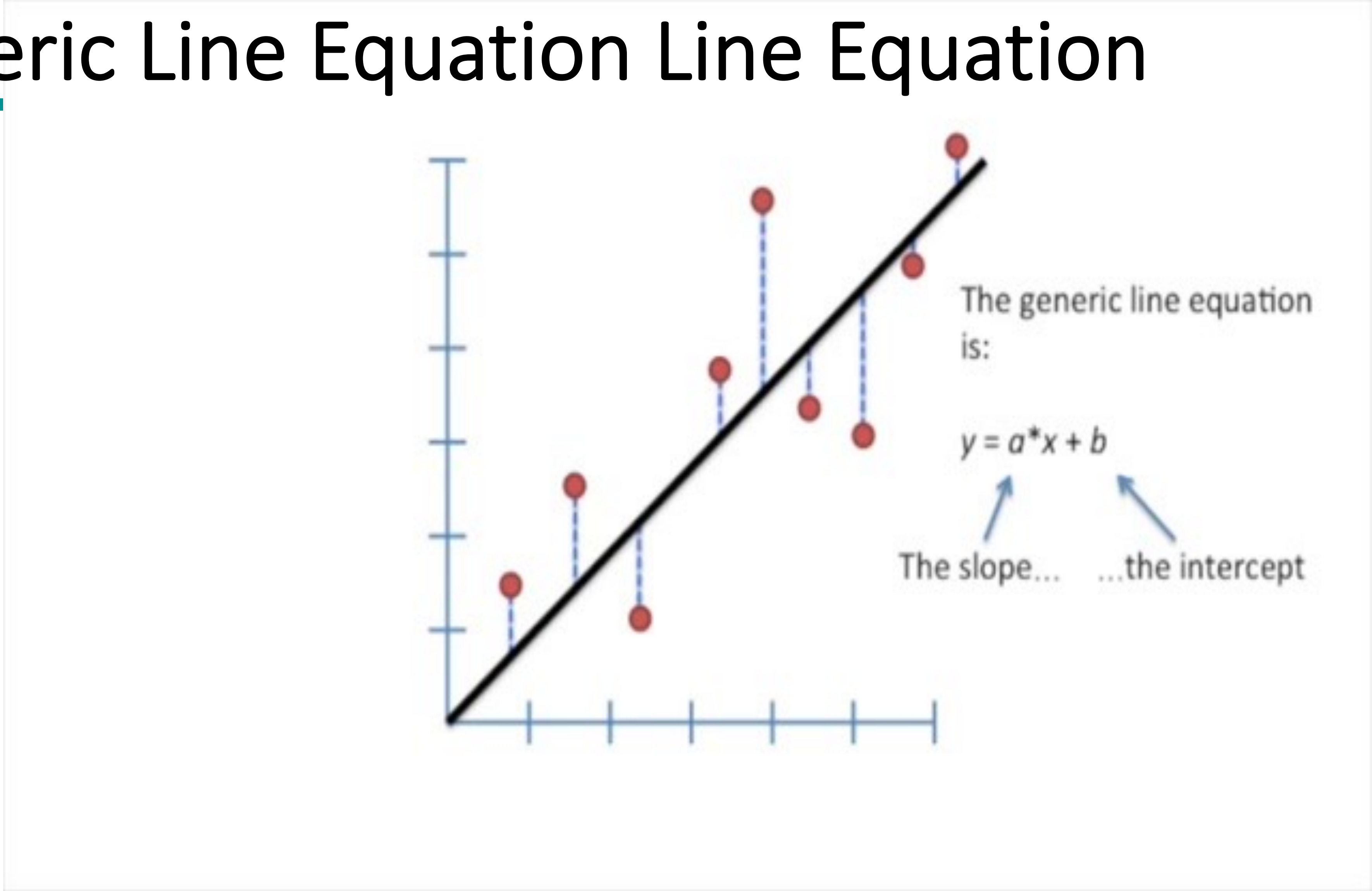
Does this fit improve if we rotate a little more?



