DATA MINING 2

Maximum Likelihood Estimation

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Contains edited slides from StatQuest

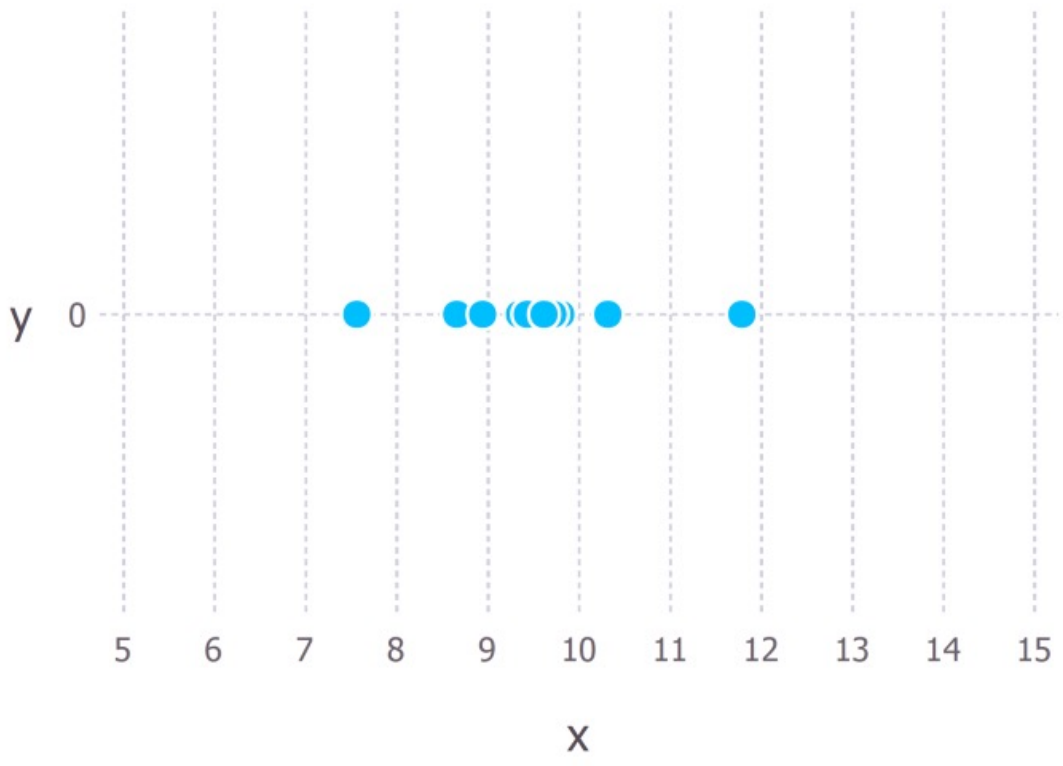
UNIVERSITÀ DI PISA

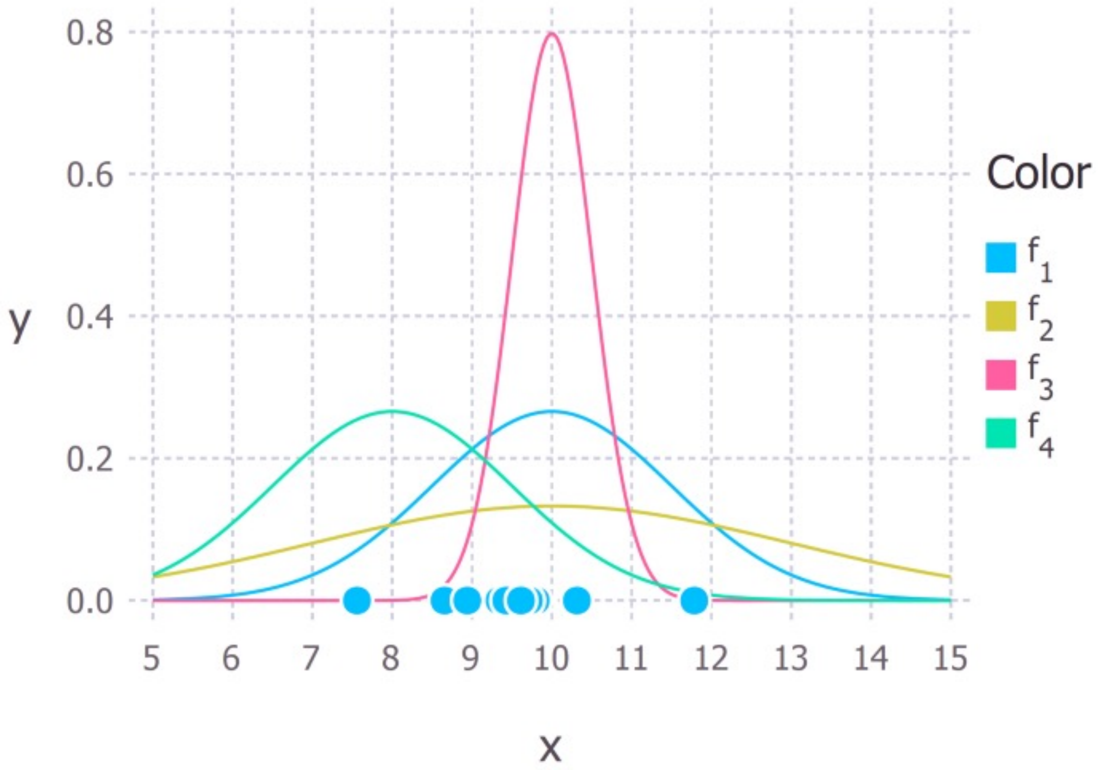
Intuition

- Maximum Likelihood Estimation (MLE) is a method that determines values for the parameters of a model.
- The parameter values are found such that they maximize the **likelihood** that the process described by the model produced the data that were actually observed.

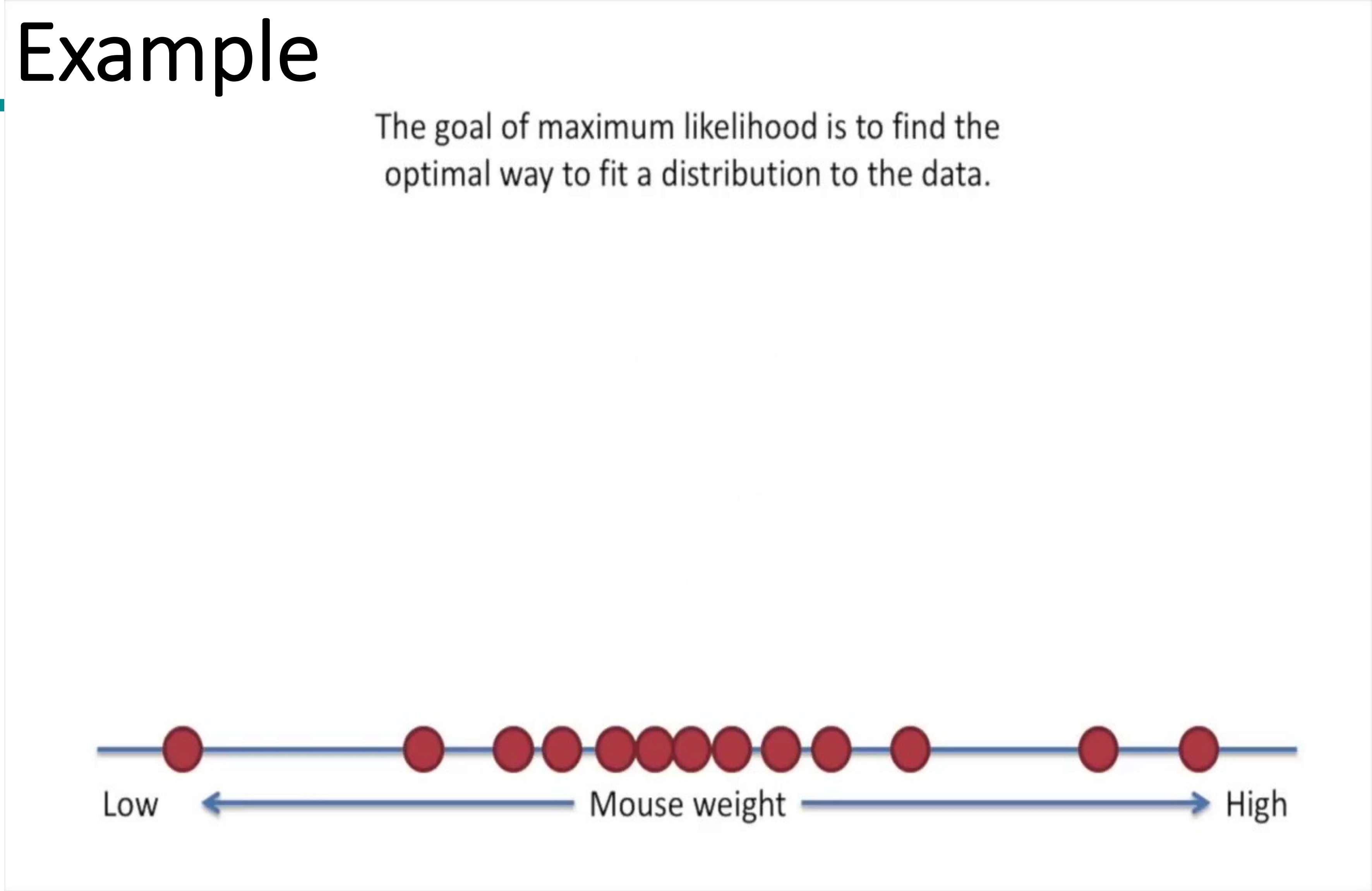
Which model fit best?

- Normal Gaussian distribution
- Parameters: *mean* and *standard deviation*



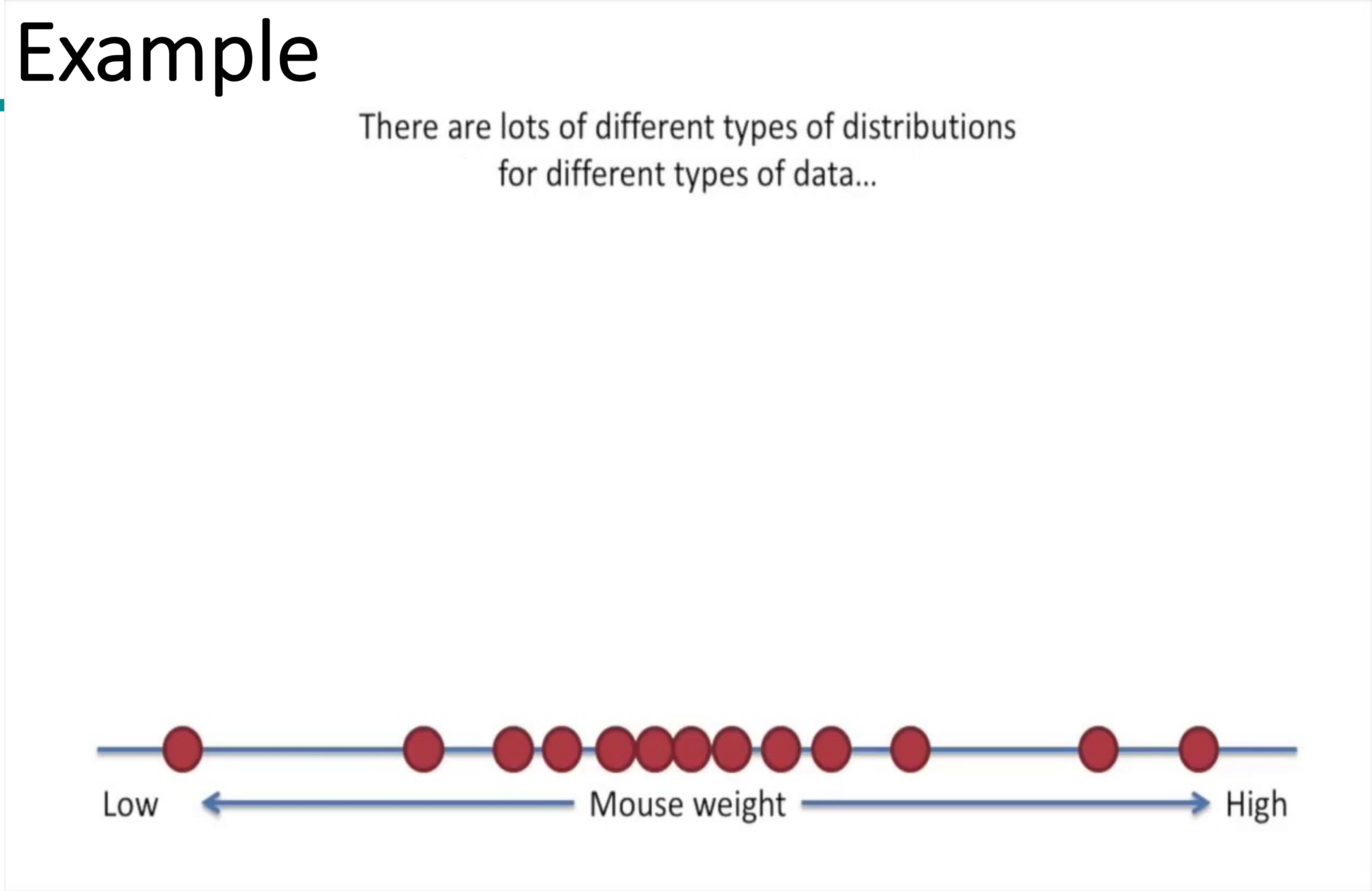


MLE Example



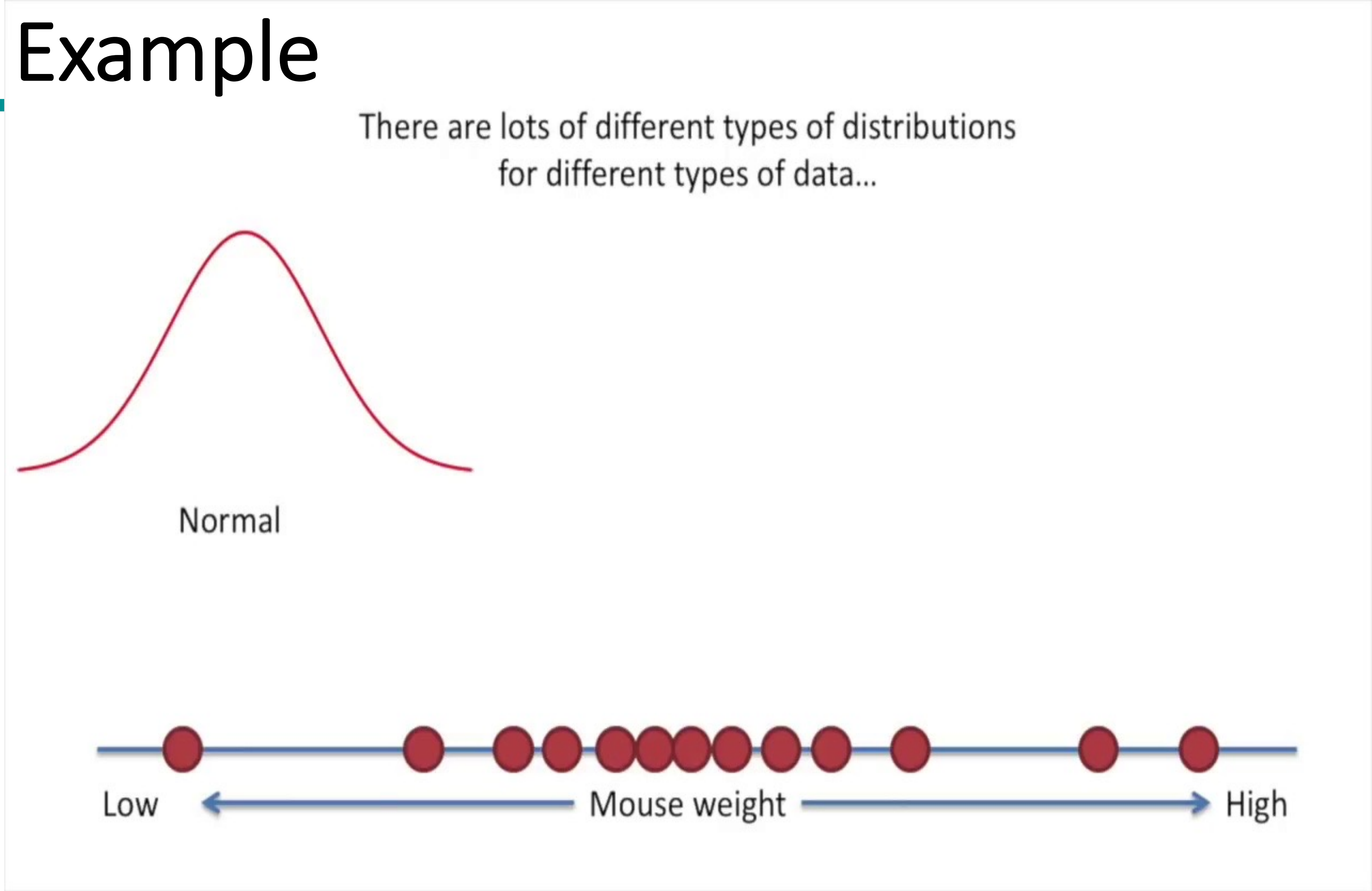
The goal of maximum likelihood is to find the optimal way to fit a distribution to the data.

MLE Example



There are lots of different types of distributions for different types of data...

