# DATA MINING 2
## Logistic Regression

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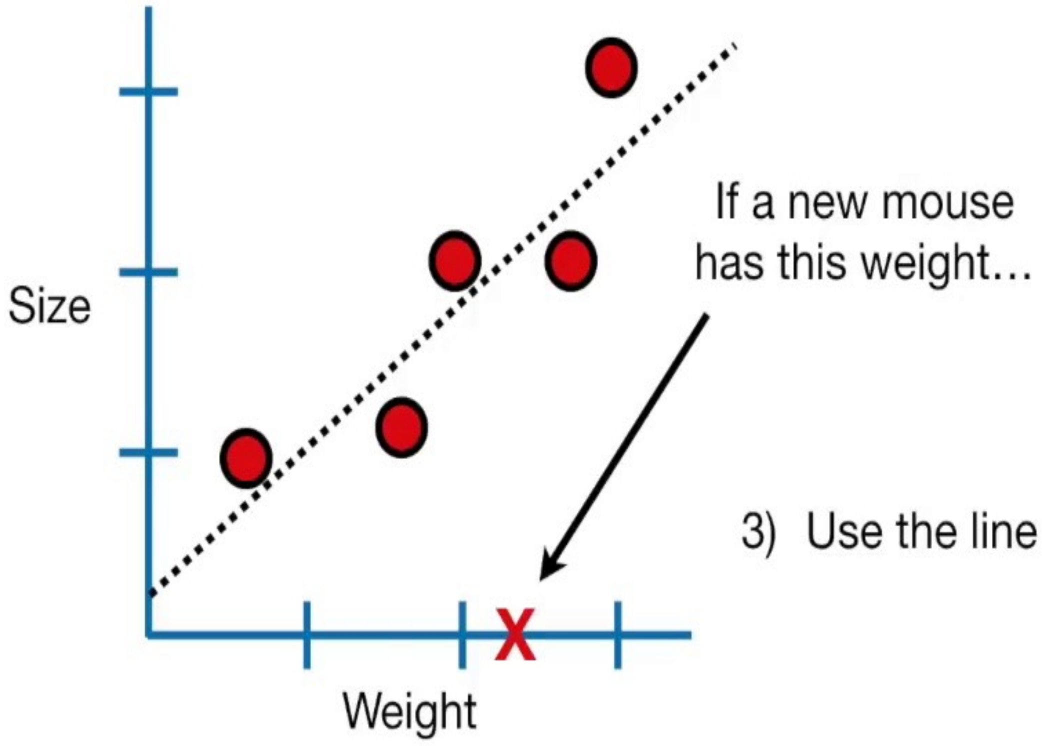
Contains edited slides from StatQuest