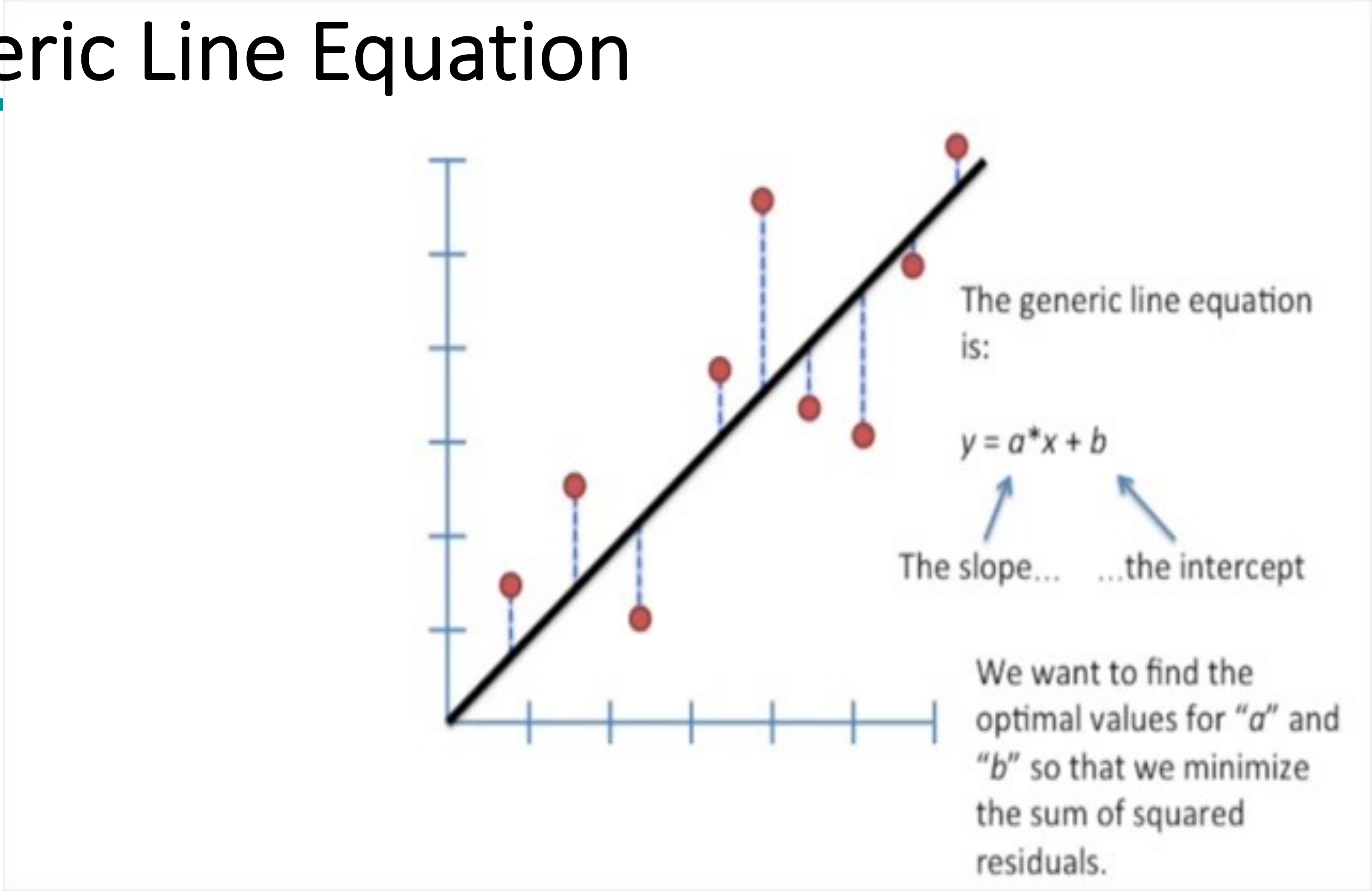
Generic Line Equation



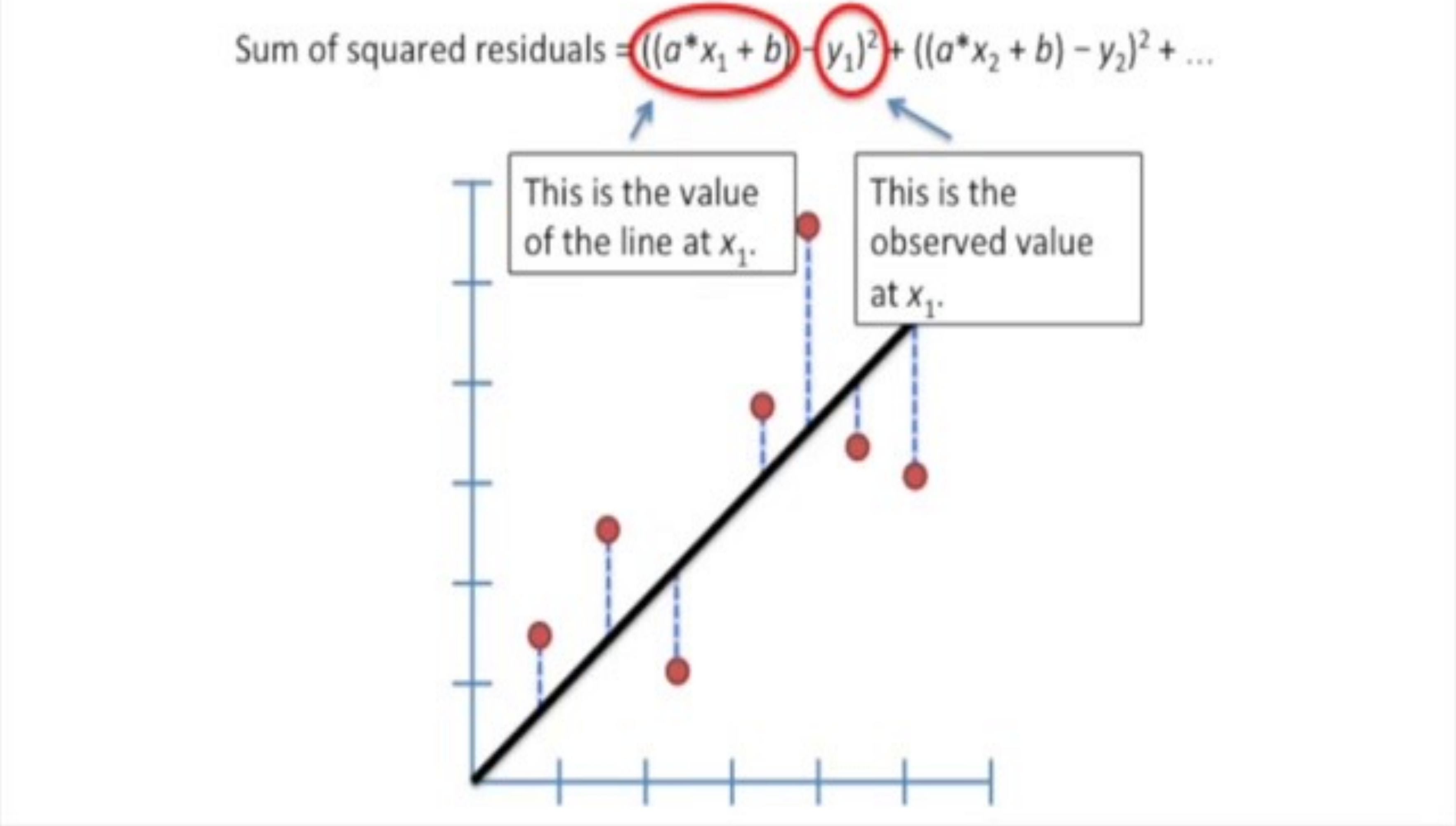
Generic Line Equation Line Equation



Generic Line Equation



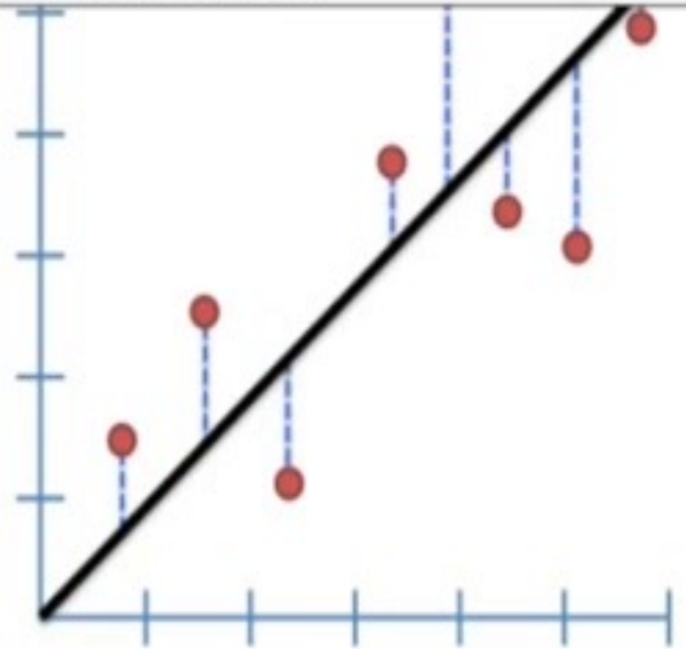
Least Square



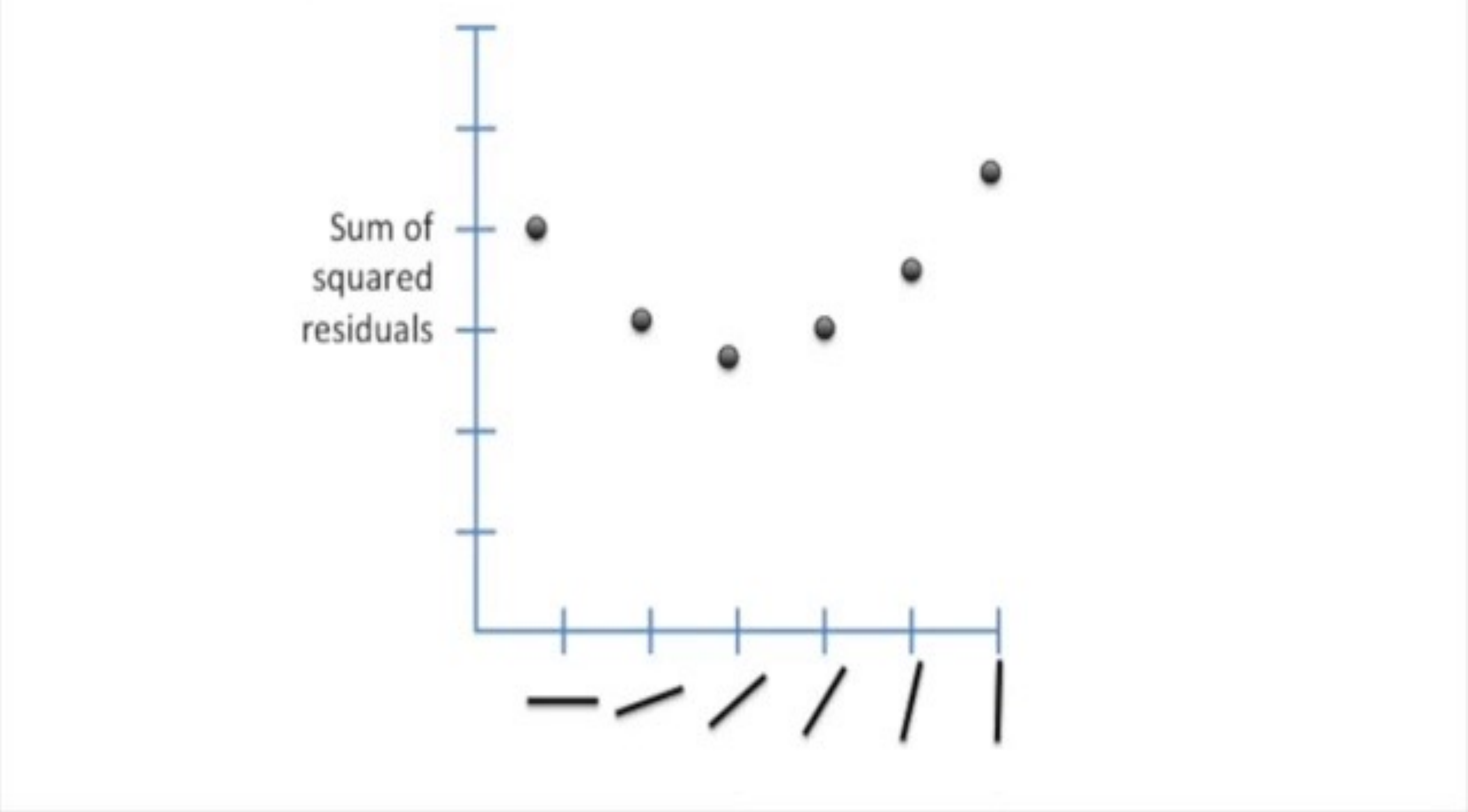
Least Square