MLE Example



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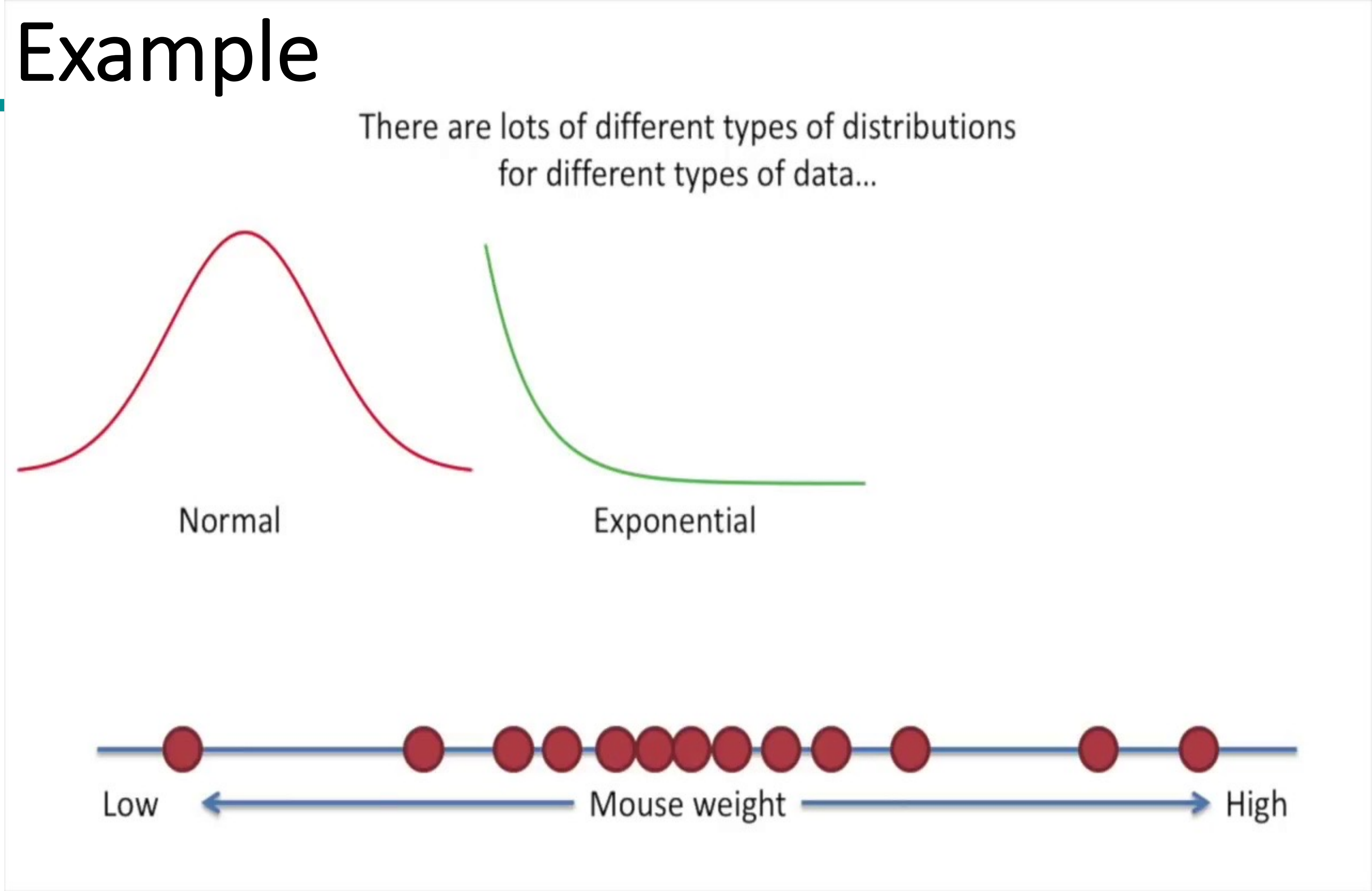
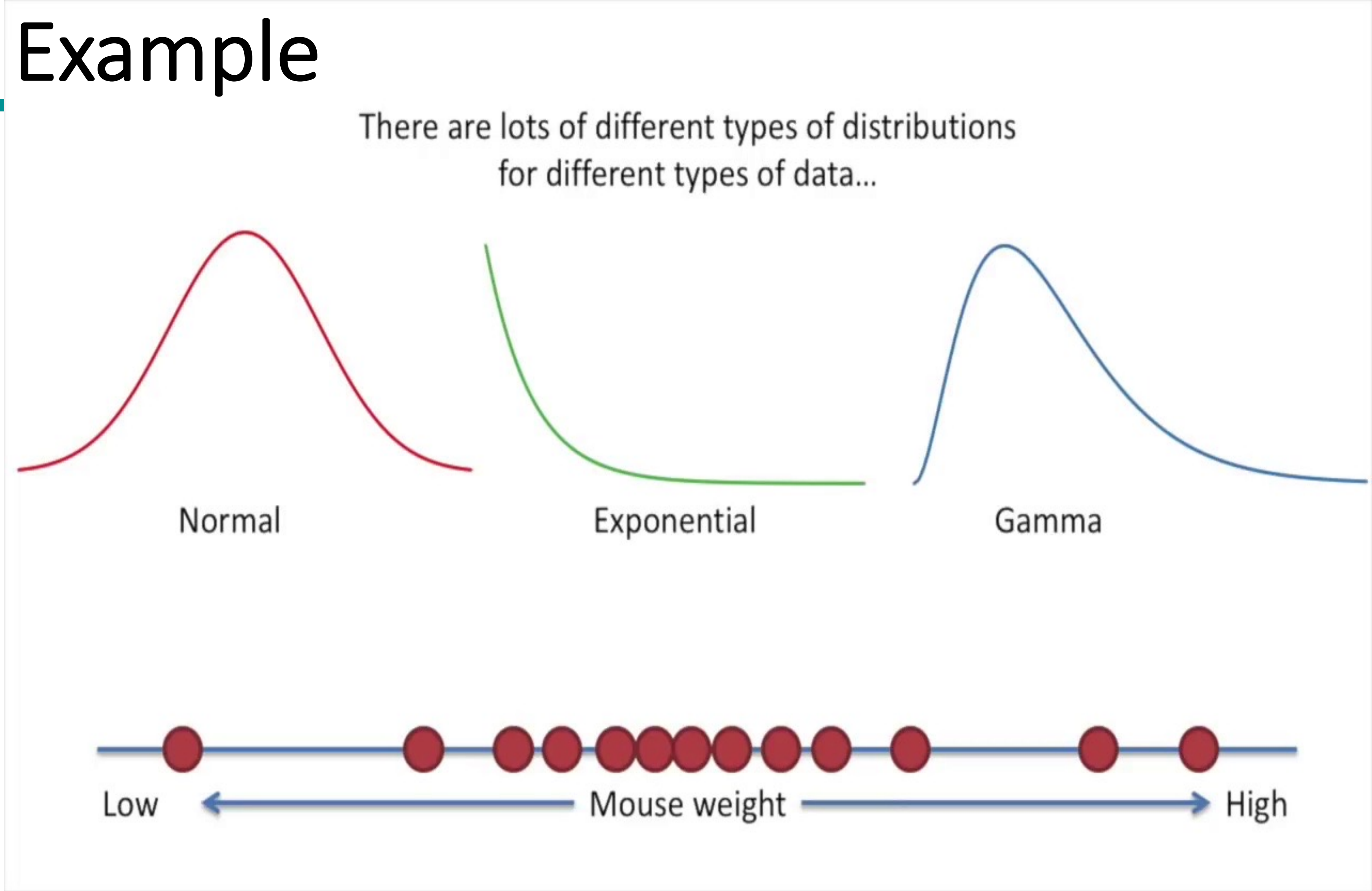


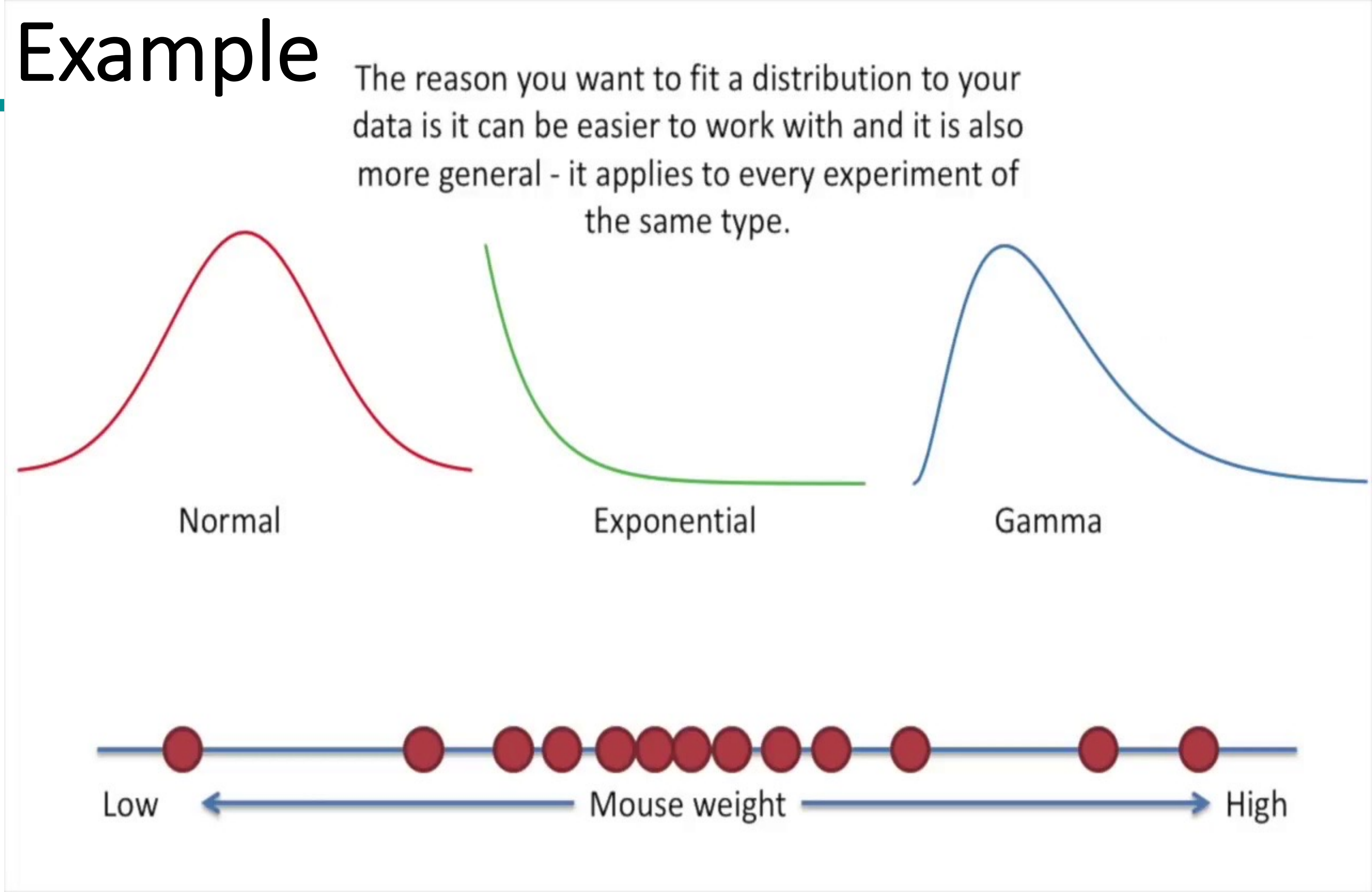
There are lots of different types of distributions for different types of data...

MLE Example



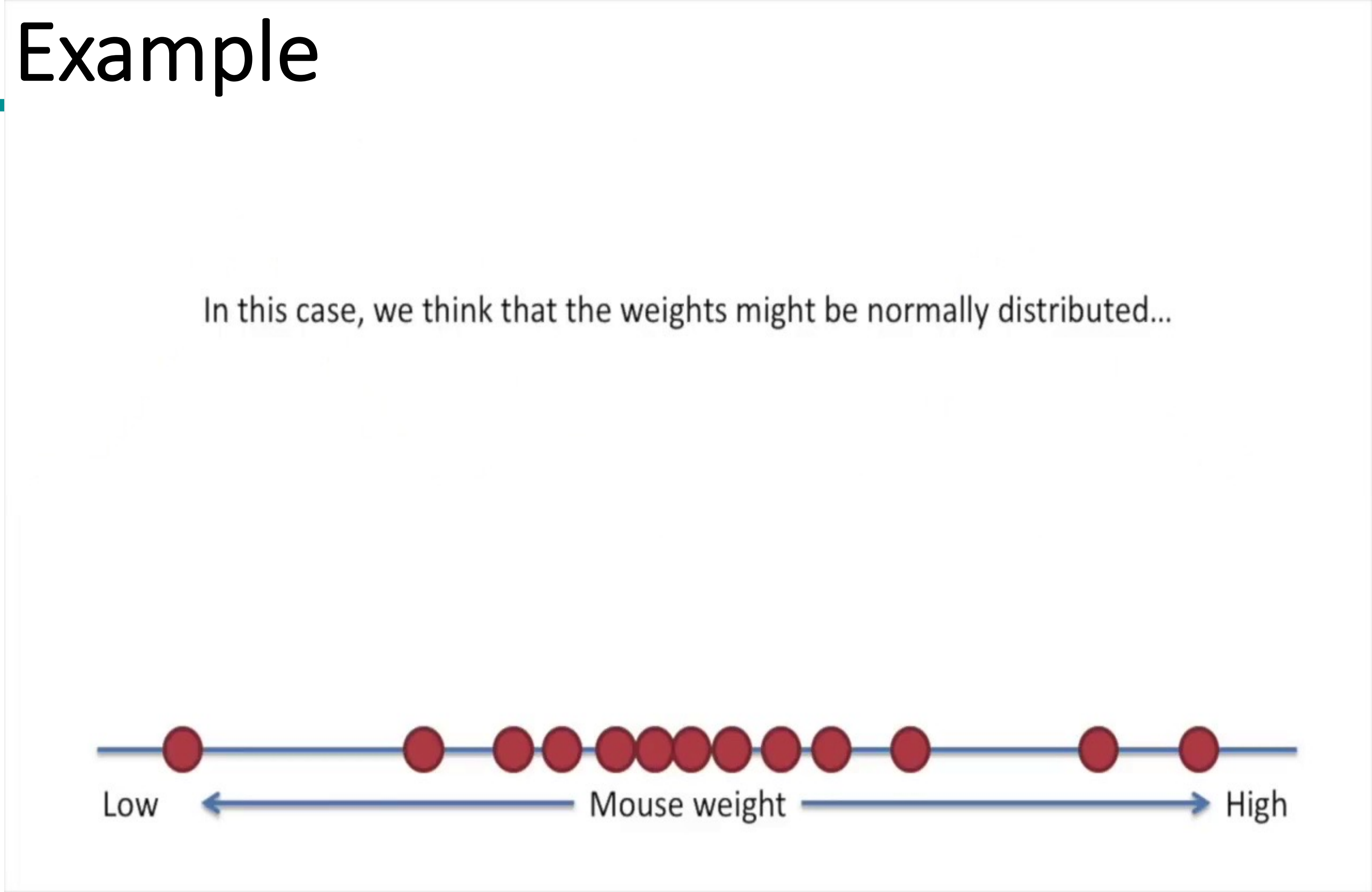
There are lots of different types of distributions for different types of data...

MLE Example



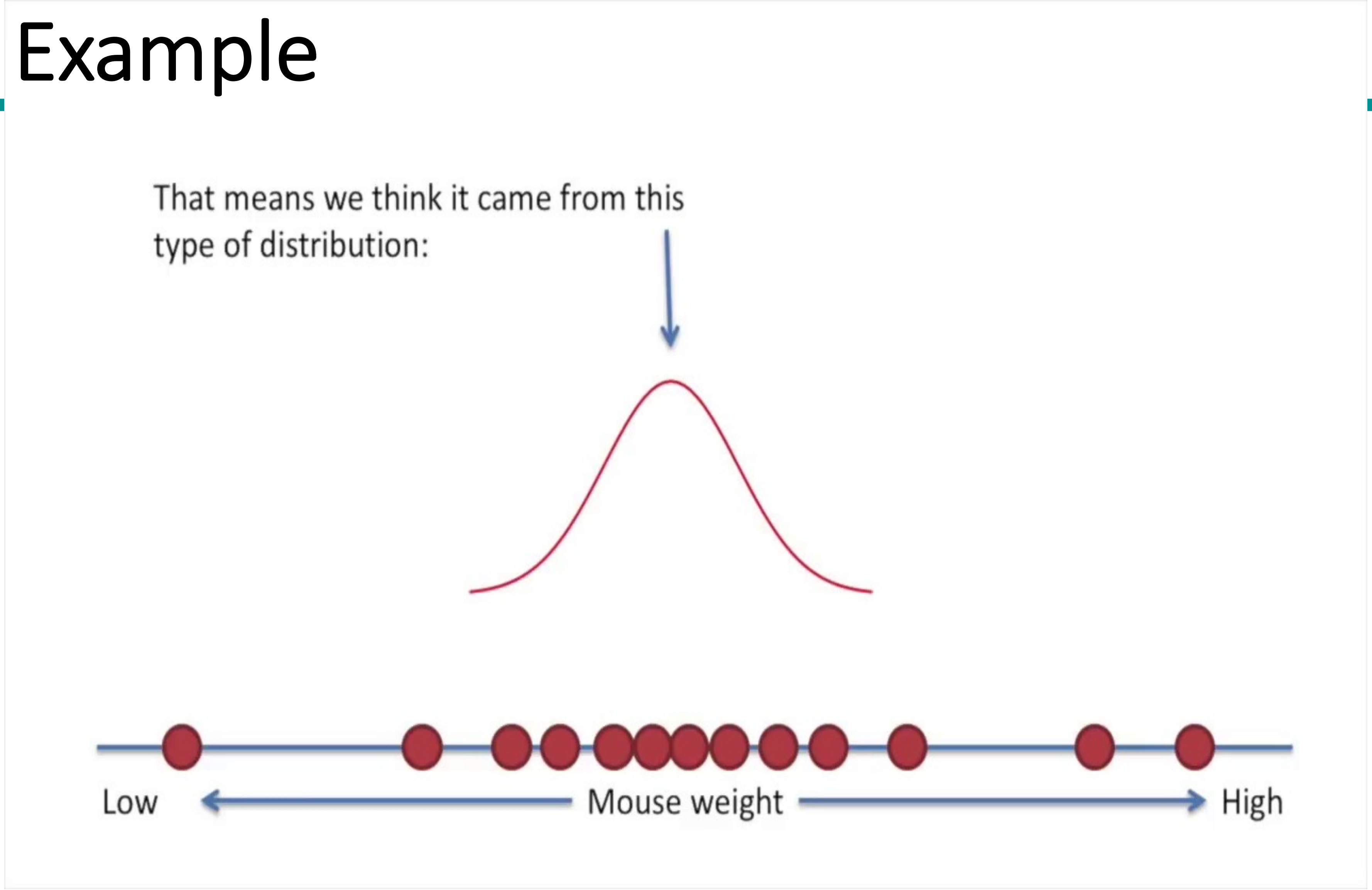
The reason you want to fit a distribution to your data is it can be easier to work with and it is also more general - it applies to every experiment of the same type.

MLE Example



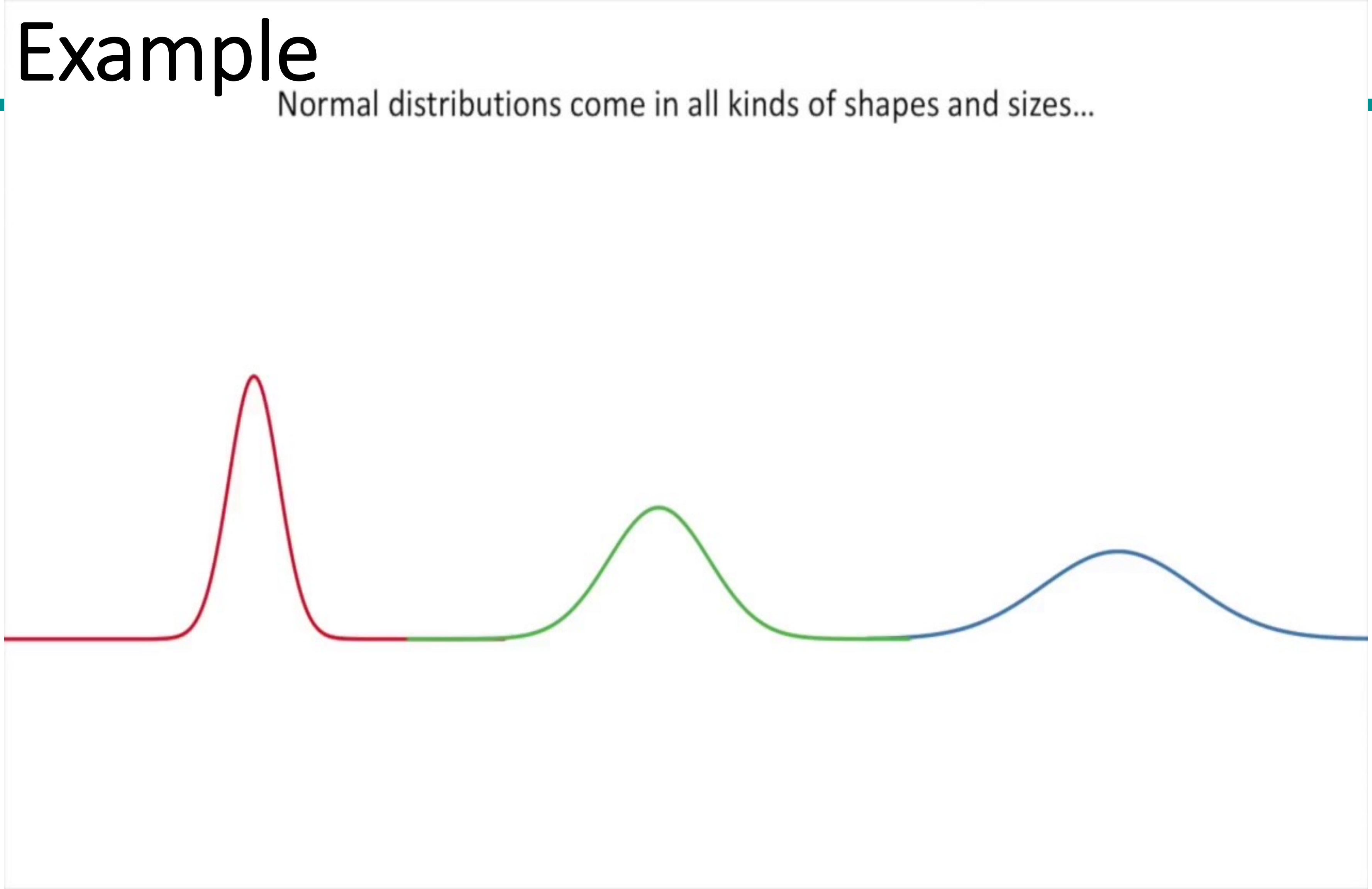
In this case, we think that the weights might be normally distributed...