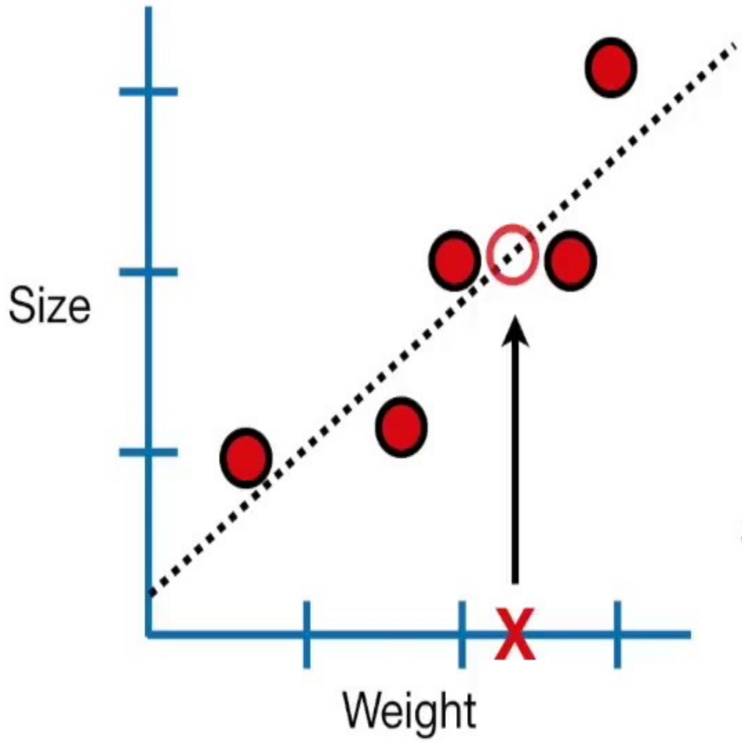
UNIVERSITÀ DI PISA

# Recalling Linear Regression



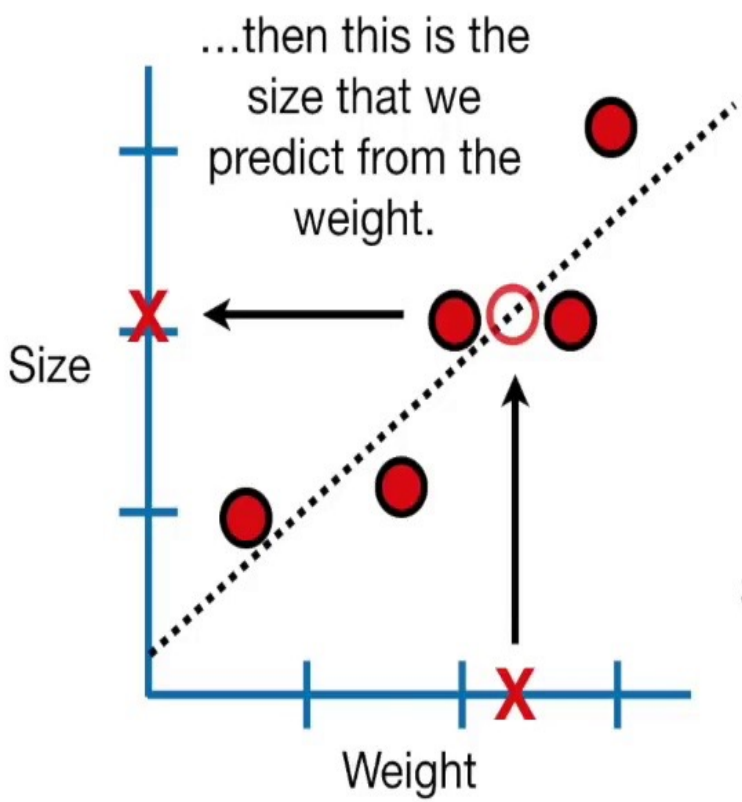

3) Use the line to predict ***size*** given ***weight***.

# Recalling Linear Regression



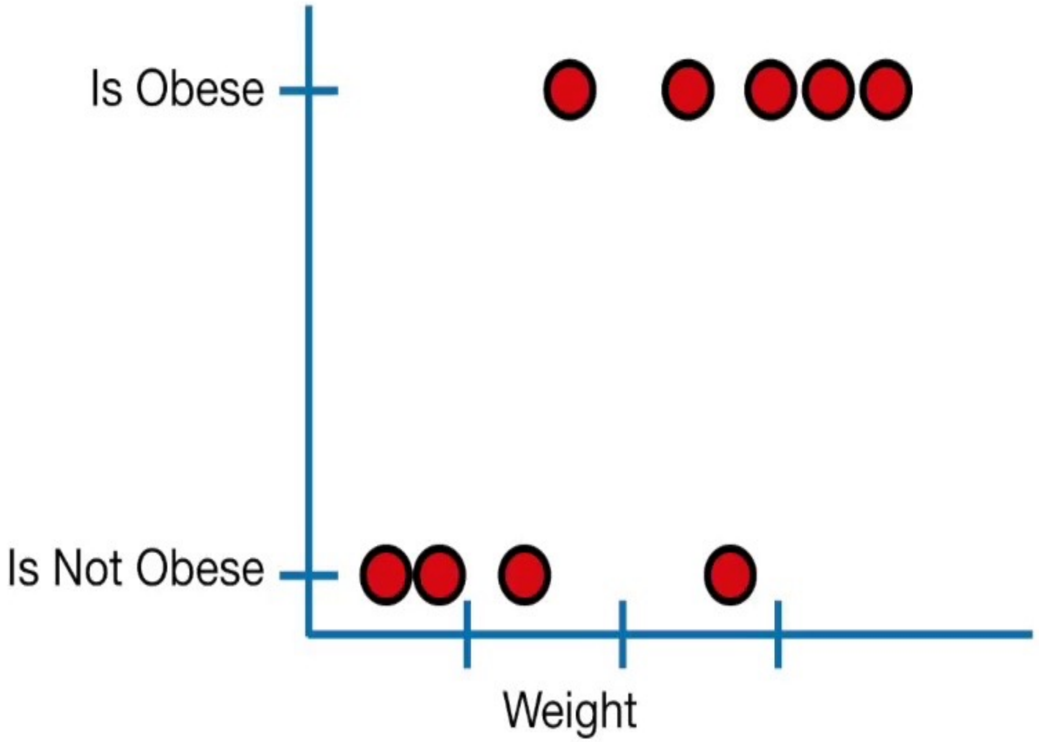

3) Use the line to predict ***size*** given ***weight***.