Sum of squared residuals = $((a^*x_1 + b) - y_1)^2 + ((a^*x_2 + b) - y_2)^2 + \ldots$

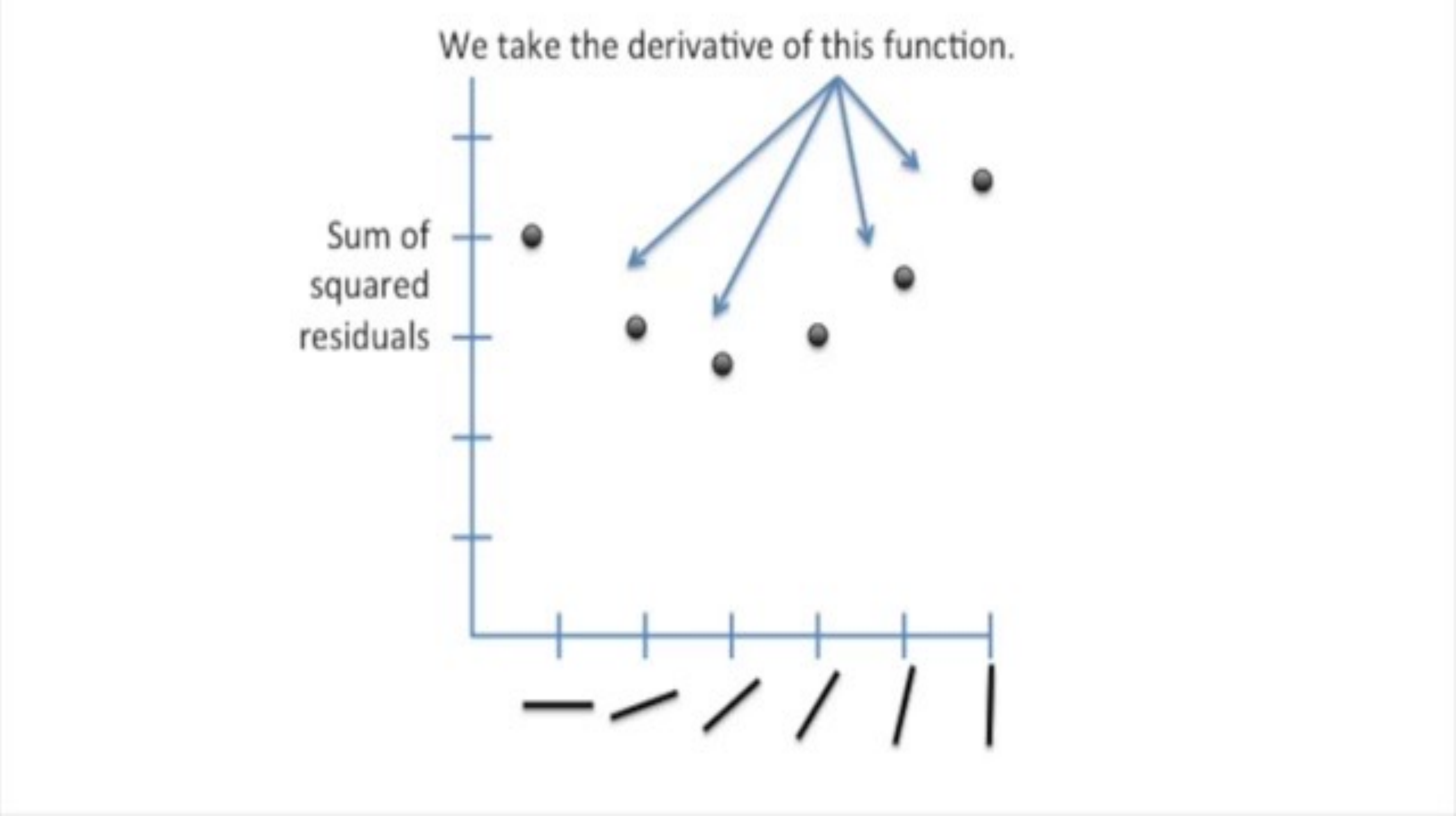
Since we want the line that will give us the smallest sum of squares, this method for finding the best values for "*a*" and "*b*" is called "Least Squares".