MLE Example



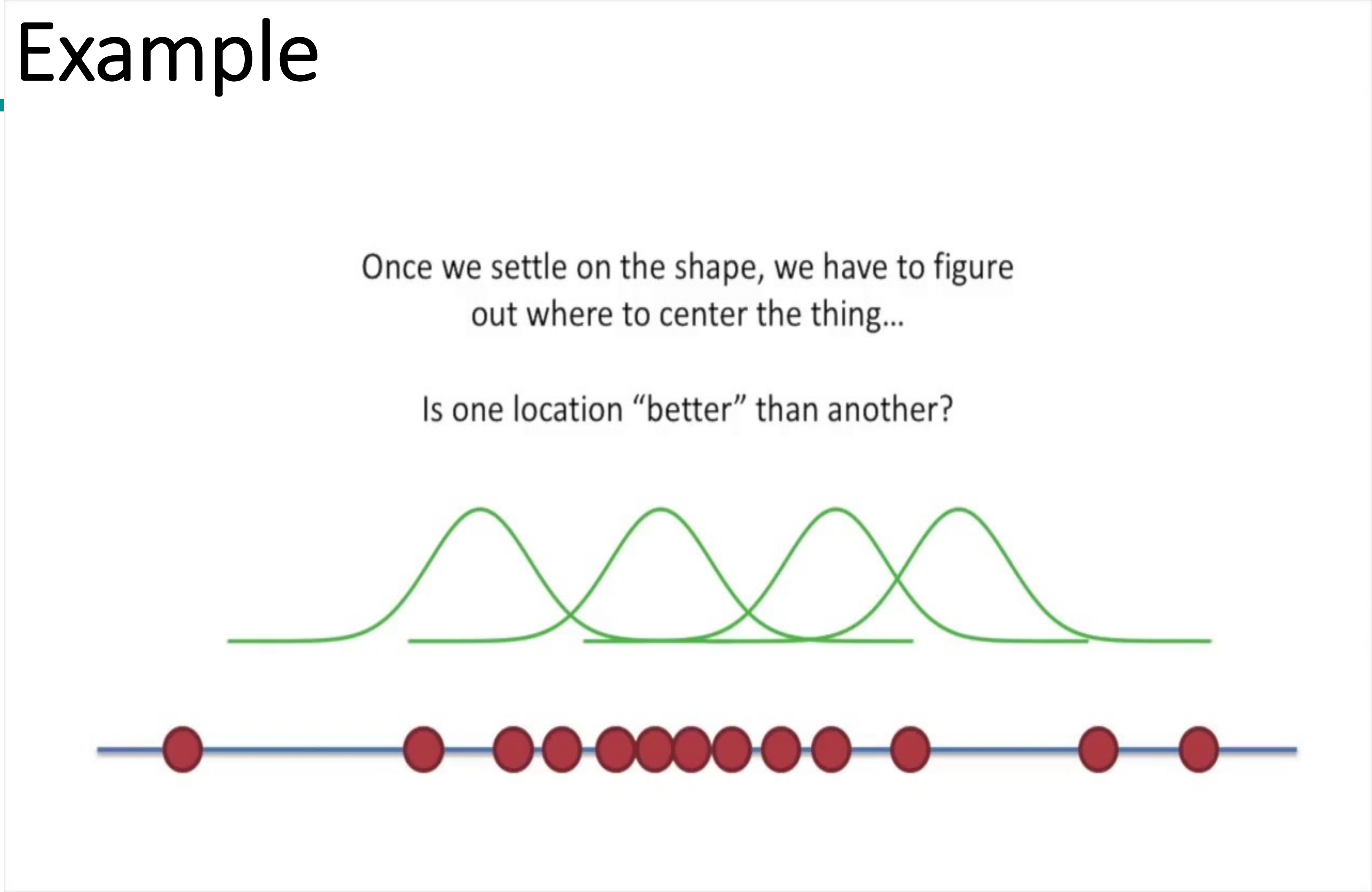
That means we think it came from this type of distribution:

MLE Example



Normal distributions come in all kinds of shapes and sizes...

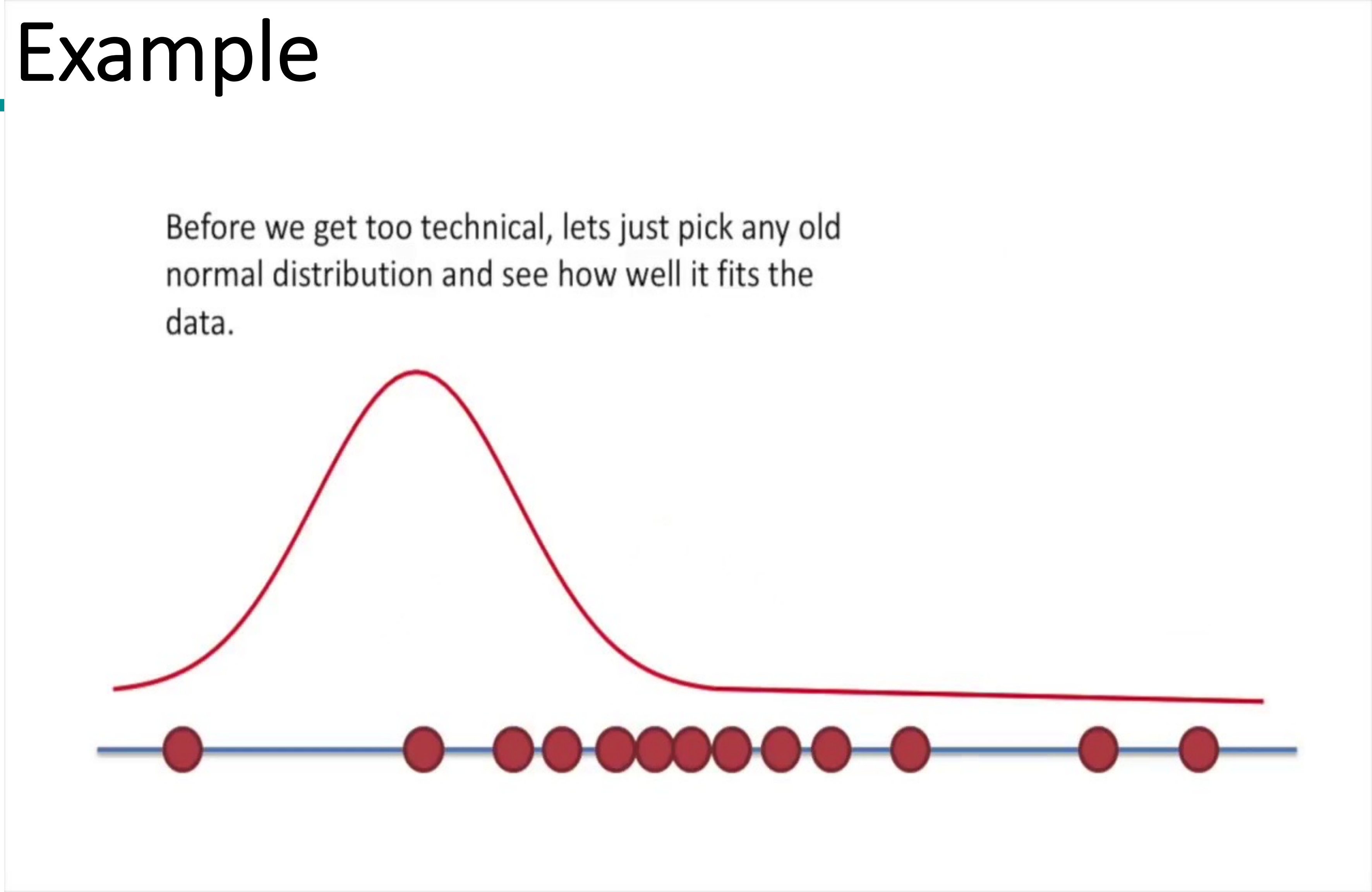
MLE Example



Once we settle on the shape, we have to figure out where to center the thing…

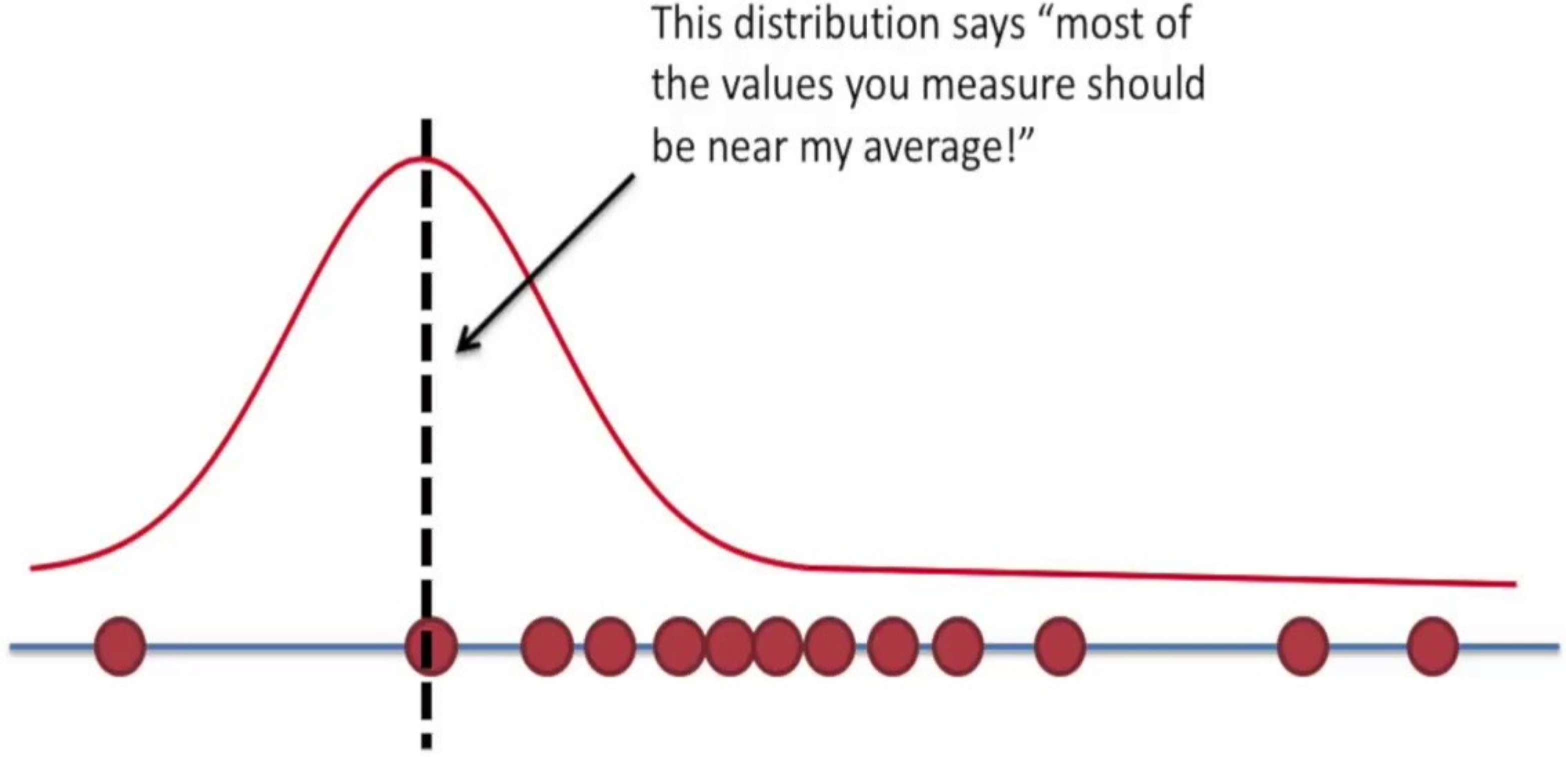
Is one location "better" than another?

MLE Example



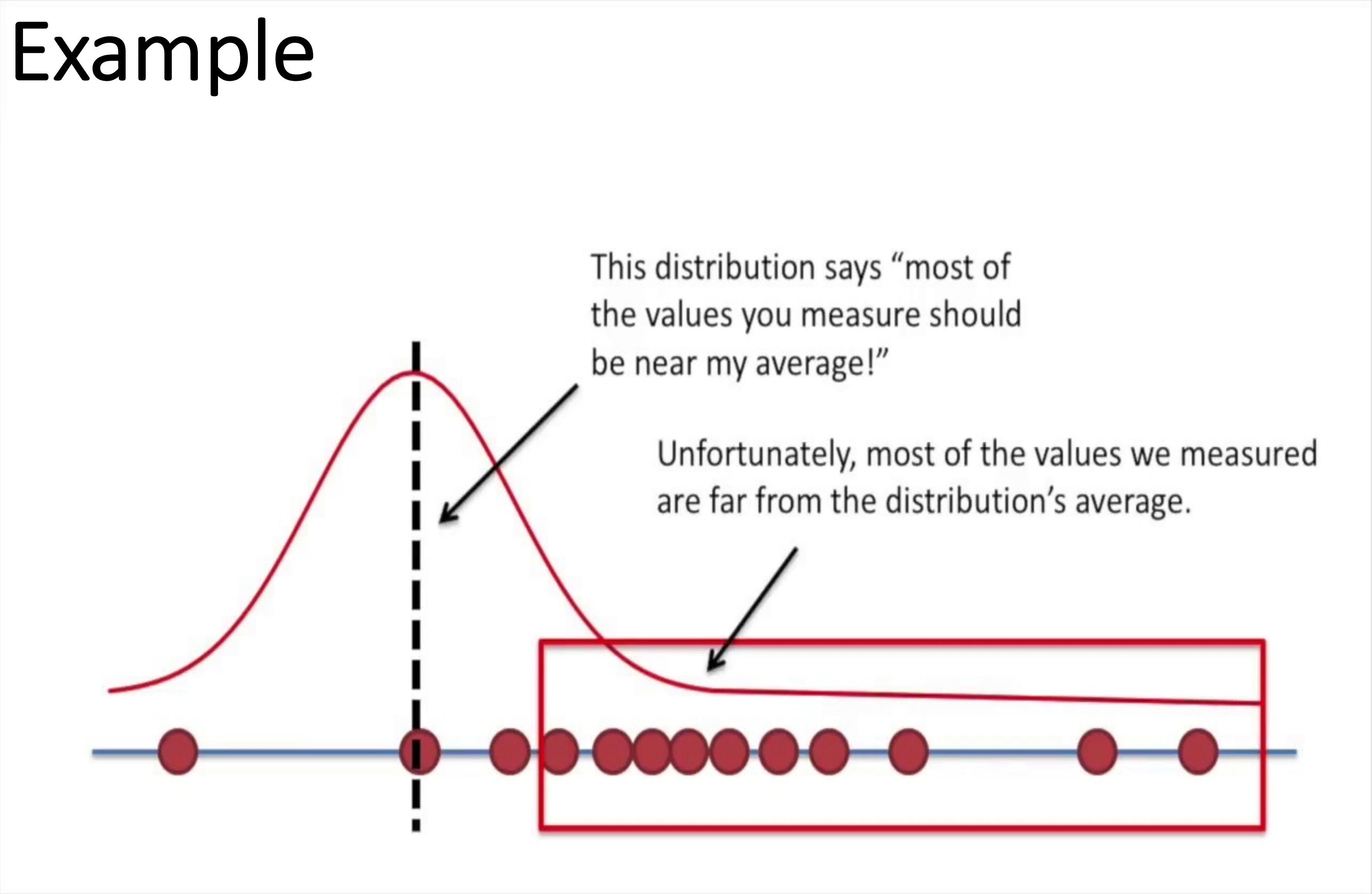
Before we get too technical, lets just pick any old normal distribution and see how well it fits the data.

MLE Example



This distribution says "most of the values you measure should be near my average!"

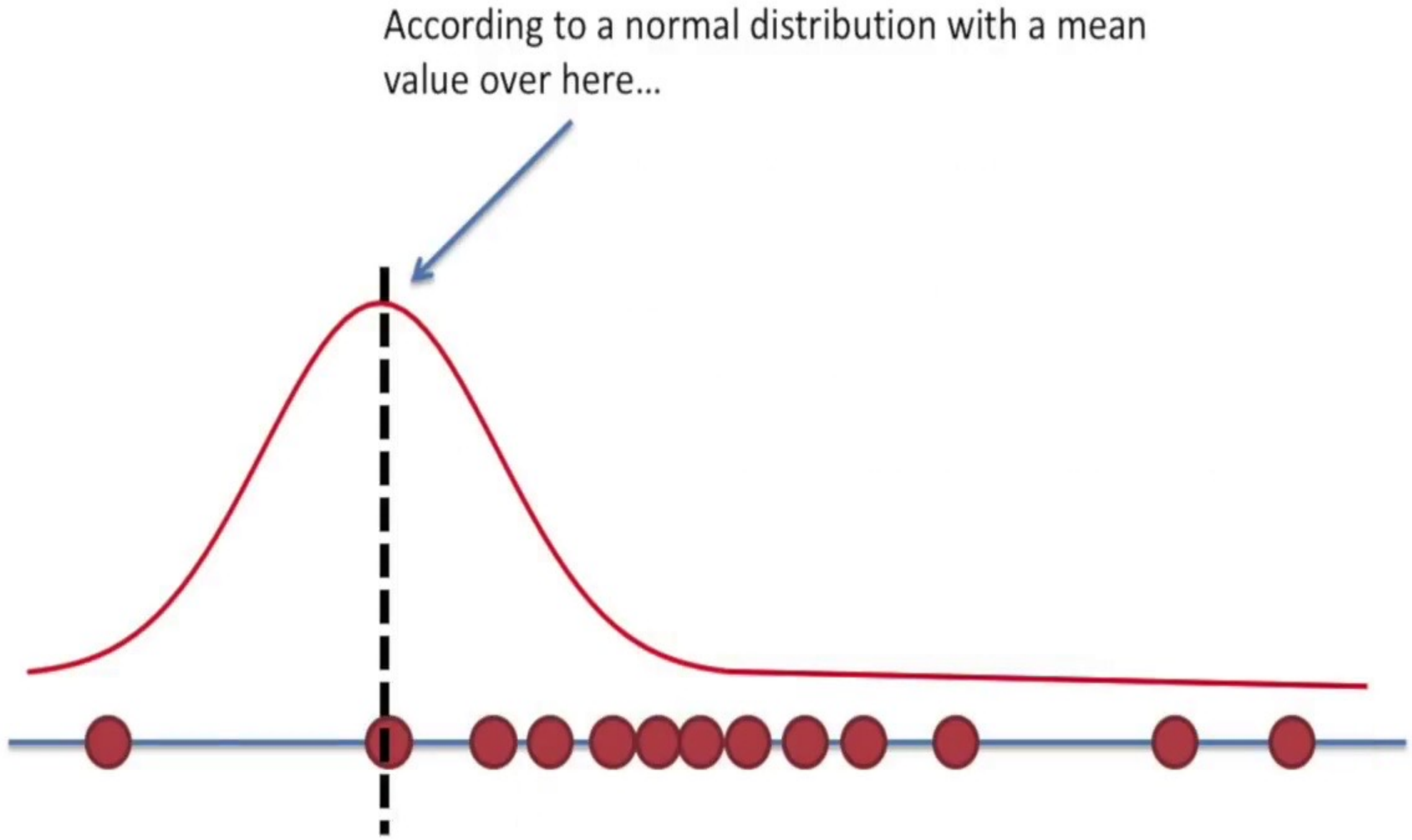
MLE Example



This distribution says "most of the values you measure should be near my average!"