# Recalling Linear Regression



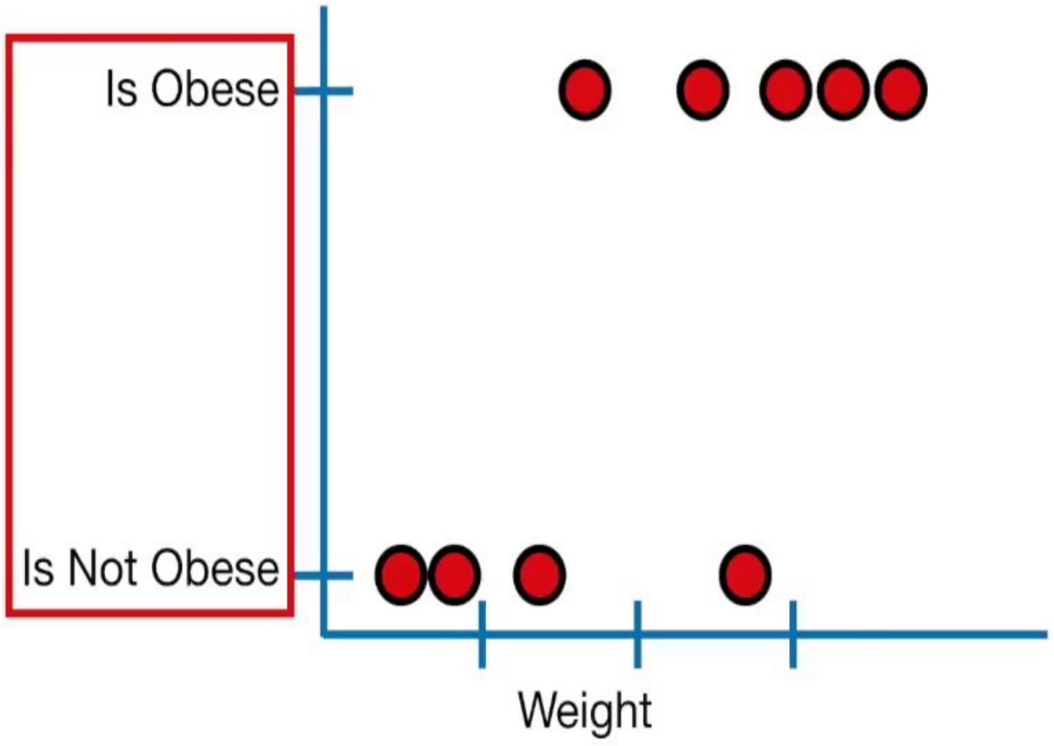

3) Use the line to predict ***size*** given ***weight***.