Sum of Squares Residuals



Finding Best Rotation



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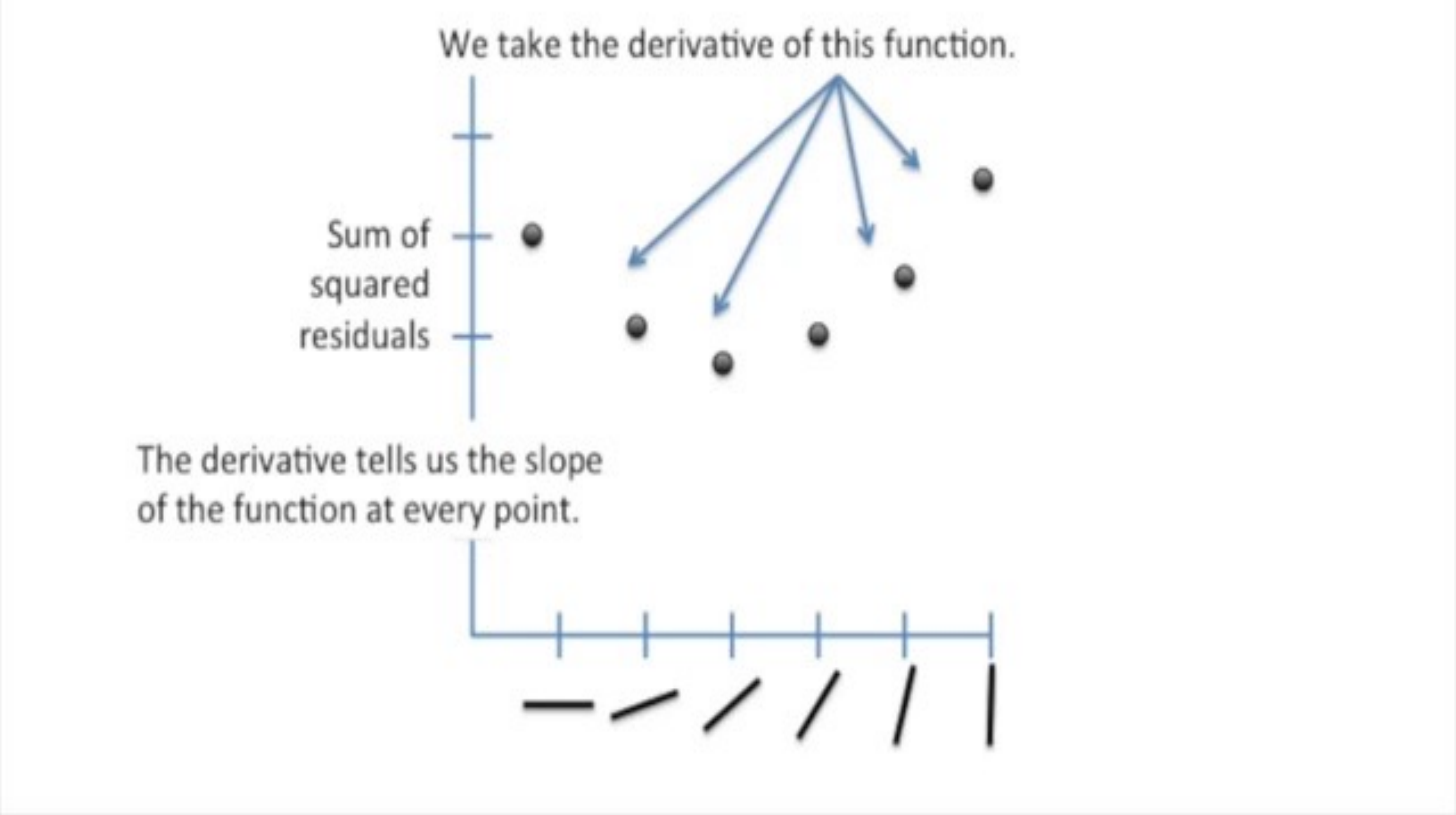