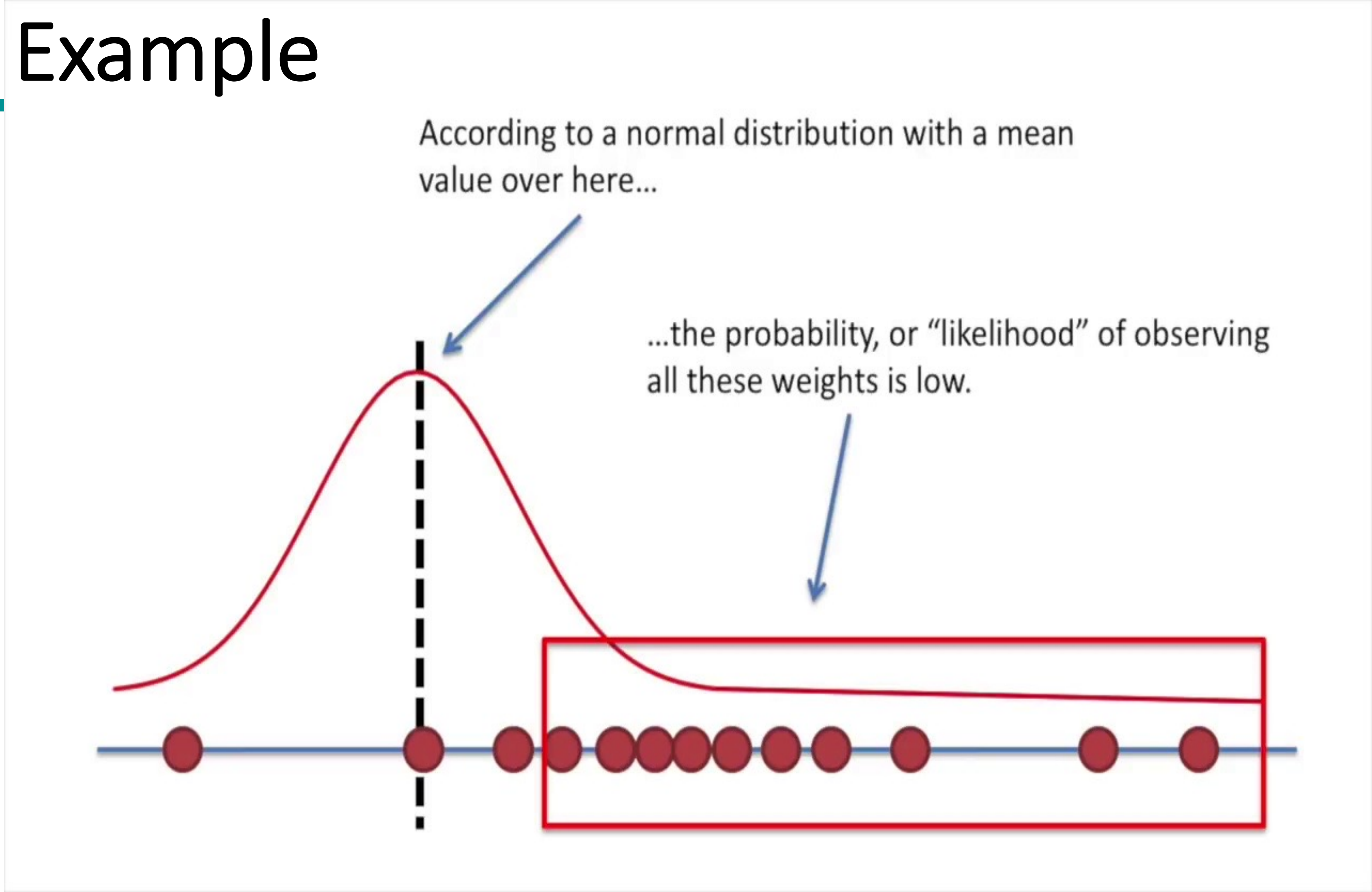
Unfortunately, most of the values we measured are far from the distribution's average.

MLE Example



According to a normal distribution with a mean value over here...

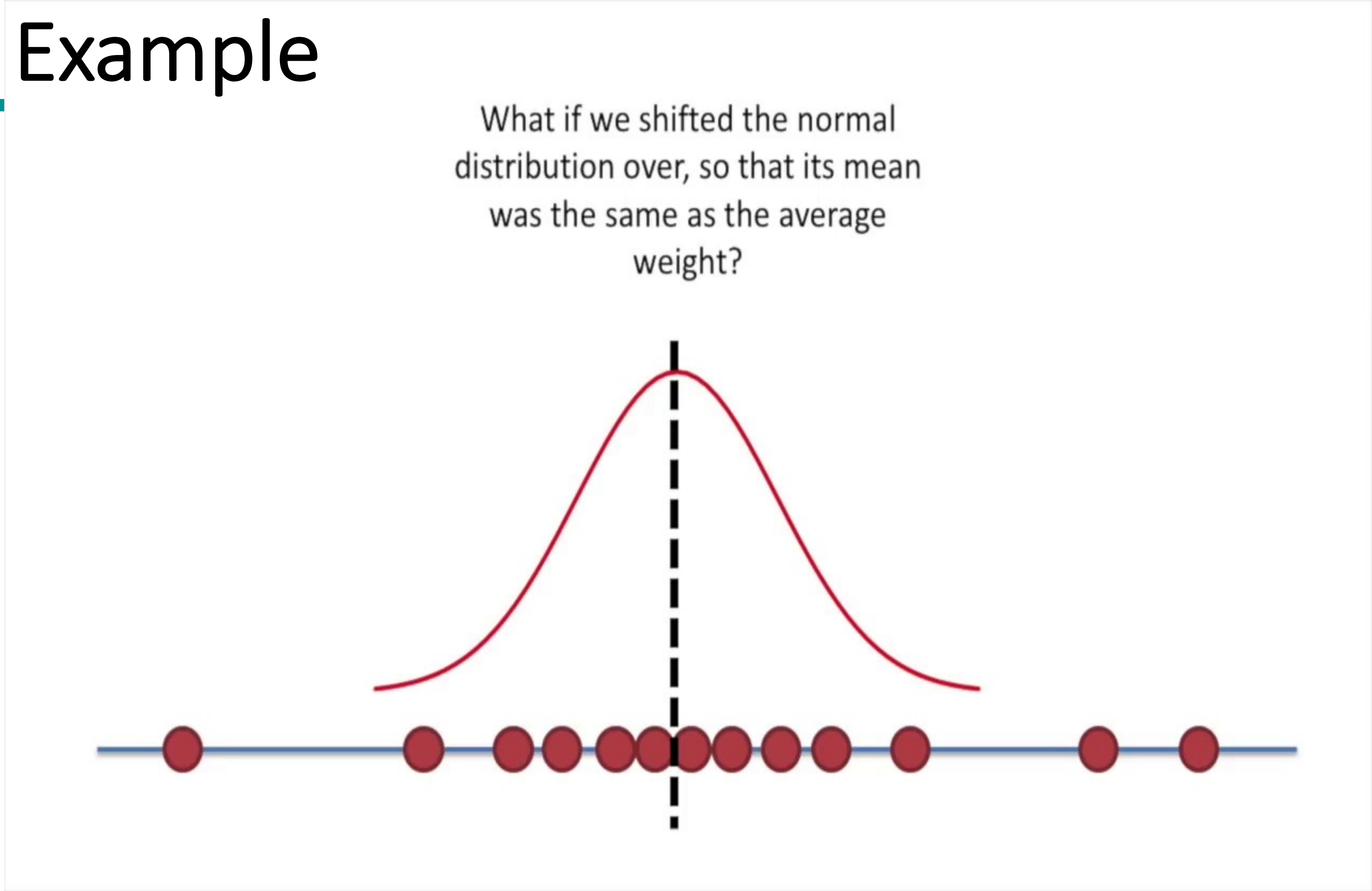
MLE Example



According to a normal distribution with a mean value over here…

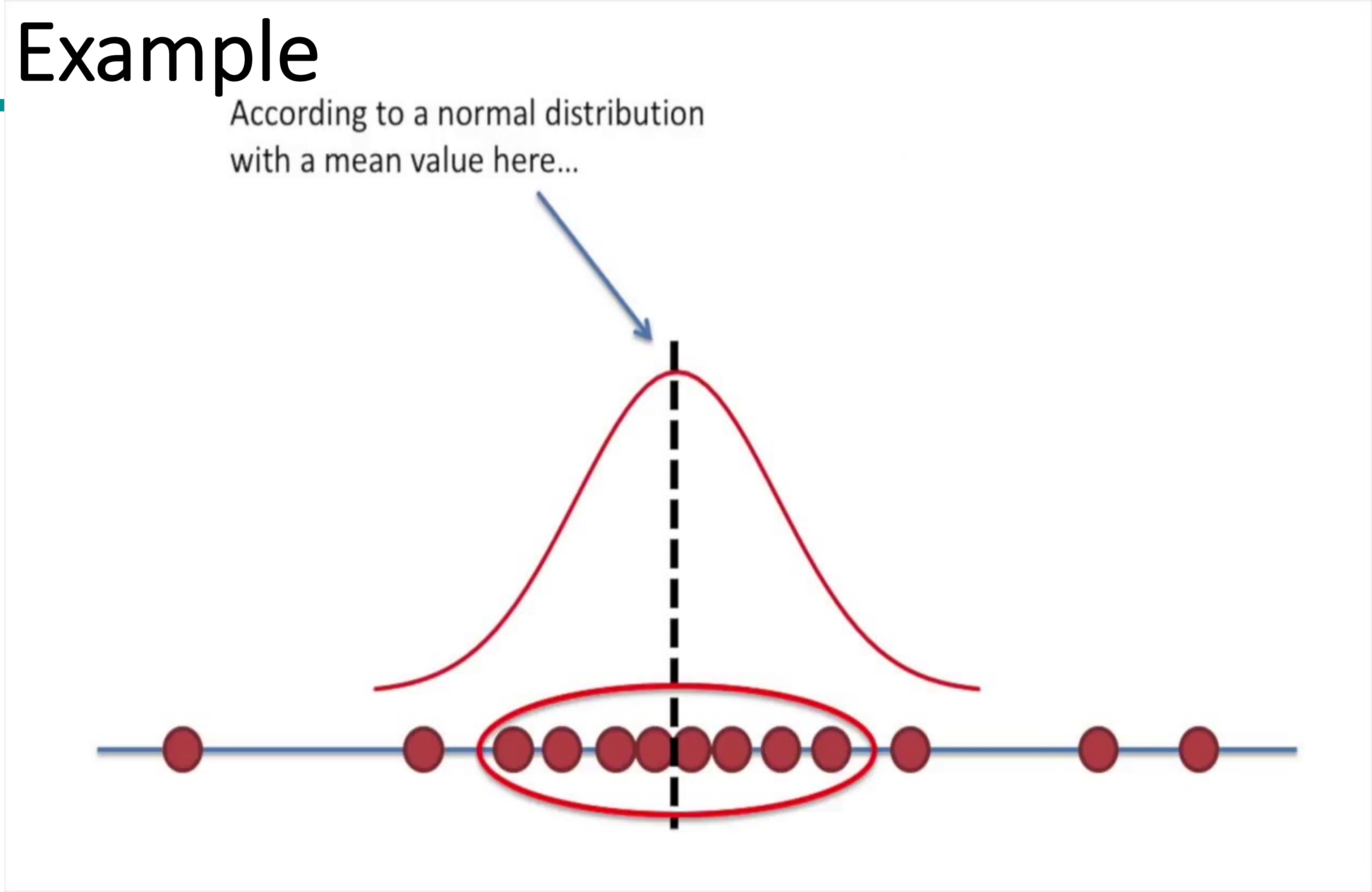
…the probability, or "likelihood" of observing all these weights is low.

MLE Example



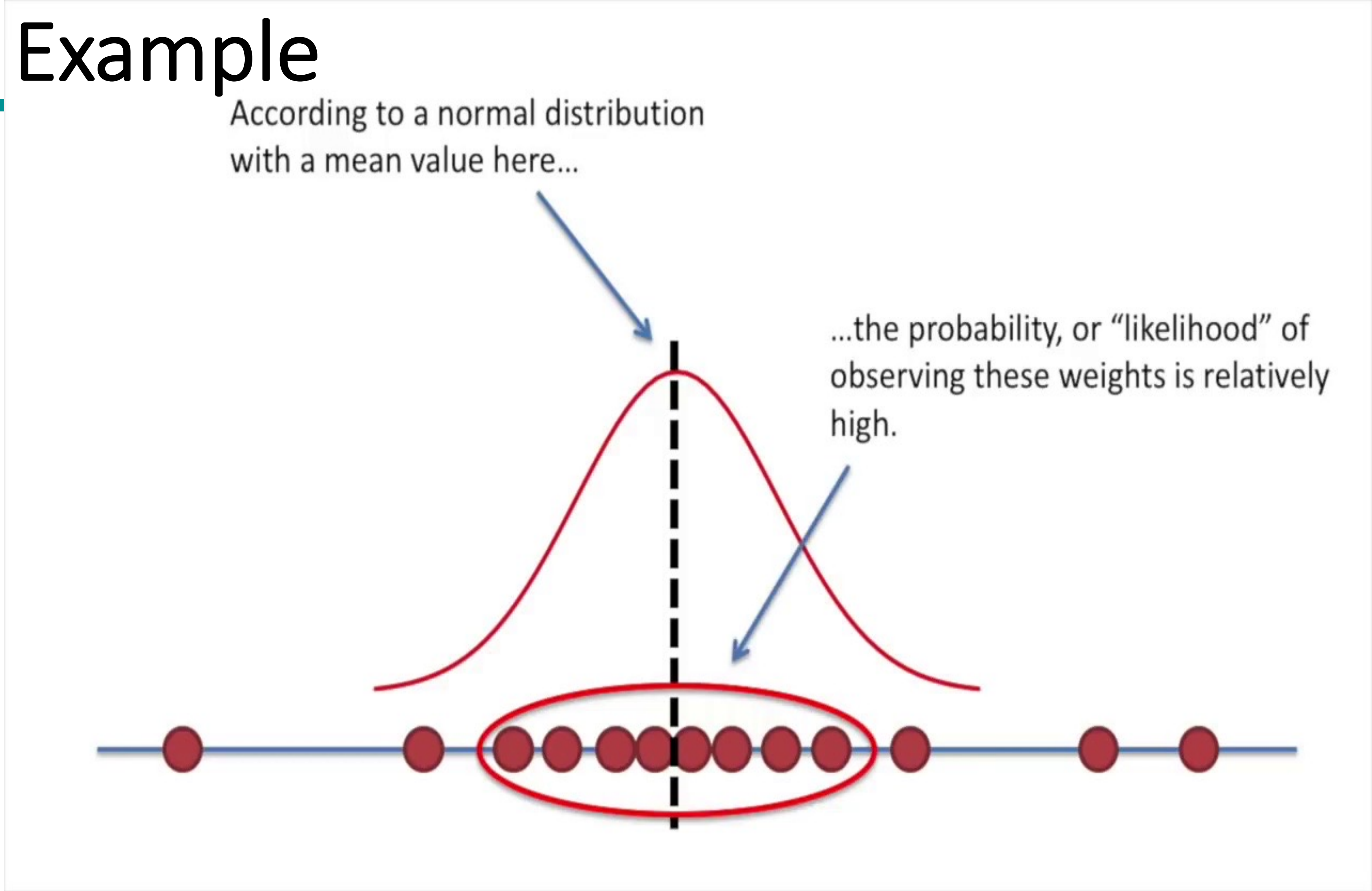
What if we shifted the normal distribution over, so that its mean was the same as the average weight?

MLE Example



According to a normal distribution with a mean value here...

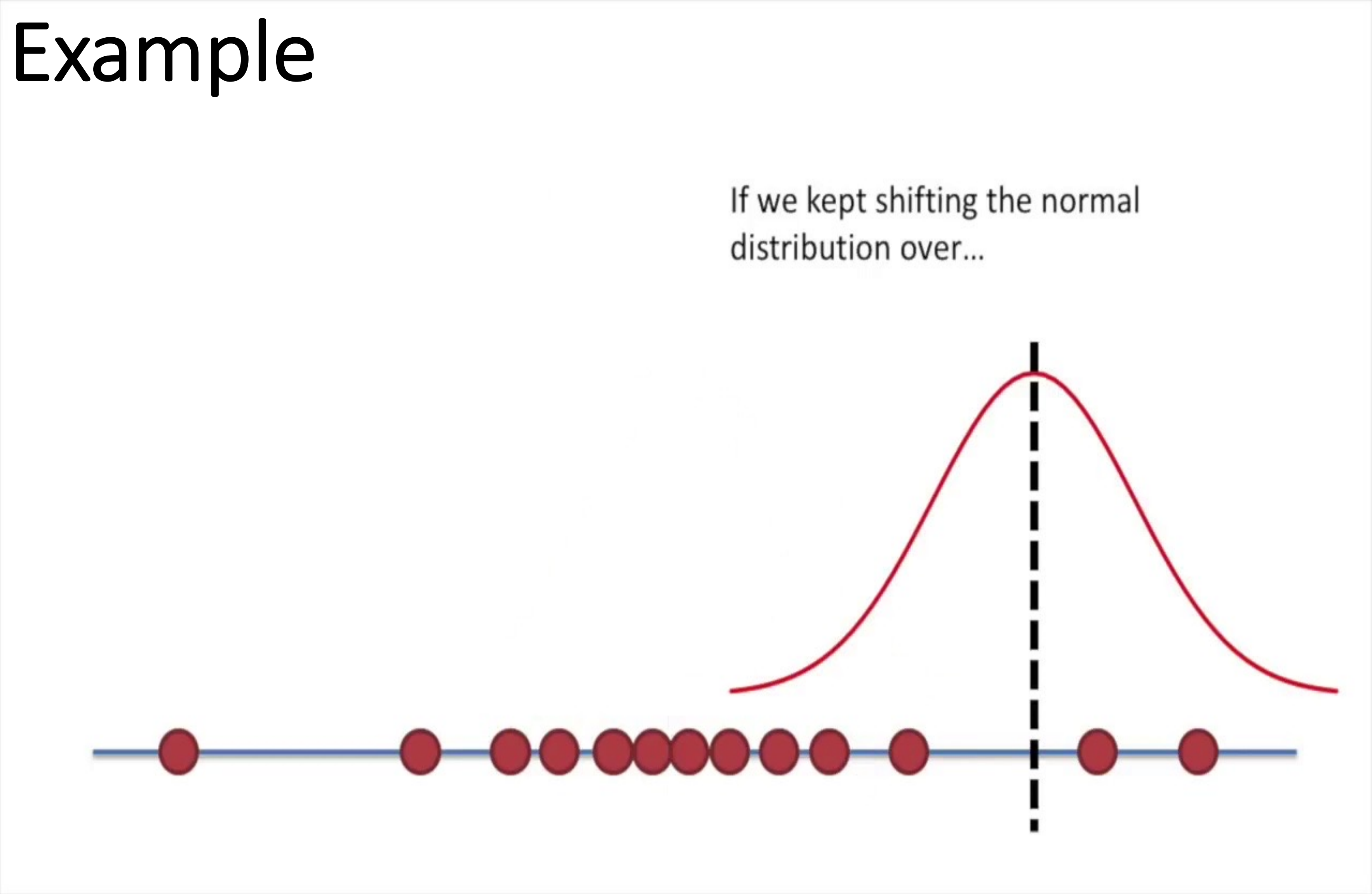
MLE Example



According to a normal distribution with a mean value here...