# Logistic Regression

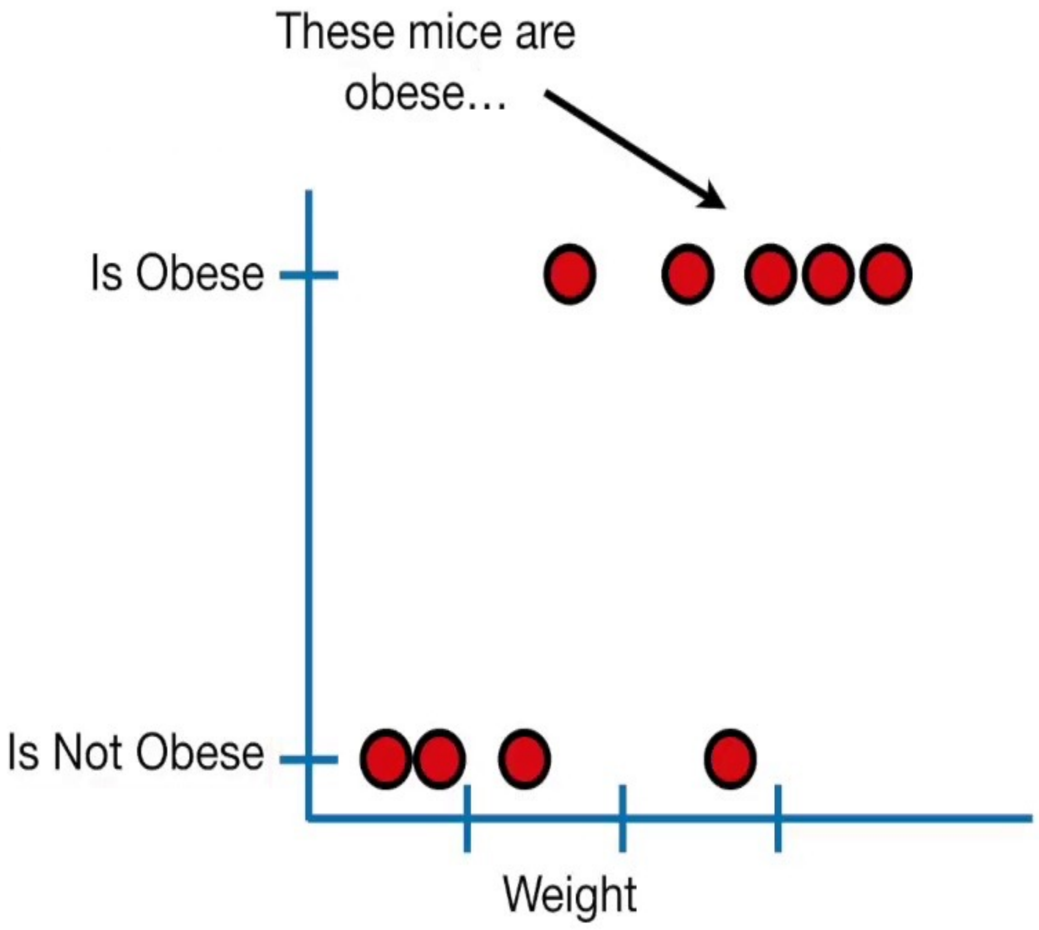


# Logistic Regression

Logistic regression predicts whether something is ***True*** or ***False***, instead of predicting something continuous like ***size***.