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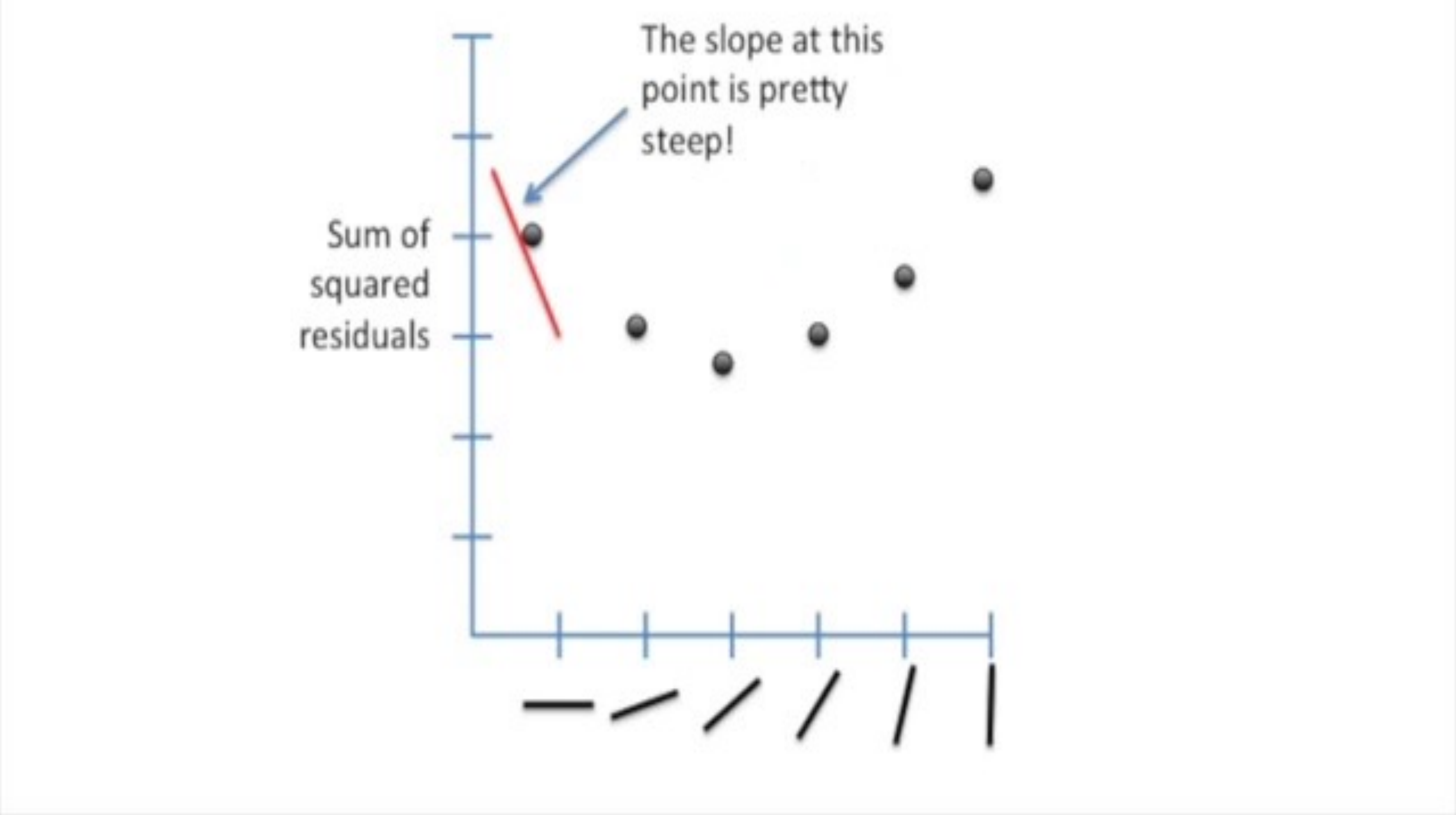

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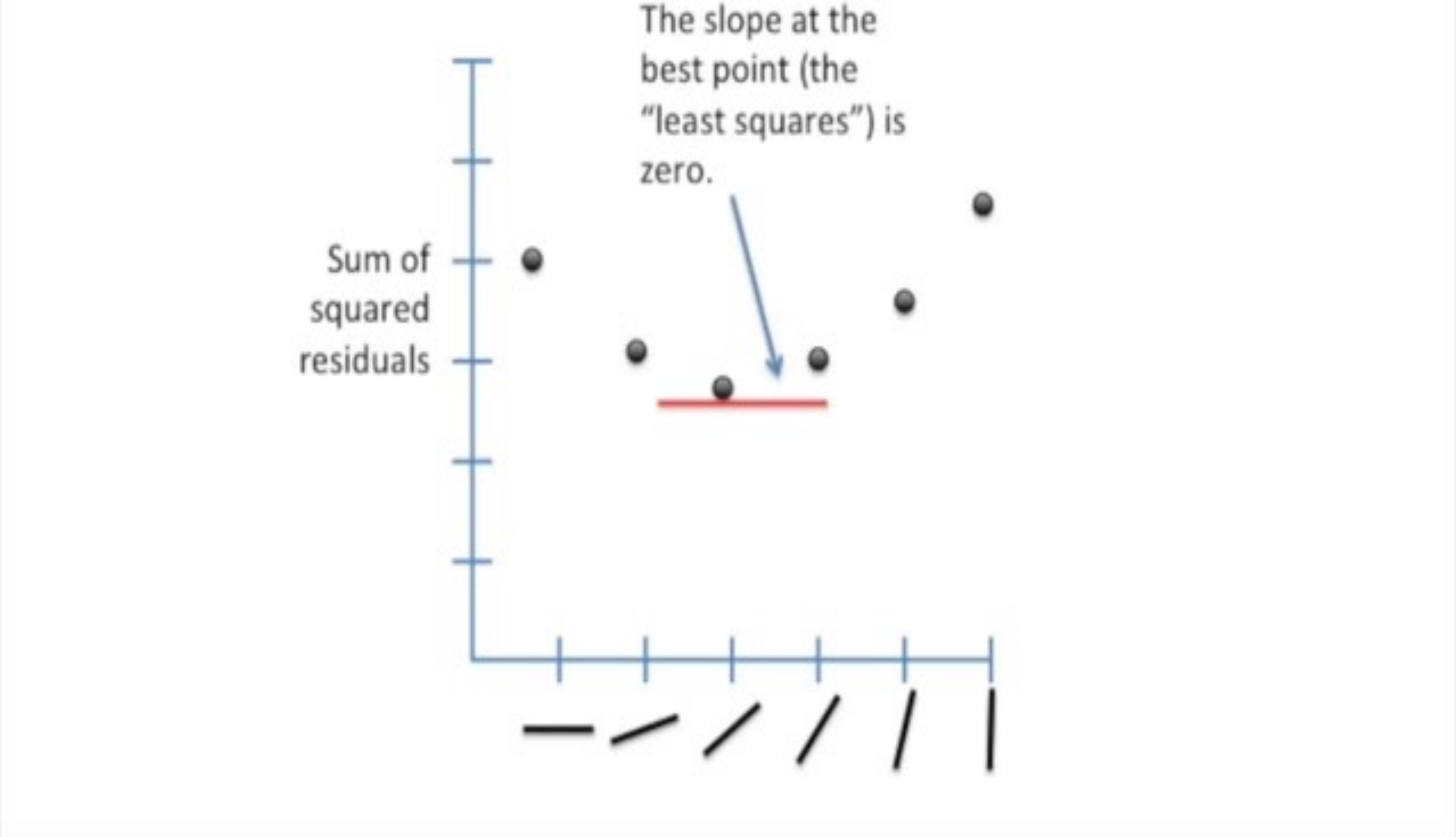