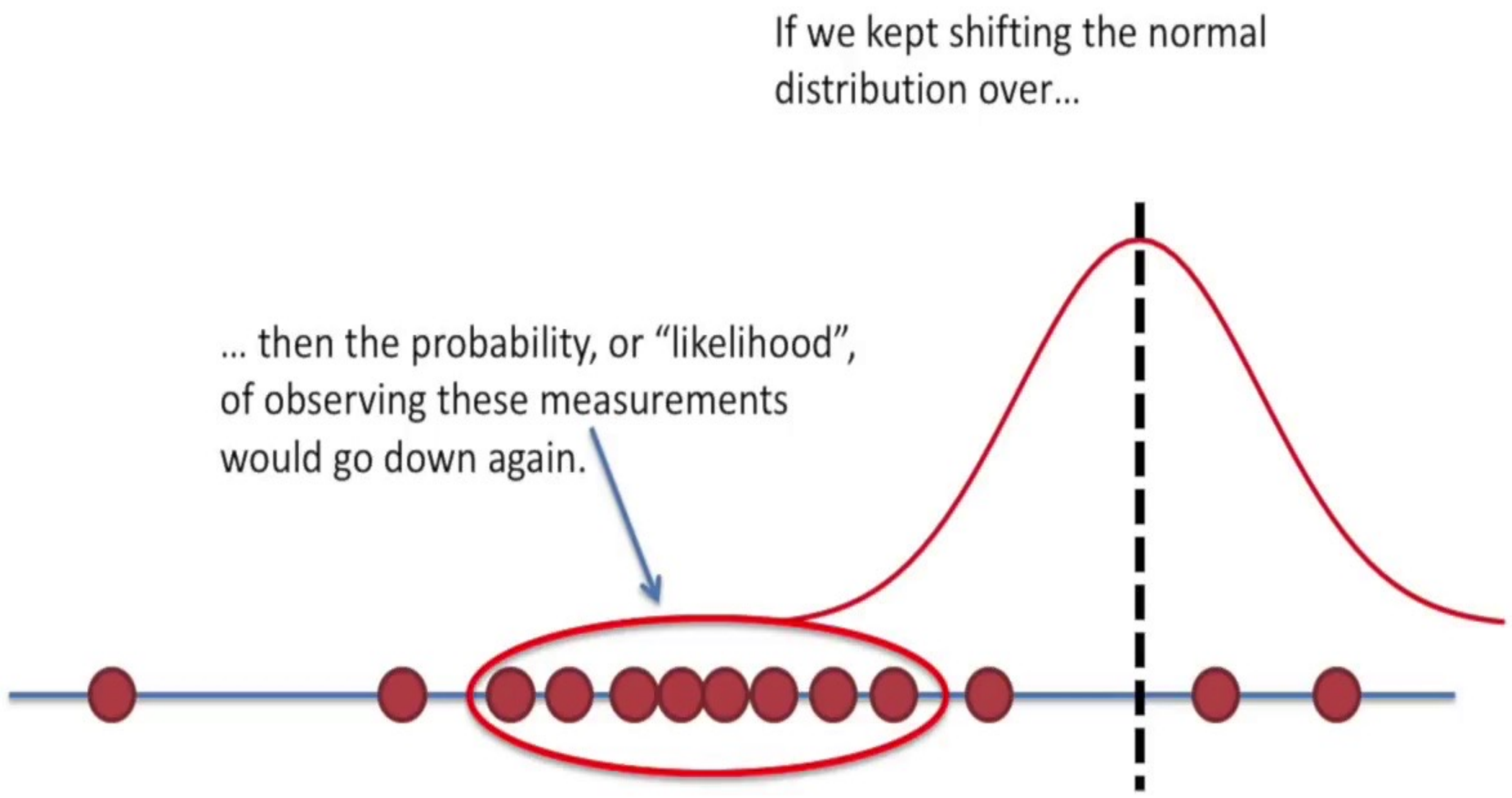
...the probability, or “likelihood” of observing these weights is relatively high.

MLE Example



If we kept shifting the normal distribution over...

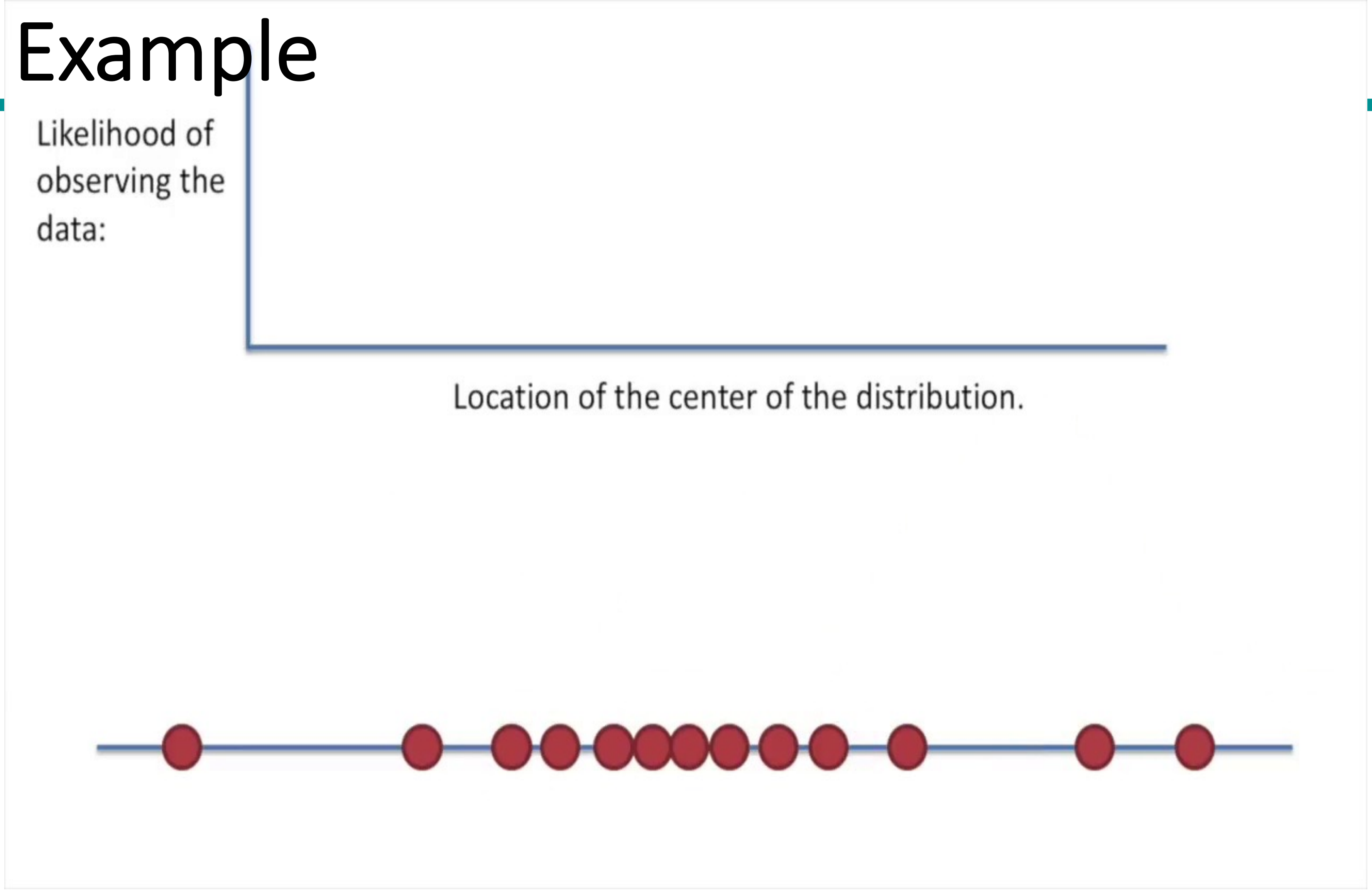
MLE Example



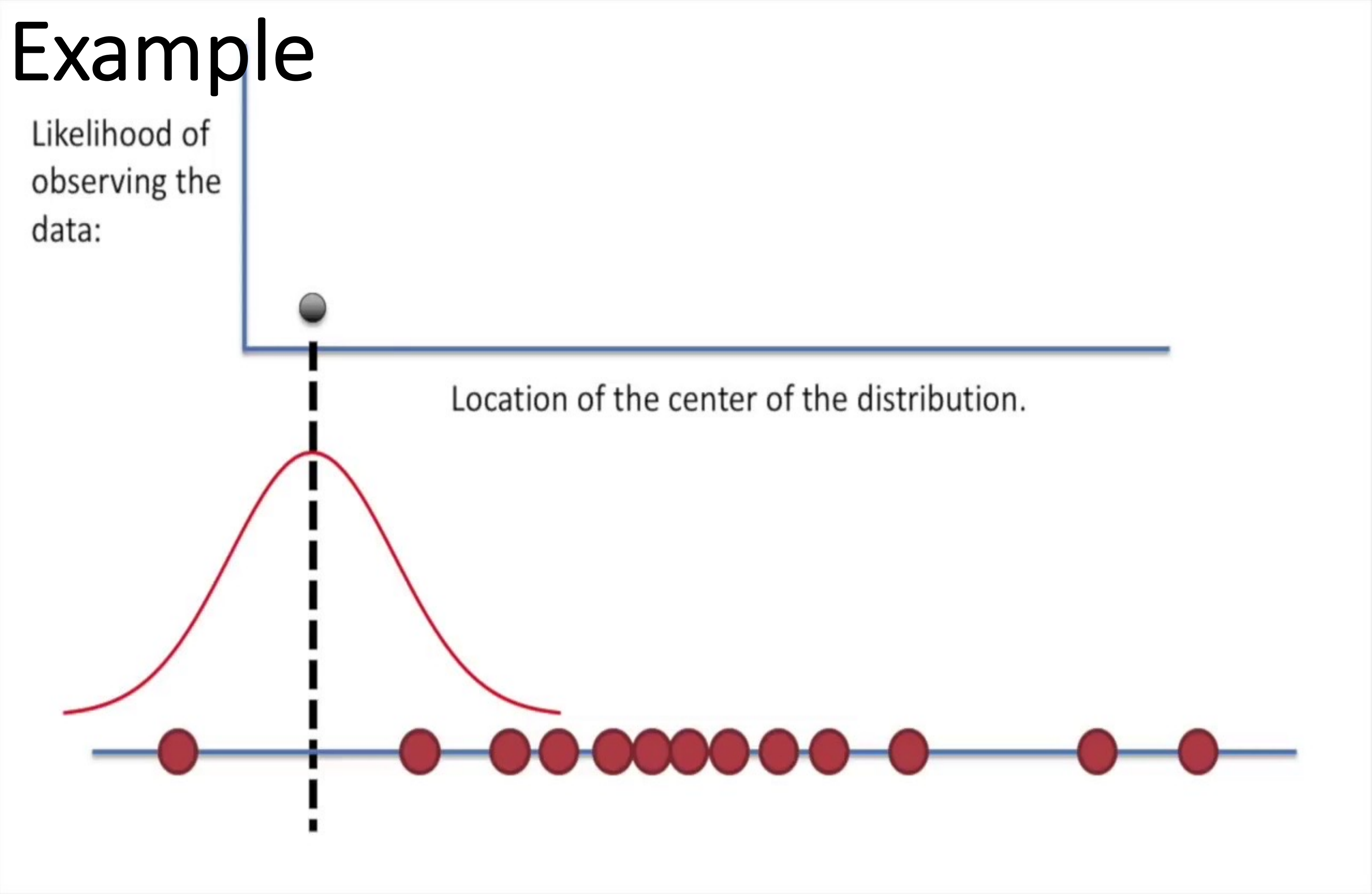
If we kept shifting the normal distribution over…

… then the probability, or “likelihood”, of observing these measurements would go down again.

MLE Example


Likelihood of observing the data:
Location of the center of the distribution.

MLE Example



Likelihood of observing the data:
Location of the center of the distribution.

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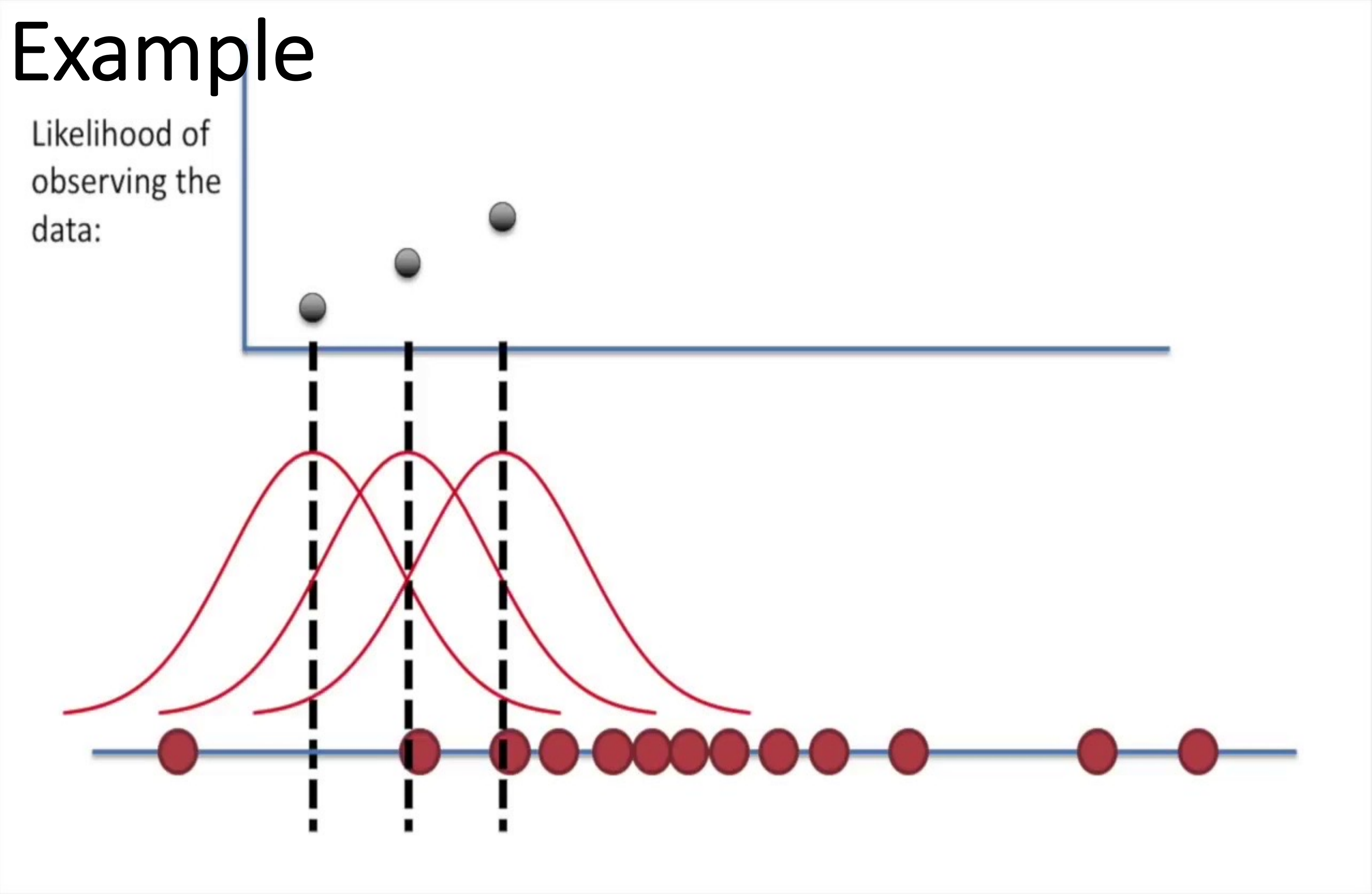

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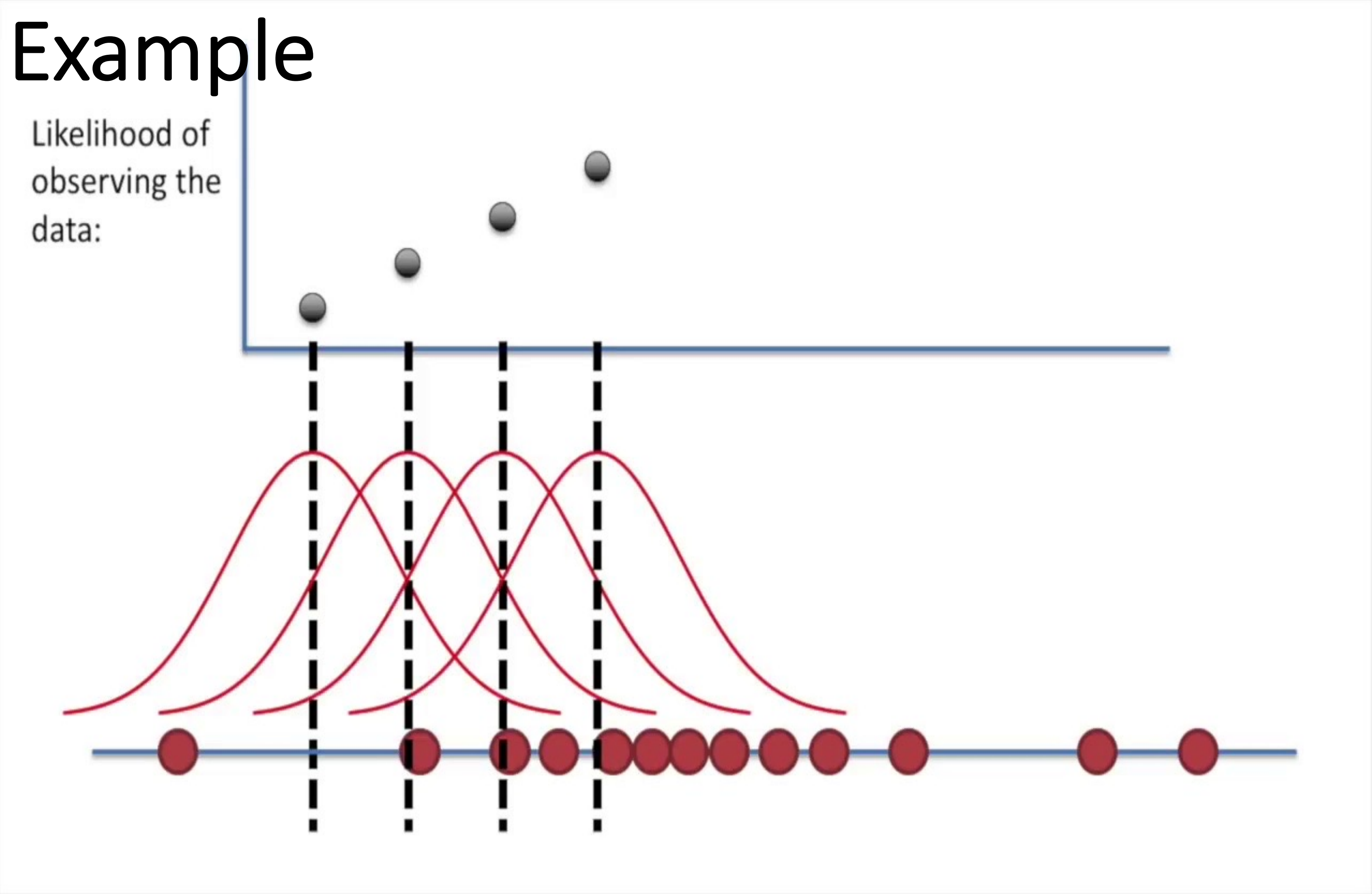

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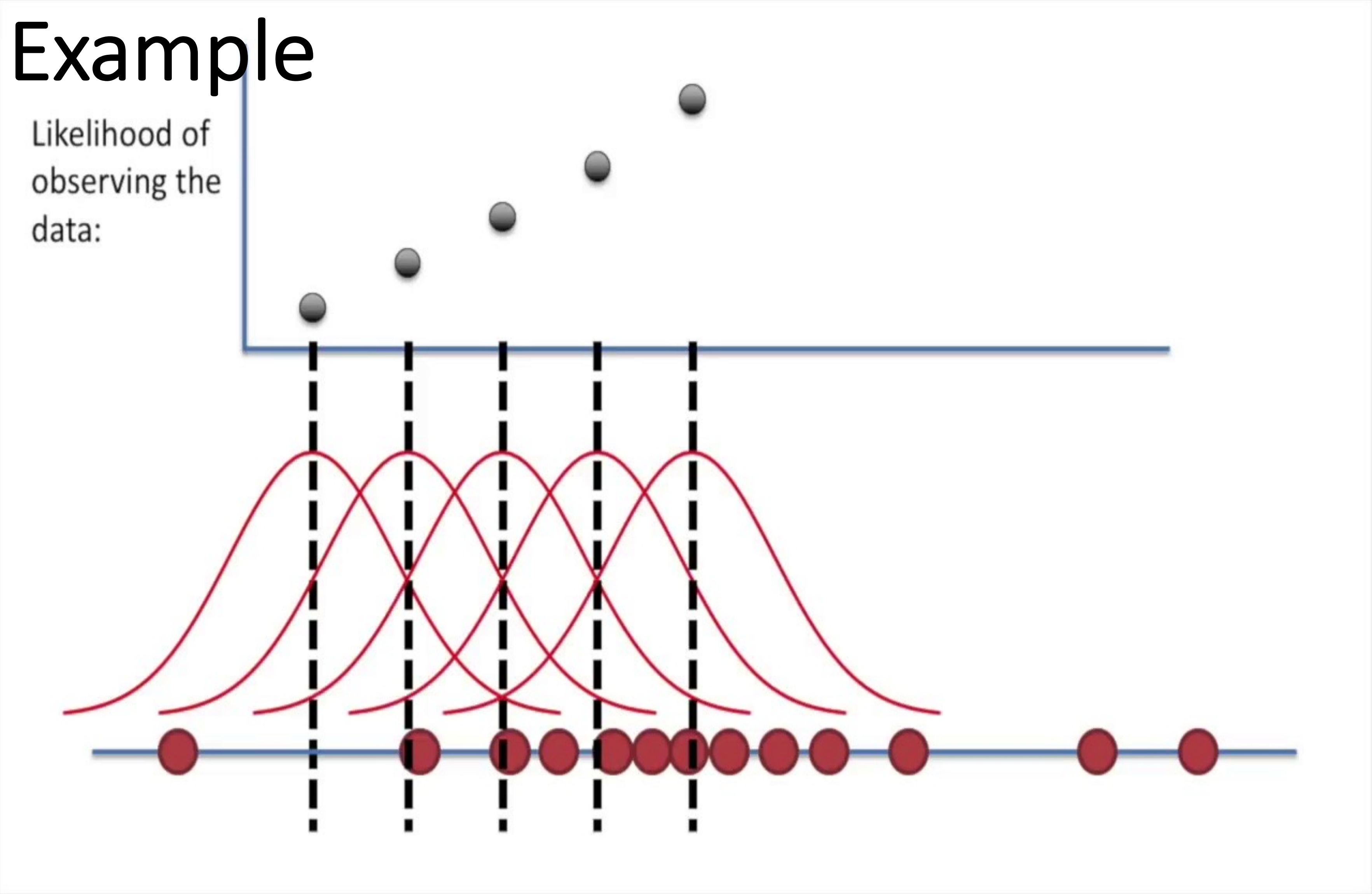

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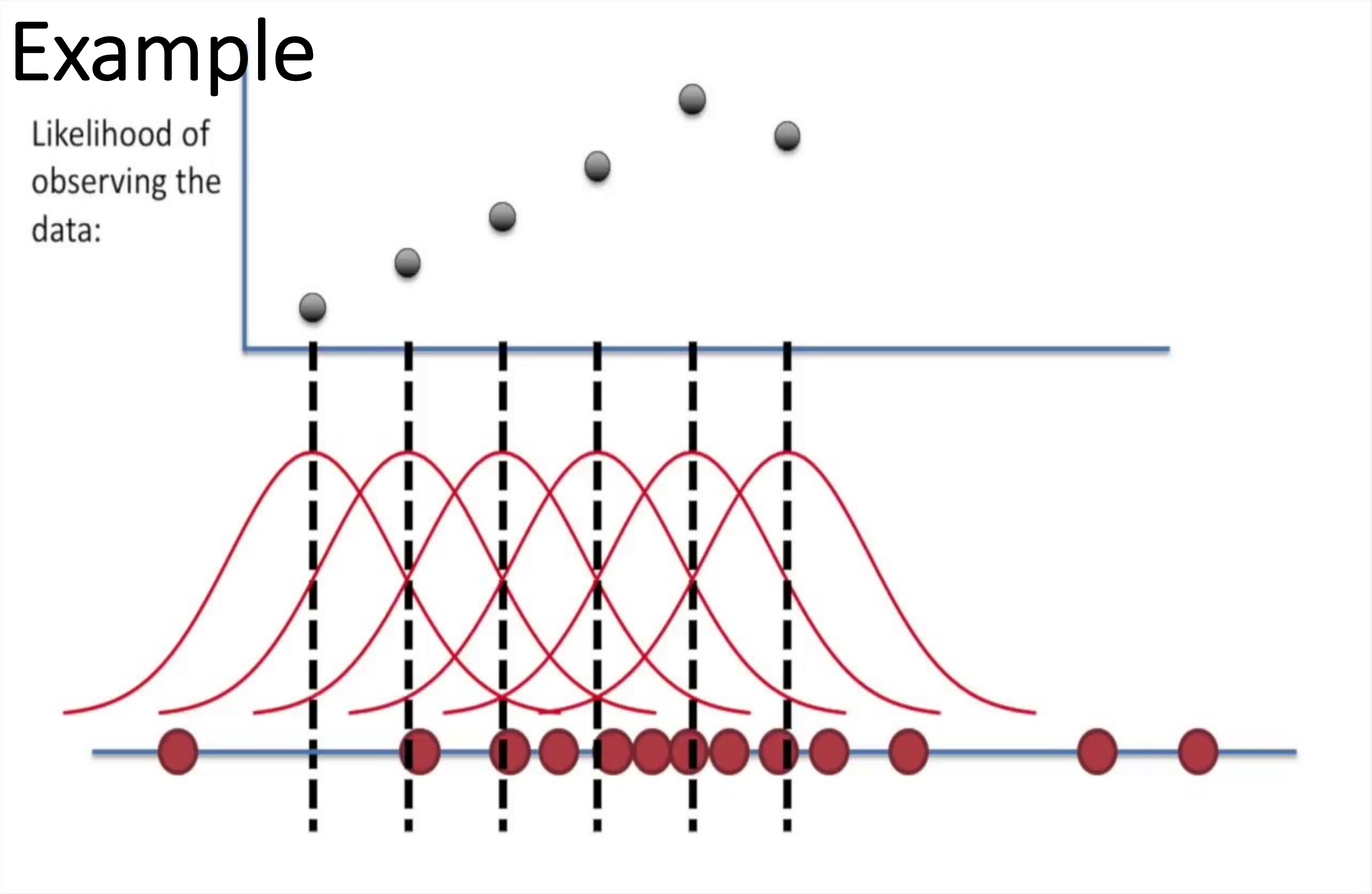

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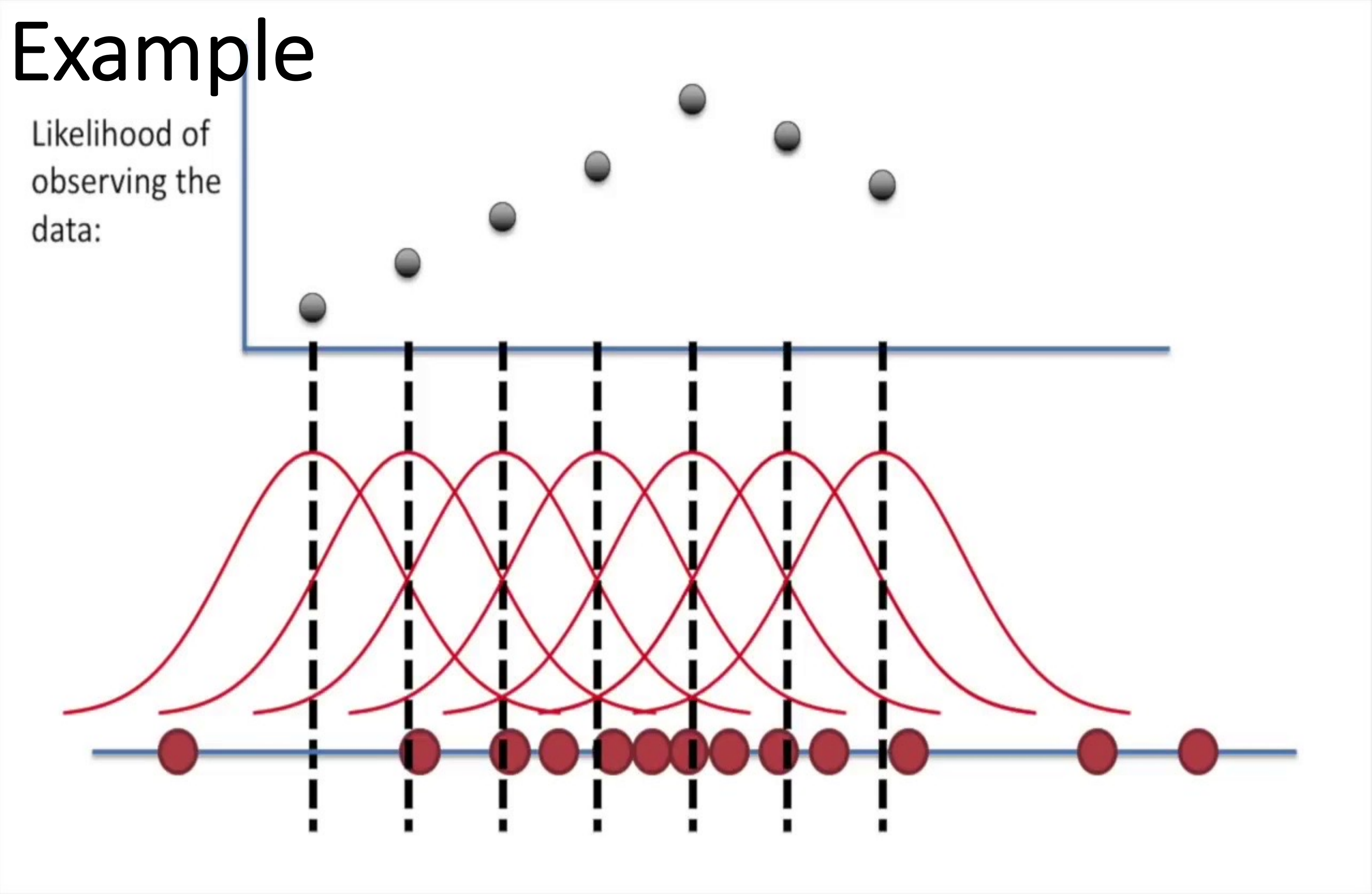

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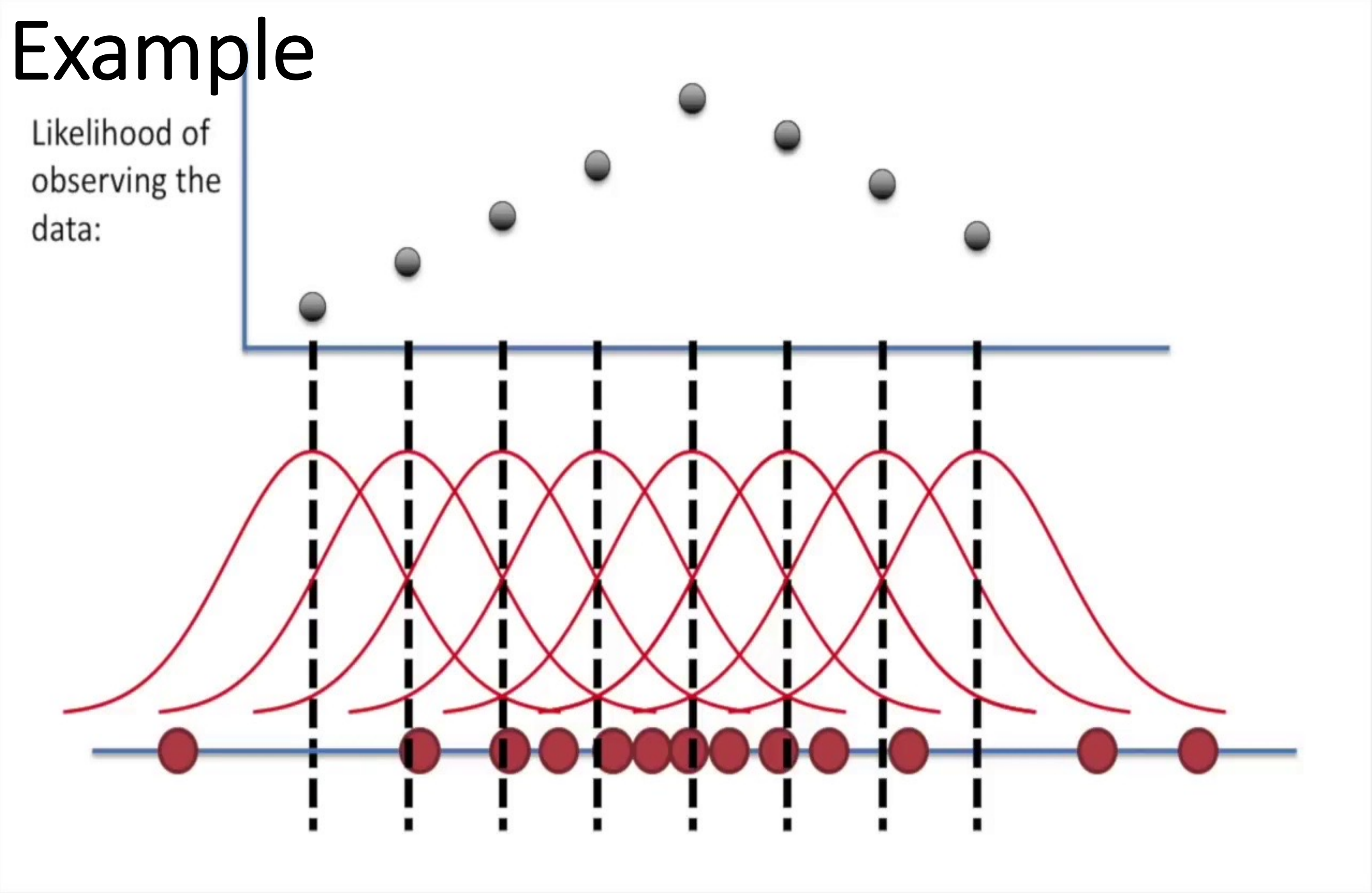

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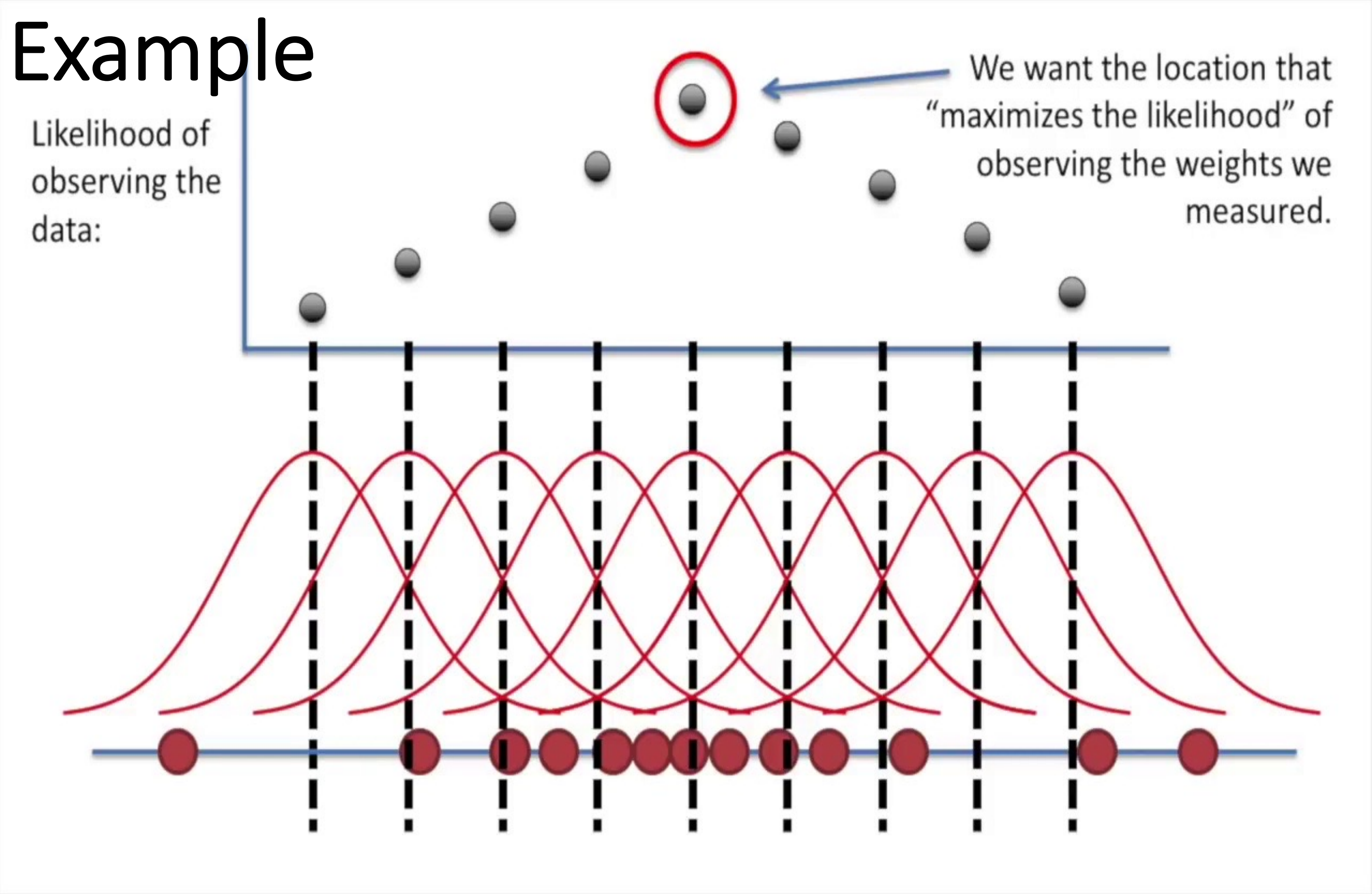
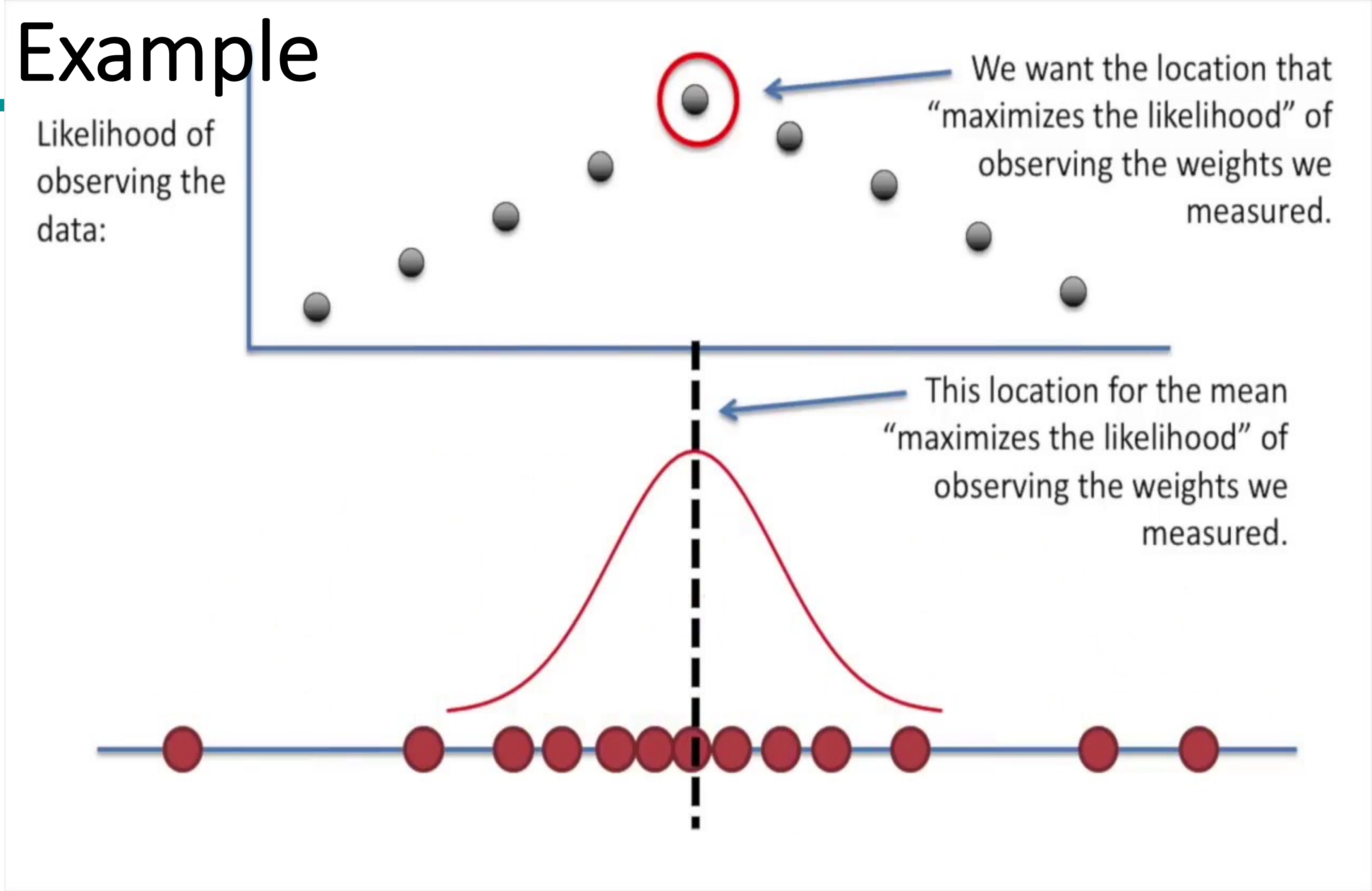


We want the location that "maximizes the likelihood" of observing the weights we measured.