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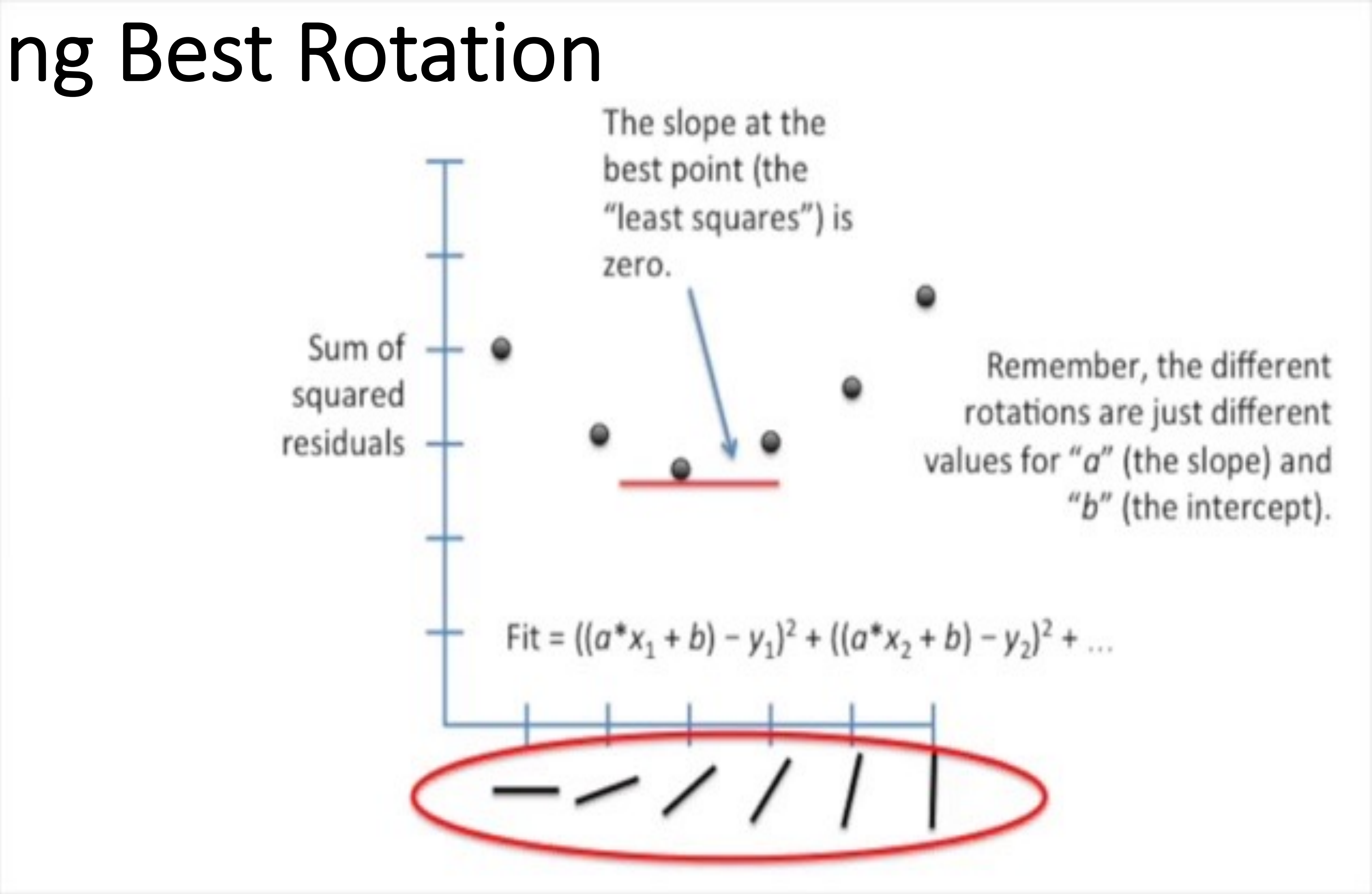
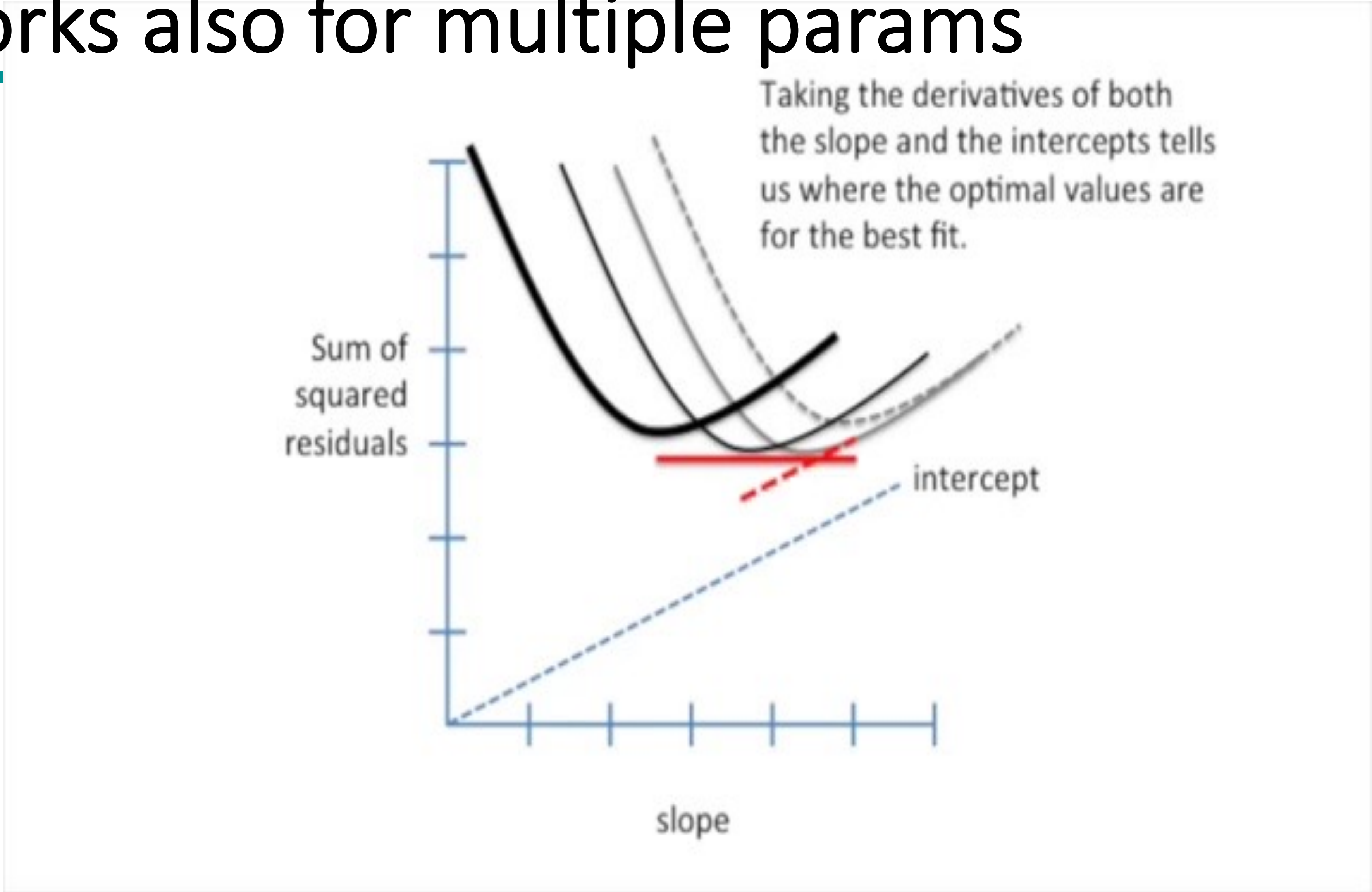