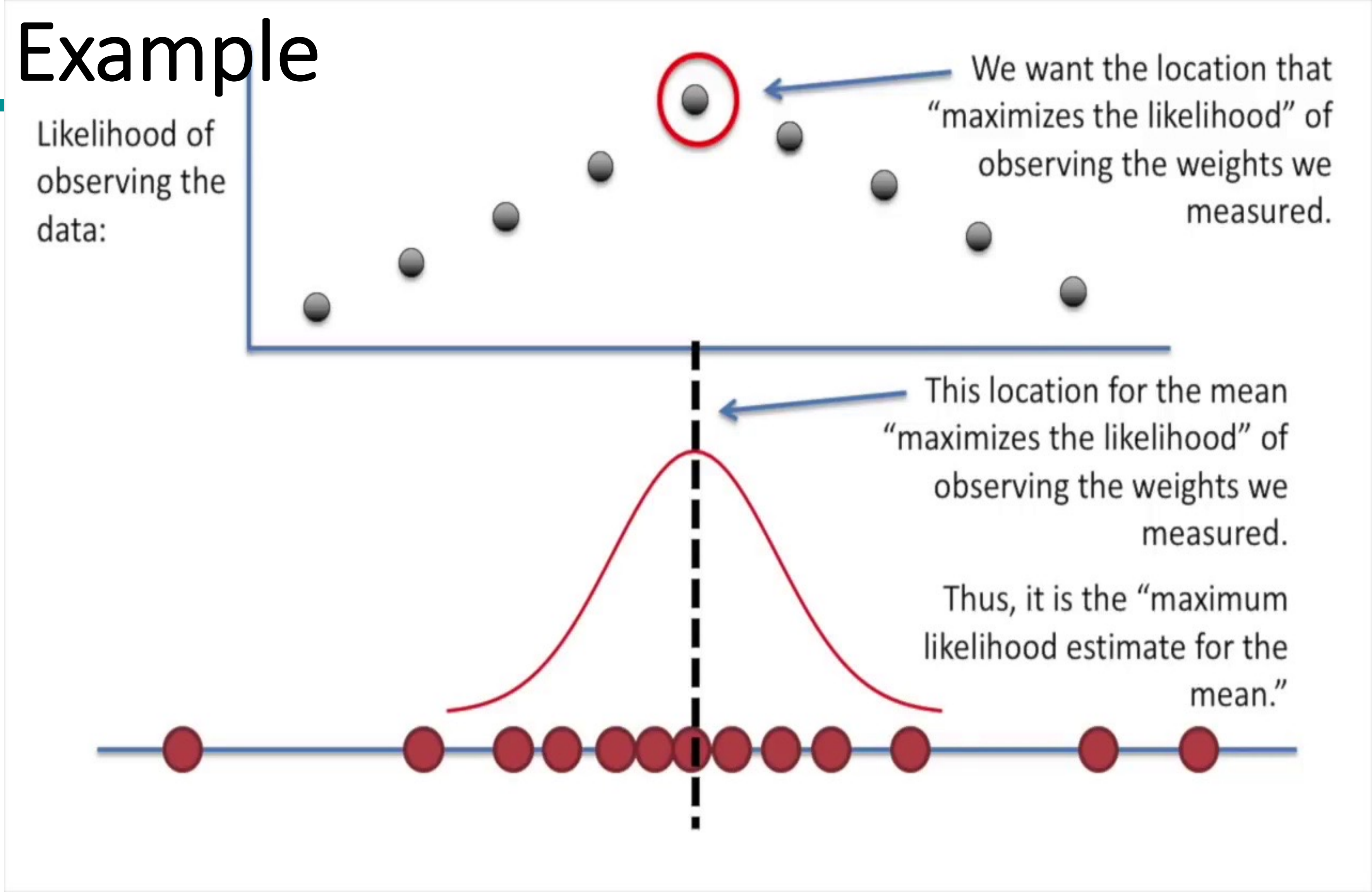
MLE Example



We want the location that "maximizes the likelihood" of observing the weights we measured.

This location for the mean "maximizes the likelihood" of observing the weights we measured.

MLE Example



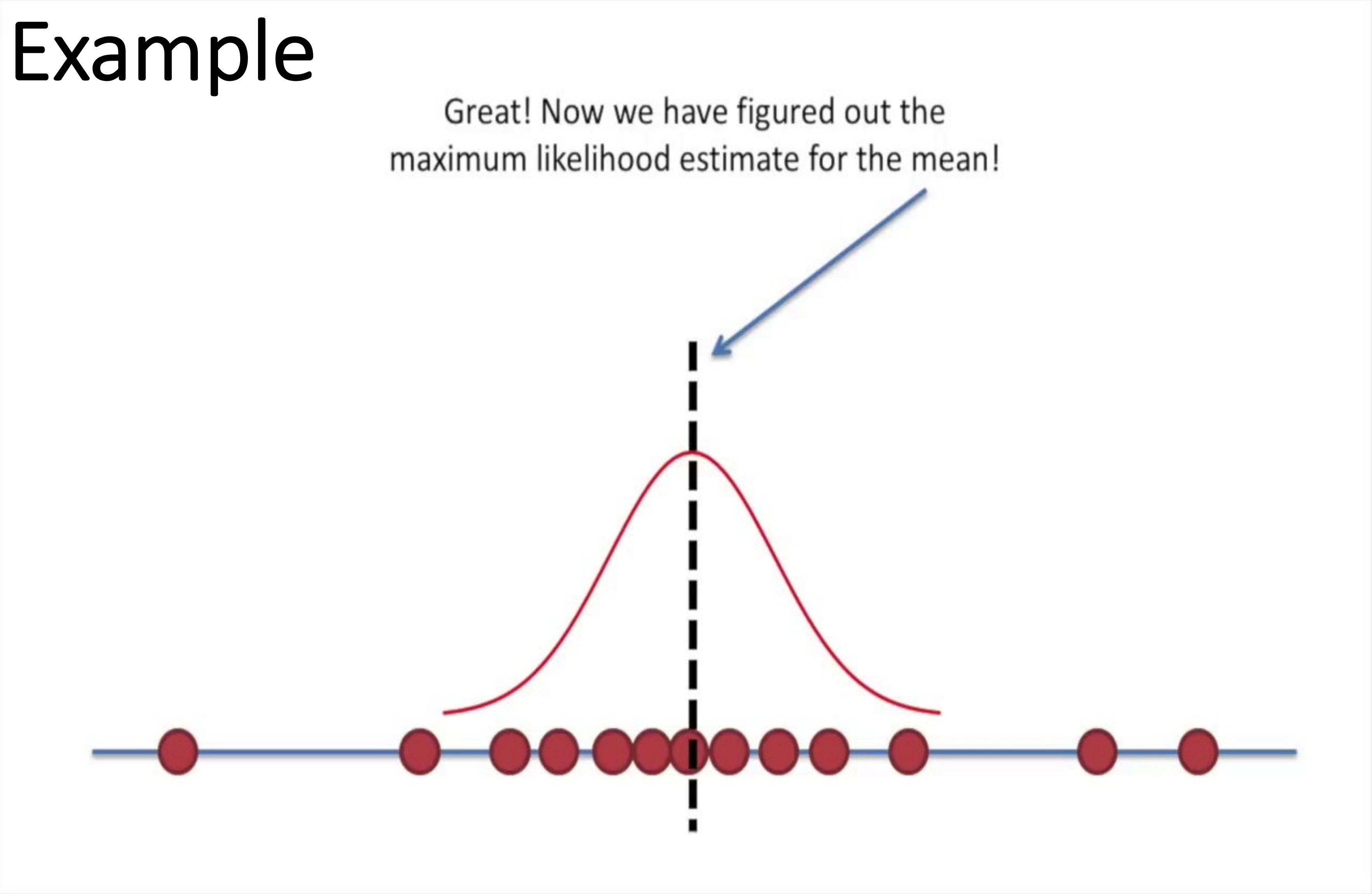
Likelihood of observing the data:

We want the location that "maximizes the likelihood" of observing the weights we measured.

This location for the mean "maximizes the likelihood" of observing the weights we measured.

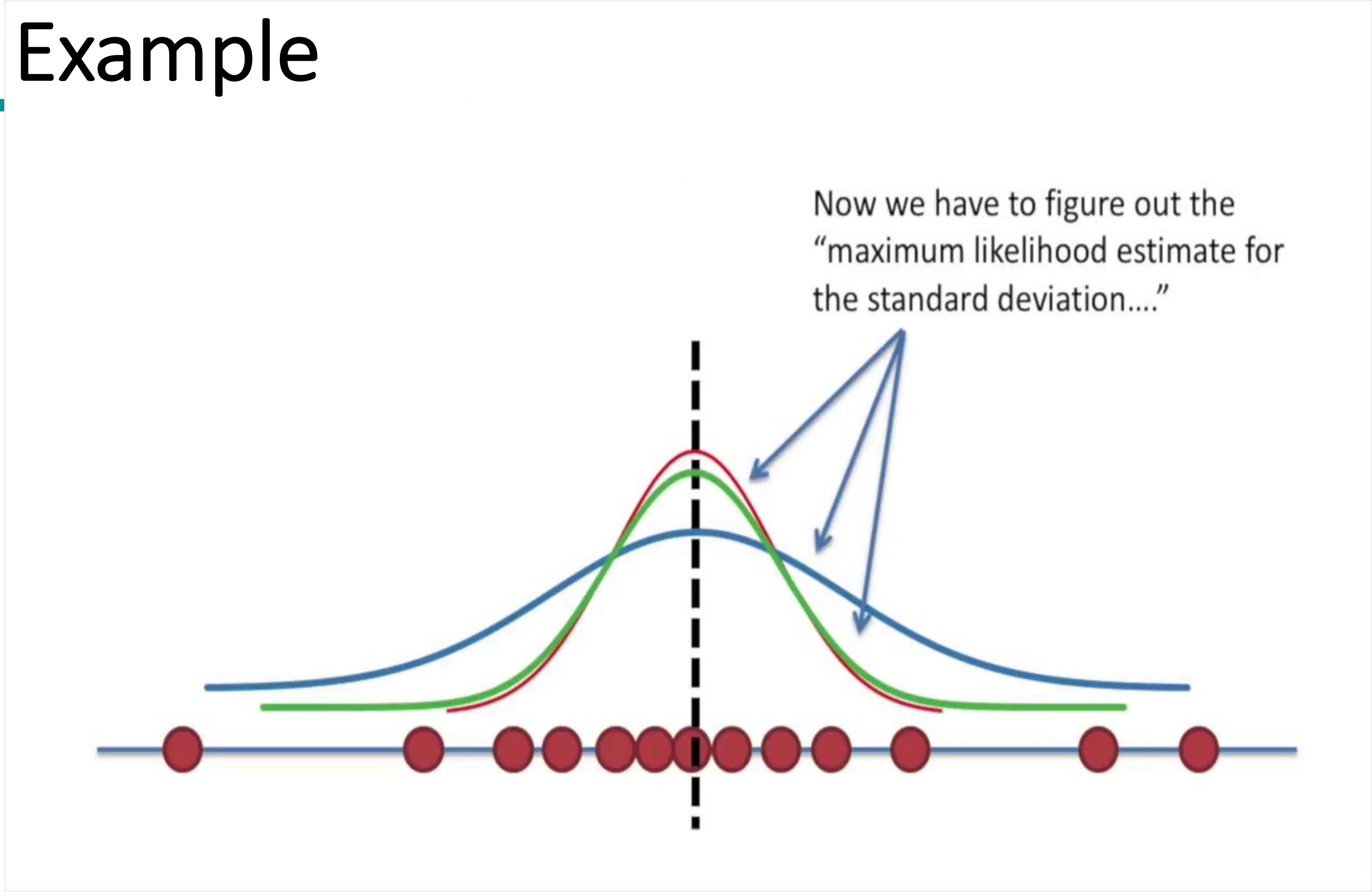
Thus, it is the "maximum likelihood estimate for the mean."

MLE Example



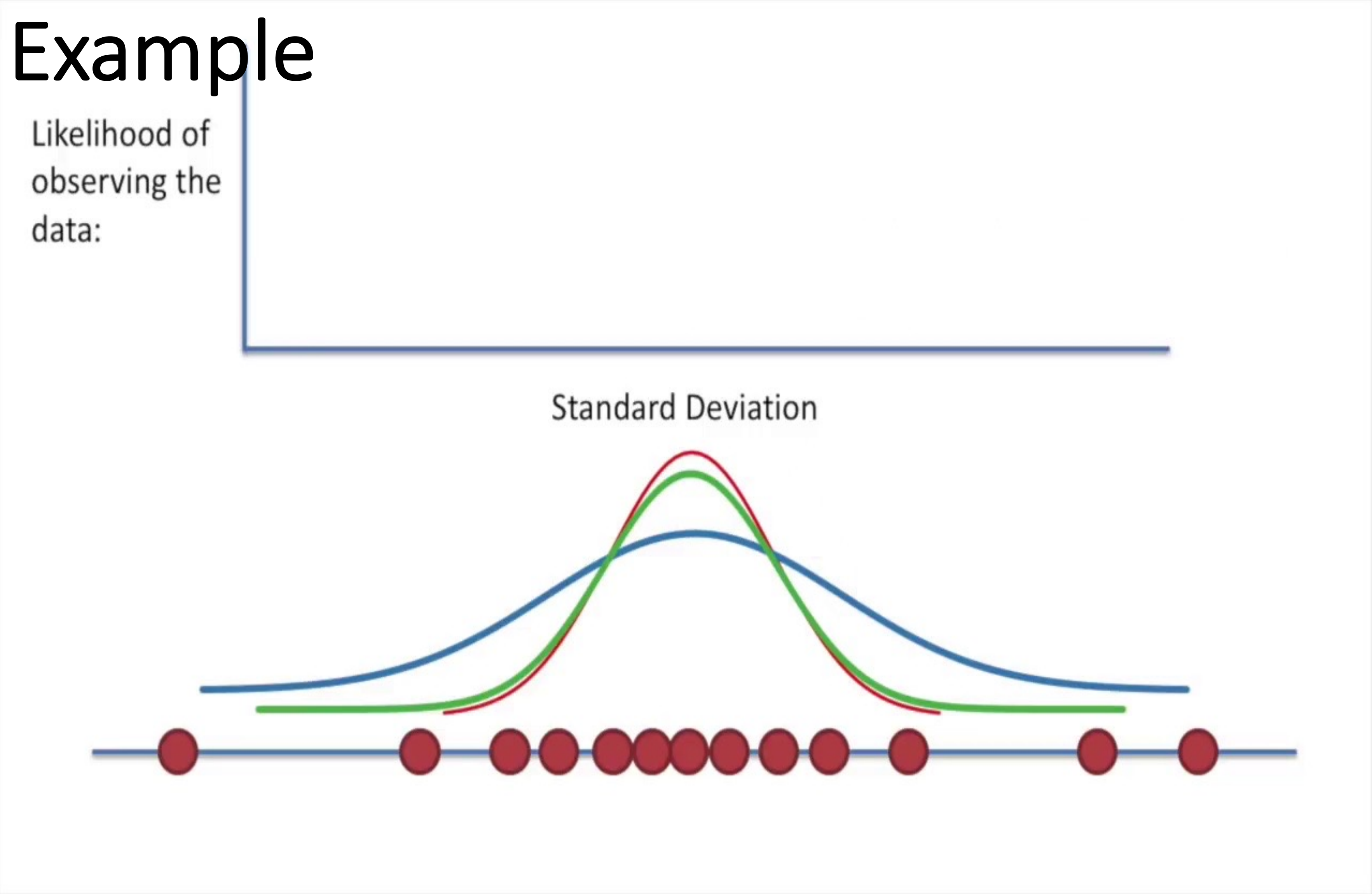
Great! Now we have figured out the maximum likelihood estimate for the mean!

MLE Example



Now we have to figure out the "maximum likelihood estimate for the standard deviation…."