It works also for multiple params



Simple Linear Regression

Dependent Variable Independent Variable

Linear Model: $$\underbrace{Y}_{\text{}} = mX + b \qquad Y = \beta_1 X + \beta_0$$

Slope Intercept (bias)

- In general, such a relationship may not hold exactly for the largely unobserved population
- We call the unobserved deviations from *Y* the errors.
- The goal is to find estimated values *m'* and *b'* for the parameters *m* and *b* which would provide the "best" fit for the data points.

Least Square Method (LSM)

- A standard approach for doing this is to apply the **method of least squares** which attempts to find the parameters *m*, *b* that minimizes the sum of squared error.
- SSE = $\sum_i (y_i - f(x_i))^2$ = $\sum_i (y_i - mx_i \ - b)^2$
- also known as the **residual sum of squares**.
- The LSM finds *m, b* by setting to zero the first partial derivative of the above function w.r.t. *m* and *b* which are therefore calculated as follows:
- $m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) - (\sum x)^2)$
- $b = (\sum y - m \sum x) / n$

- An alternative to find *m, b*, typically adopted in case of multivariate regression is the Gradient Descent method (see next lectures)

$m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) - (\sum x)^2)$
$b = (\sum y - m \sum x) / n$

LSM - Example

"x" Hours of Sunshine	"y" Ice Creams Sold
2	4
3	5
5	7
7	10
9	15

Let us find the best **m** (slope) and **b** (y-intercept) that suits that data
$y = mx + b$

$m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) - (\sum x)^2)$
$b = (\sum y - m \sum x) / n$

LSM - Example

x	y	x^2	xy
2	4	4	8
3	5	9	15
5	7	25	35
7	10	49	70
9	15	81	135

Step 1: Calculate x^2 and xy

$m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) - (\sum x)^2)$
$b = (\sum y - m \sum x) / n$

LSM - Example

x	y	x^2	xy
2	4	4	8
3	5	9	15
5	7	25	35
7	10	49	70
9	15	81	135
Σx: 26	**Σy: 41**	**Σx^2: 168**	**Σxy: 263**

Step 2: Sum all the columns

$m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) – (\sum x)^2)$
$b = (\sum y - m \sum x) / n$

LSM - Example

x	y	x^2	xy
2	4	4	8
3	5	9	15
5	7	25	35
7	10	49	70
9	15	81	135
Σx: 26	**Σy: 41**	**Σx^2: 168**	**Σxy: 263**

$$\mathbf{m} = \frac{N\,\Sigma(xy) - \Sigma x\,\Sigma y}{N\,\Sigma(x^2) - (\Sigma x)^2}$$

$$= \frac{5 \times 263 - 26 \times 41}{5 \times 168 - 26^2}$$

$$= \frac{1315 - 1066}{840 - 676}$$

$$= \frac{249}{164} = 1{,}5183...$$

$$\mathbf{b} = \frac{\Sigma y - m\,\Sigma x}{N}$$

$$= \frac{41 - 1{,}5183 \times 26}{5}$$

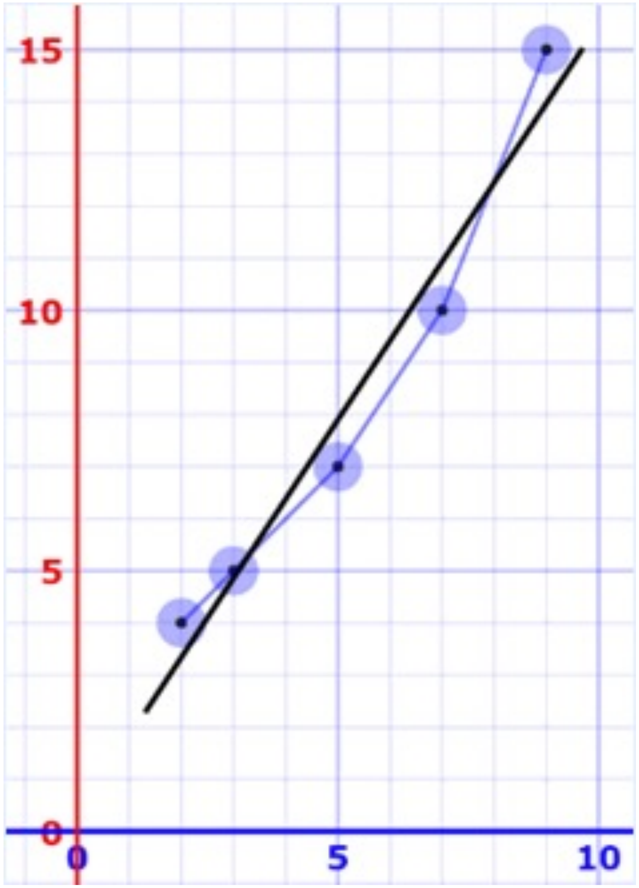
$$= 0{,}3049...$$

Step 3: Calcualte the slope and the intercept with N = 5

$m = (n \sum(xy) - \sum x \sum y) / (n \sum(x^2) - (\sum x)^2)$
$b = (\sum y - m \sum x) / n$

LSM - Example

x	y	y = 1,518x + 0,305	error
2	4	3,34	−0,66
3	5	4,86	−0,14
5	7	7,89	0,89
7	10	10,93	0,93
9	15	13,97	−1,03



Step 4: test y = 1,518x + 0,305

If x = 8 then we expect to sell 12,45 ice creams

Alternative Fitting Methods

- Linear regressions fitted using gradient descent can benefit from some regularizations.
- However, they can be fitted in other ways, such as by minimizing a penalized version of the least squares cost function as in **ridge regression** (L2-norm penalty) and **lasso** (L1-norm penalty).
- **Tikhonov** regularization, also known as *ridge regression*, is a method of regularization of ill-posed problems particularly useful to mitigate the multicollinearity, which commonly occurs in models with large numbers of parameters.
- **Lasso** (least absolute shrinkage and selection operator) performs both variable selection and regularization in order to enhance the prediction accuracy and interpretability of the statistical model it produces.

Multicollinearity: is a phenomenon in which one predictor variable in a multiple regression model can be linearly predicted from the others with a substantial degree of accuracy. In this situation, the coefficient estimates of the multiple regression may change erratically in response to small changes in the model or the data.