MLE Example



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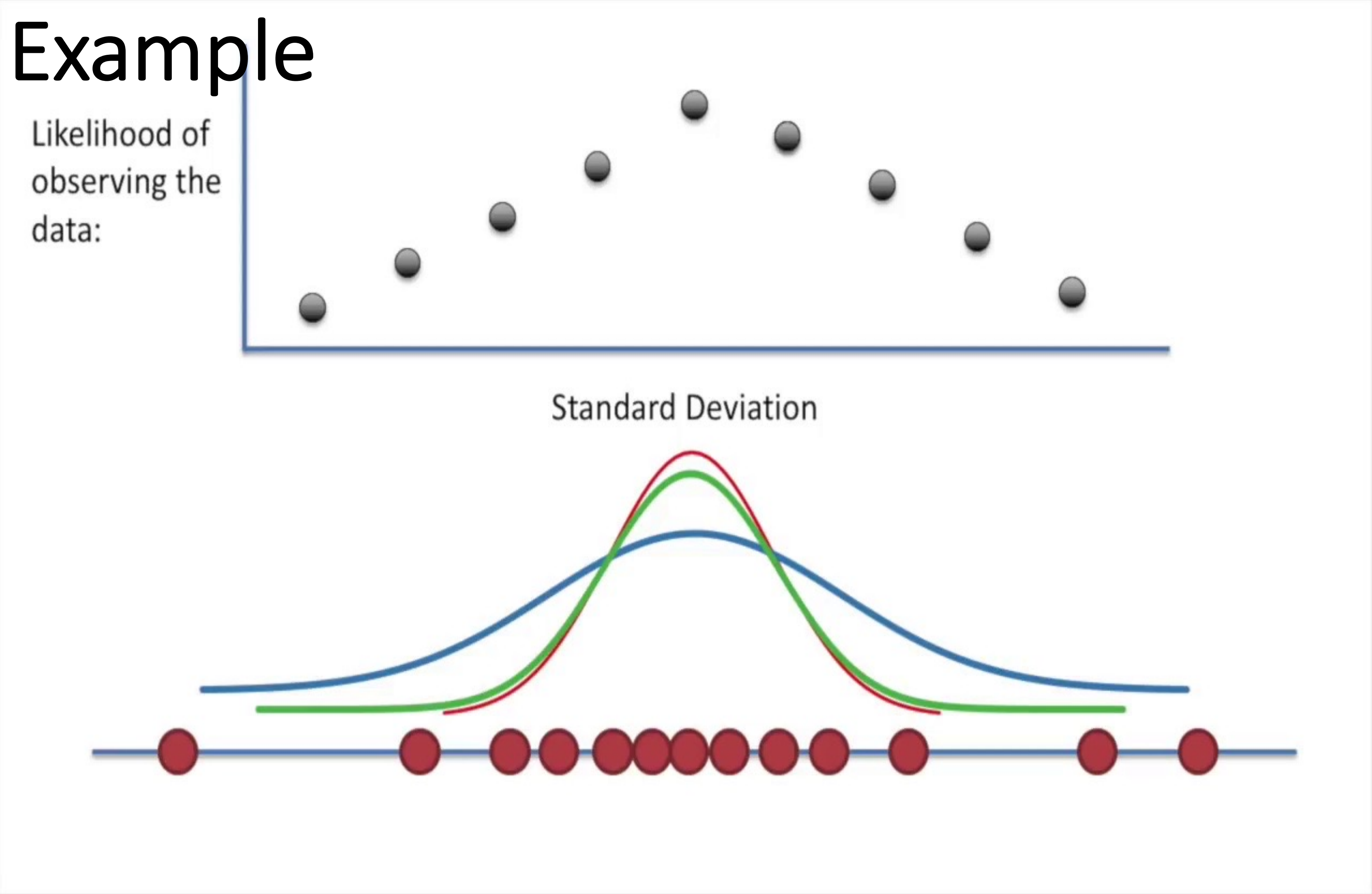

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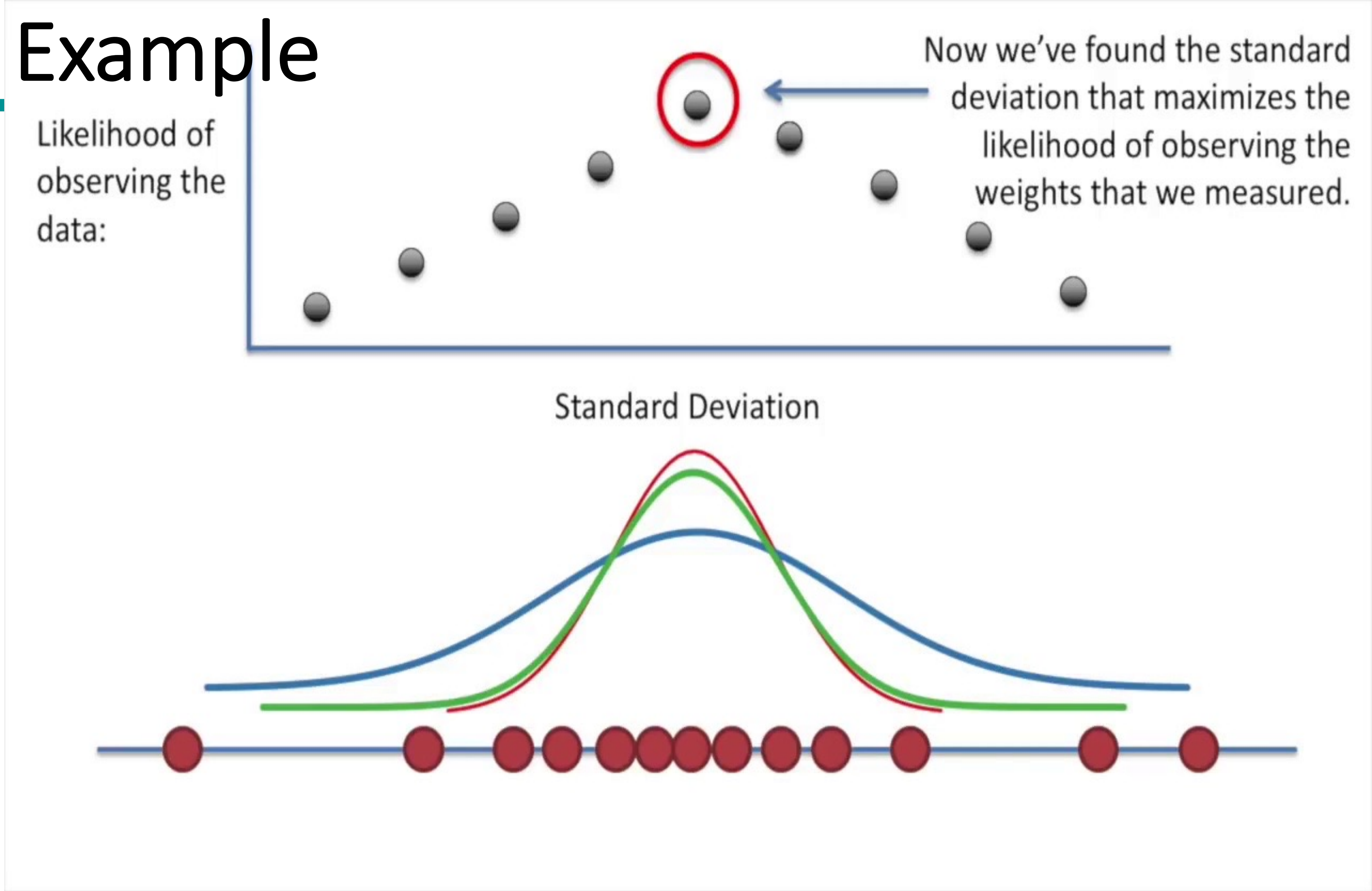
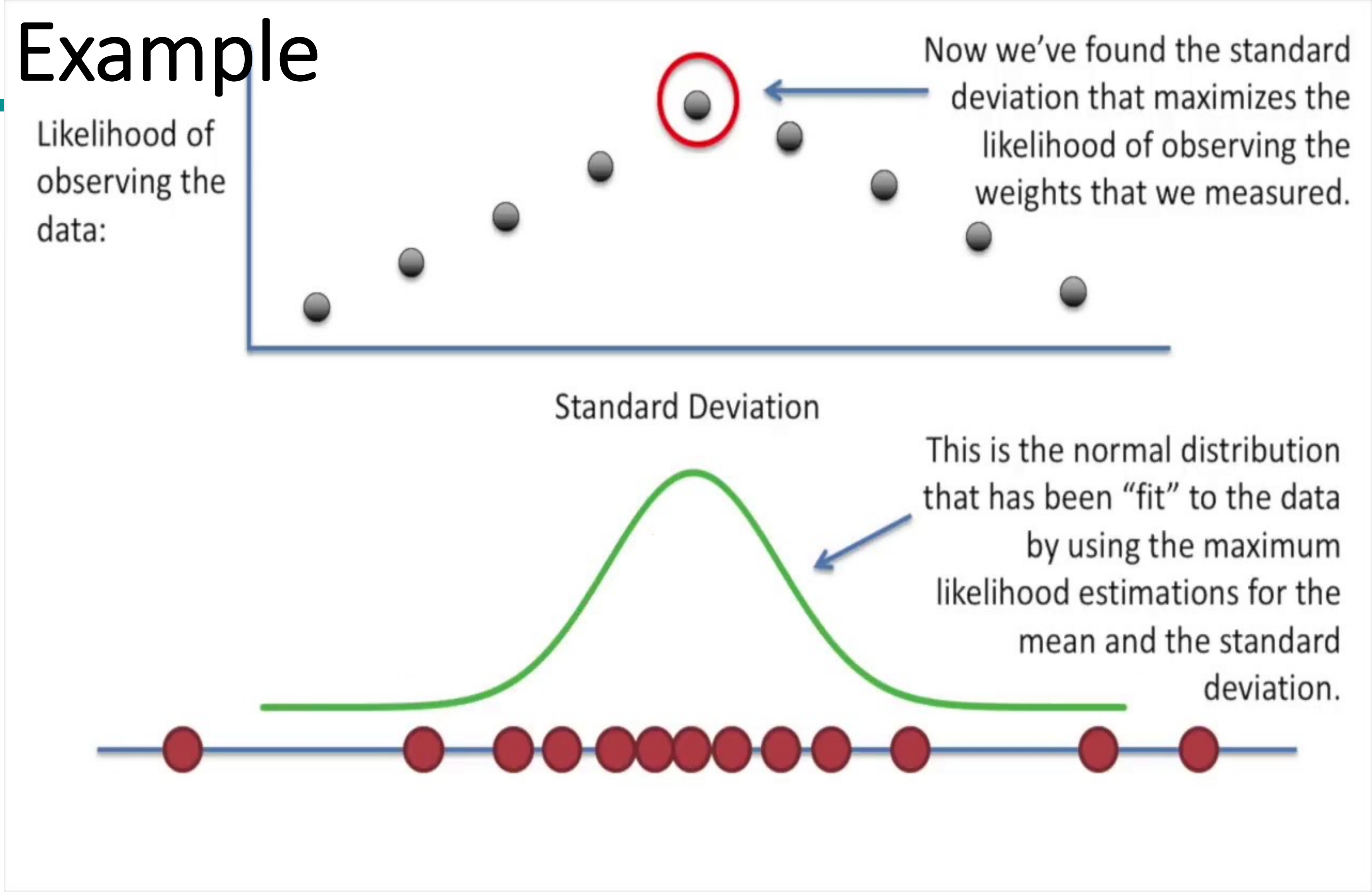


Now we've found the standard deviation that maximizes the likelihood of observing the weights that we measured.

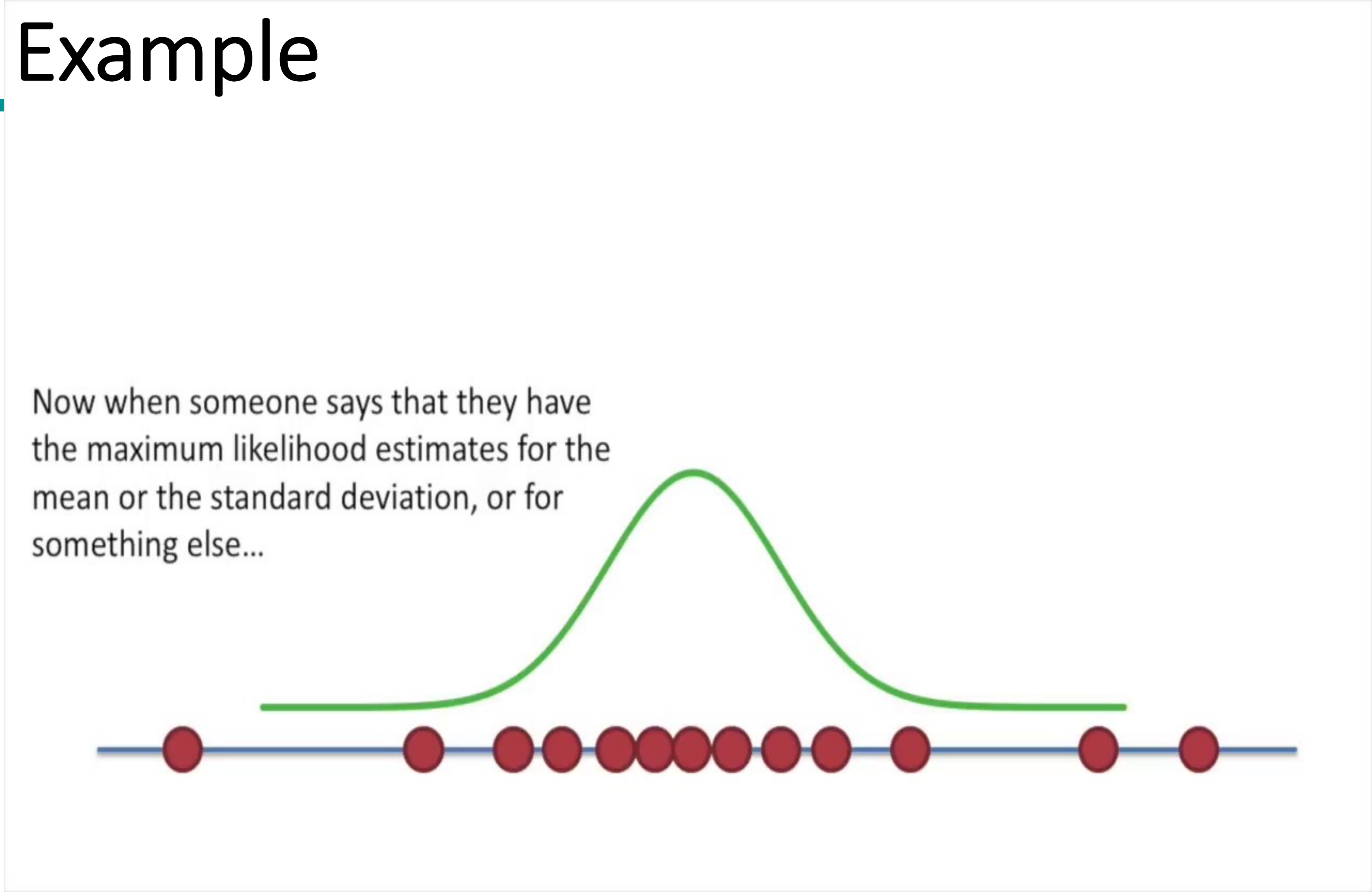
MLE Example



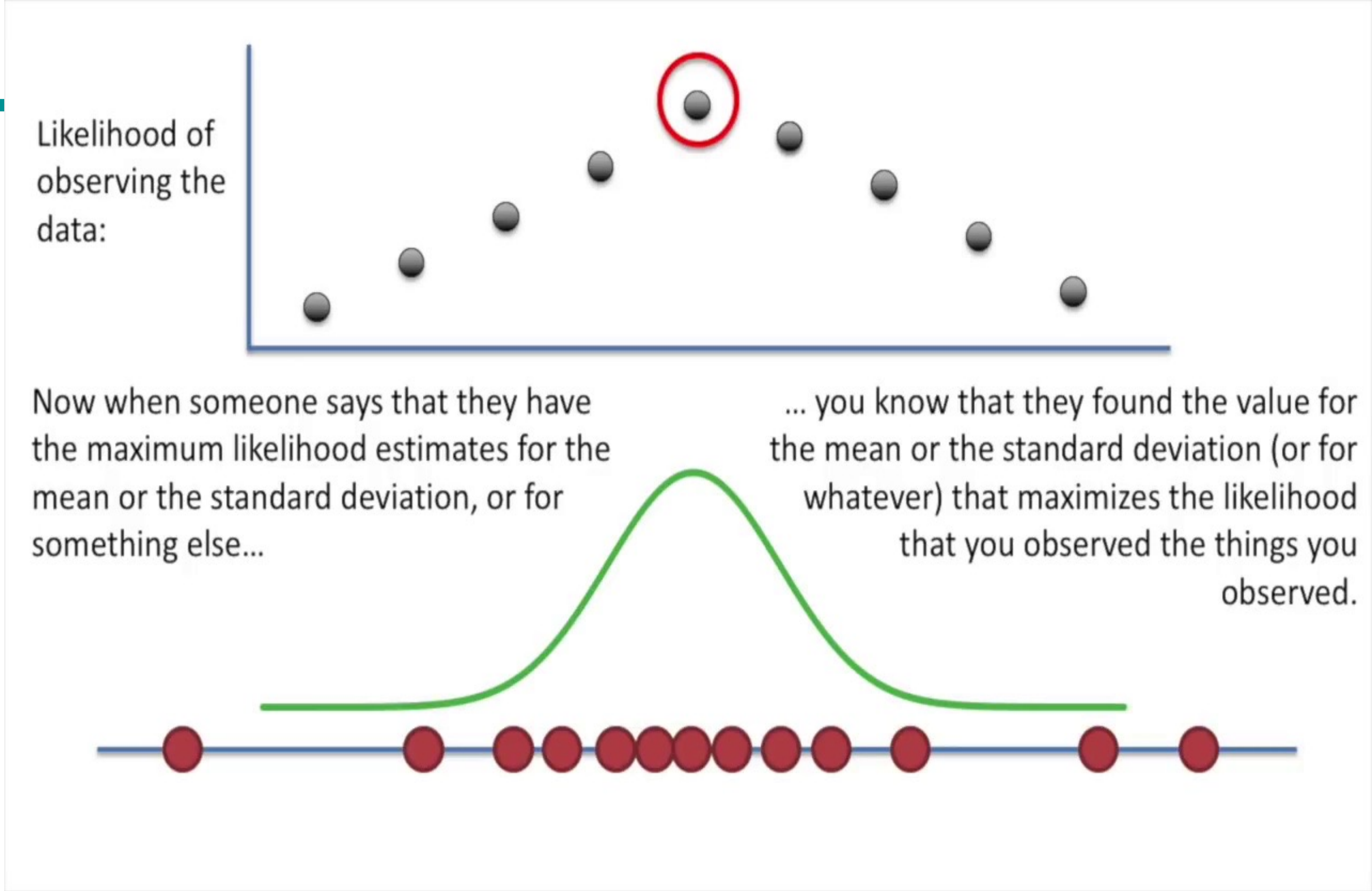
Now we've found the standard deviation that maximizes the likelihood of observing the weights that we measured.

This is the normal distribution that has been "fit" to the data by using the maximum likelihood estimations for the mean and the standard deviation.

MLE Example



Now when someone says that they have the maximum likelihood estimates for the mean or the standard deviation, or for something else…



Now when someone says that they have the maximum likelihood estimates for the mean or the standard deviation, or for something else…

… you know that they found the value for the mean or the standard deviation (or for whatever) that maximizes the likelihood that you observed the things you observed.

Calculating the MLE

- Example: we have three data points 9, 9.5, 11
- We want to calculated the total probability of observing all the data, i.e. the joint probability distribution of all observed data points.
- Assumption: each data point is generated independently from the others.
- If the events are independent, then the total probability of observing all the data is the product of observing each data point individually (i.e. the product of the marginal probabilities).

Calculating the MLE

- Probability of observing a single data point *x*

$$P(x; \underset{\uparrow}{\mu}, \underset{\uparrow}{\sigma}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Parameters

- Example:

$$P(9, 9.5, 11; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(9-\mu)^2}{2\sigma^2}\right) \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(9.5-\mu)^2}{2\sigma^2}\right) \\ \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(11-\mu)^2}{2\sigma^2}\right)$$

The Log Likelihood

- Maximum is found by differentiation, i.e., find the derivative of the function w.r.t. a variable, set it to zero and find the required value.
- Since the previous expression is not easy to differentiate, we simplify the calculus considering the natural logarithm of the expression.

$$\begin{aligned}\ln(P(x;\mu,\sigma)) = \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{(9-\mu)^2}{2\sigma^2} + \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{(9.5-\mu)^2}{2\sigma^2} \\ + \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{(11-\mu)^2}{2\sigma^2}\end{aligned}$$

$$\ln(P(x;\mu,\sigma)) = -3\ln(\sigma) - \frac{3}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\left[(9-\mu)^2 + (9.5-\mu)^2 + (11-\mu)^2\right]$$

The Log Likelihood

- This expression can be easily differentiated to find the maximum.

$$\frac{\partial \ln(P(x;\mu,\sigma))}{\partial \mu} = \frac{1}{\sigma^2}\left[9 + 9.5 + 11 - 3\mu\right].$$

$$\mu = \frac{9 + 9.5 + 11}{3} = 9.833$$

- The same can be done for the standard deviation.