Linear Regression Models Objective Functions

- Simple $\beta_0 + \beta_1 x \ - y$
- Multiple $\beta_0 + \sum_i (y_i - \beta_i x_i)^2$
- Ridge $\beta_0 + \sum_i (y_i - \beta_i x_i)^2 \ + \lambda \sum_j {\beta_j}^2$
- Lasso $\beta_0 + \sum_i (y_i - \beta_i x_i)^2 \ + \lambda \sum_j |\beta_j|$

regularization

Evaluating Regression

- **Coefficient of determination** R^2
  - is the proportion of the variance in the dependent variable that is predictable from the independent variable(s)

$$R^2(y,\hat{y}) = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

hat means predicted

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}\epsilon_i^2$

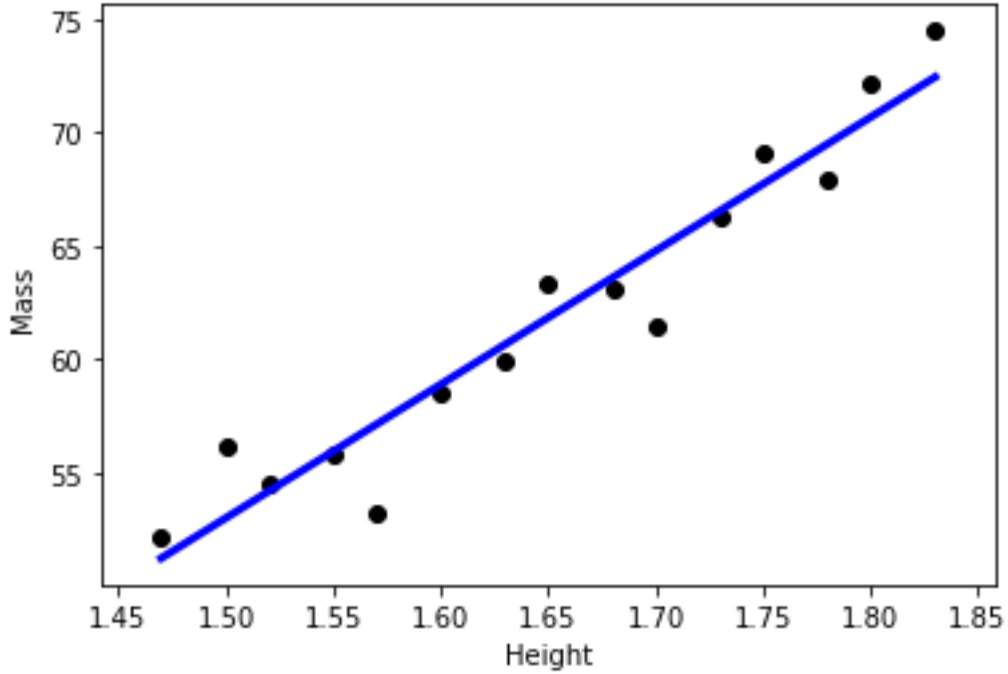
- **Mean Squared/Absolute Error** MSE/MAE
  - a risk metric corresponding to the expected value of the squared (quadratic)/absolute error or loss

$$\text{MSE}(y,\hat{y}) = \frac{1}{n_{\text{samples}}}\sum_{i=0}^{n_{\text{samples}}-1}(y_i - \hat{y}_i)^2 \quad \text{MAE}(y,\hat{y}) = \frac{1}{n_{\text{samples}}}\sum_{i=0}^{n_{\text{samples}}-1}|y_i - \hat{y}_i|$$

Example

- Height (m): 1.47, 1.50, 1.52, 1.55, 1.57, 1.60, 1.63, 1.65, 1.68, 1.70, 1.73, 1.75, 1.78, 1.80, 1.83
- Mass (kg): 52.21, 56.12, 54.48, 55.84, 53.20, 58.57, 59.93, 63.29, 63.11, 61.47, 66.28, 69.10, 67.92, 72.19, 74.46

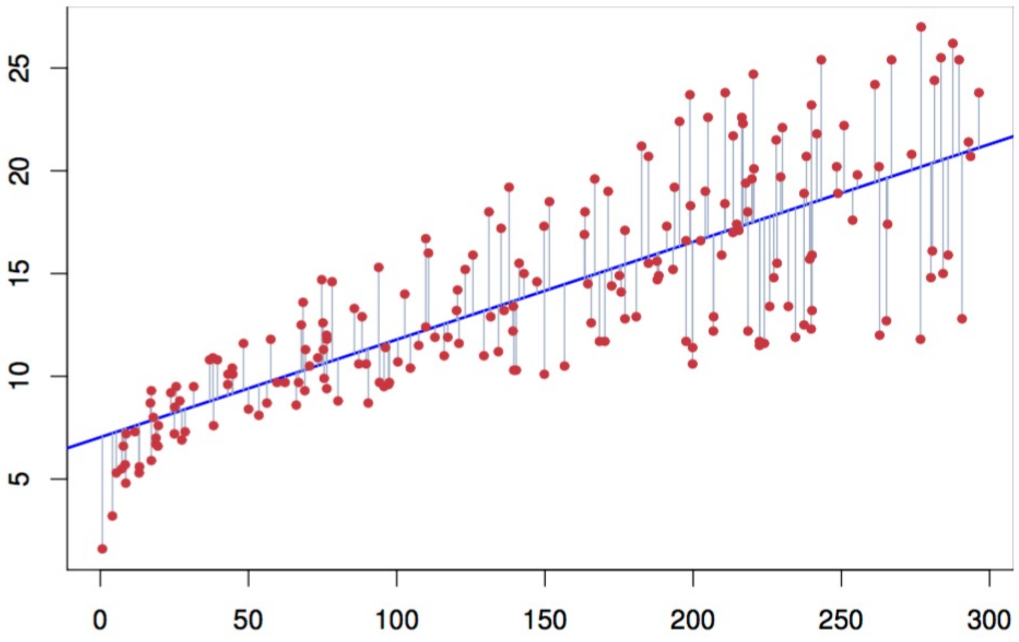
- Intercept: -35.30454824113264
- Coefficient: 58.87472632
- R^2: 0.93
- MSE: 3.40
- MAE: 1.43



Linear Regression Recap

- Linear regression is used to fit a linear model to data where the dependent variable is continuous.
- Given a set of points *(X_i,Y_i)*, we wish to find a linear function (or line in 2 dimensions) that "goes through" these points.
- In general, the points are not exactly aligned.
- The objective is to find the line that best fits the points.

$$Y = \beta_1 X + \beta_0$$



References

- Regression. Appendix D. Introduction to Data Mining.