# Logistic Regression



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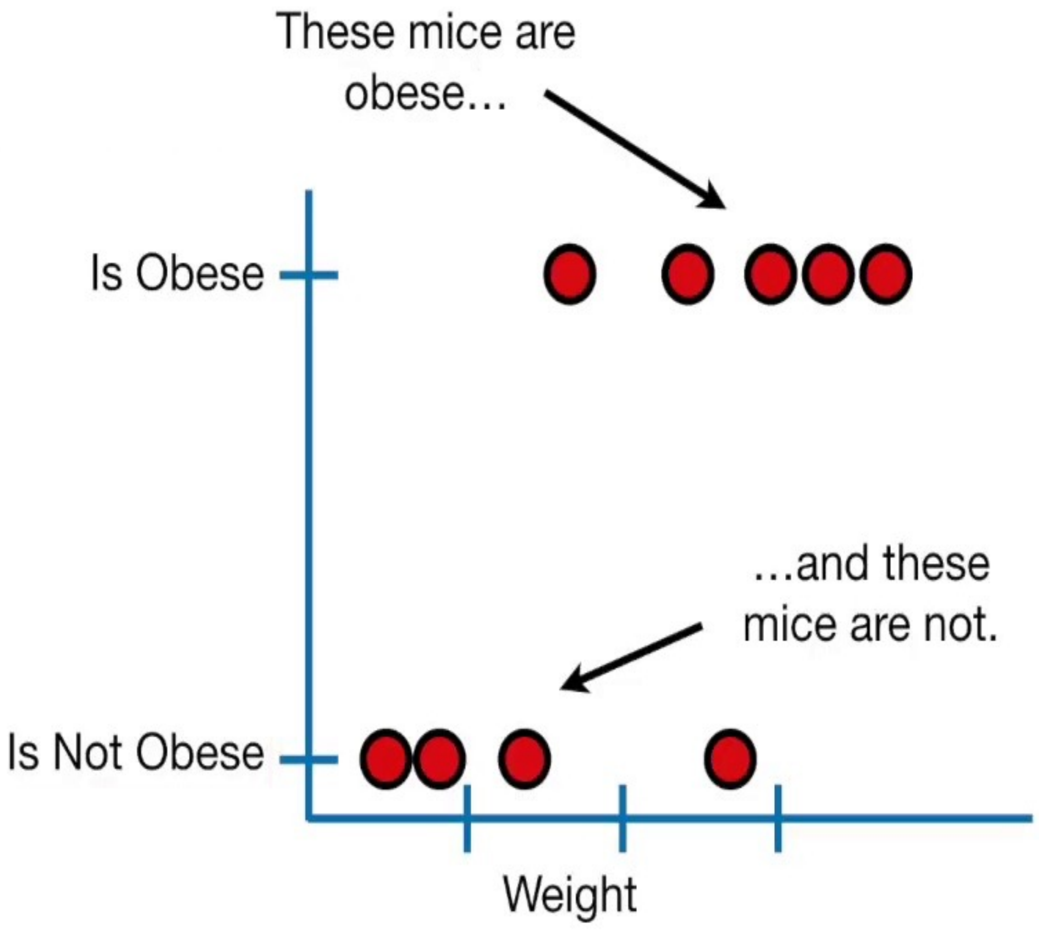
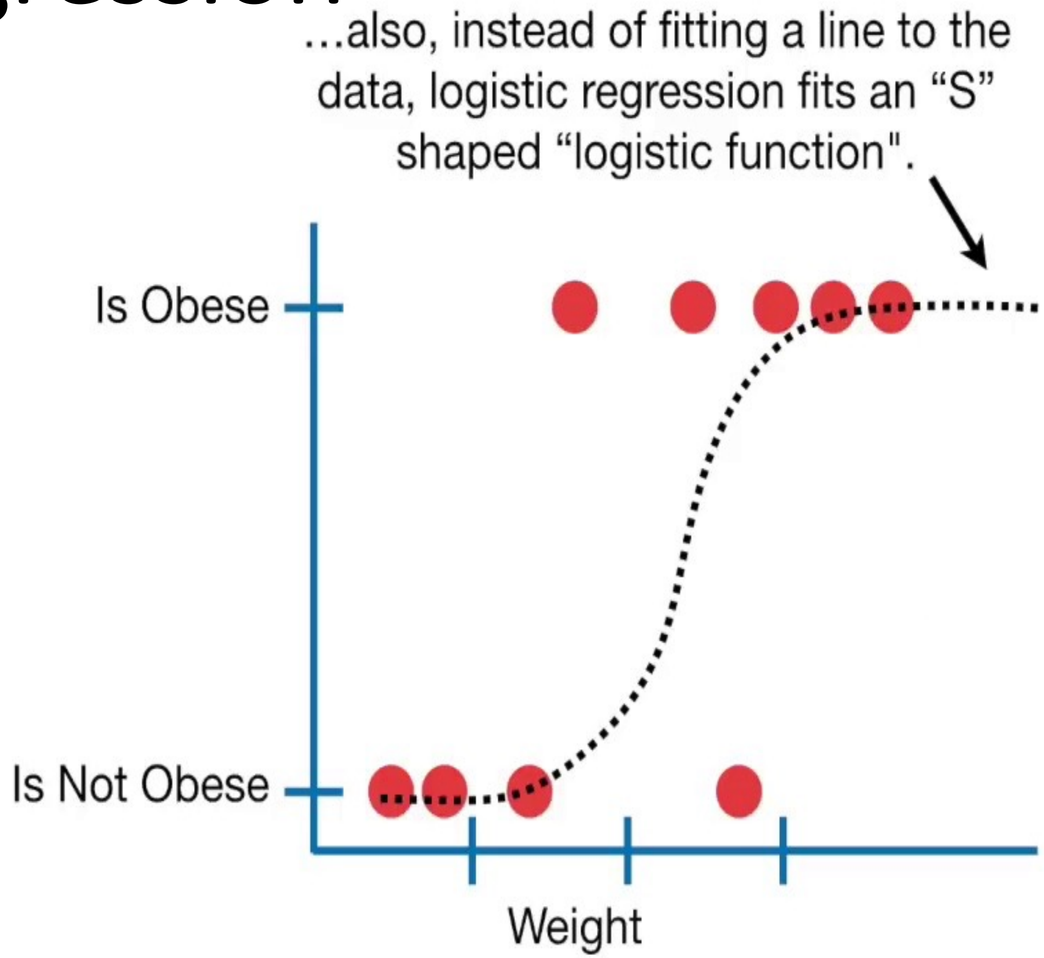
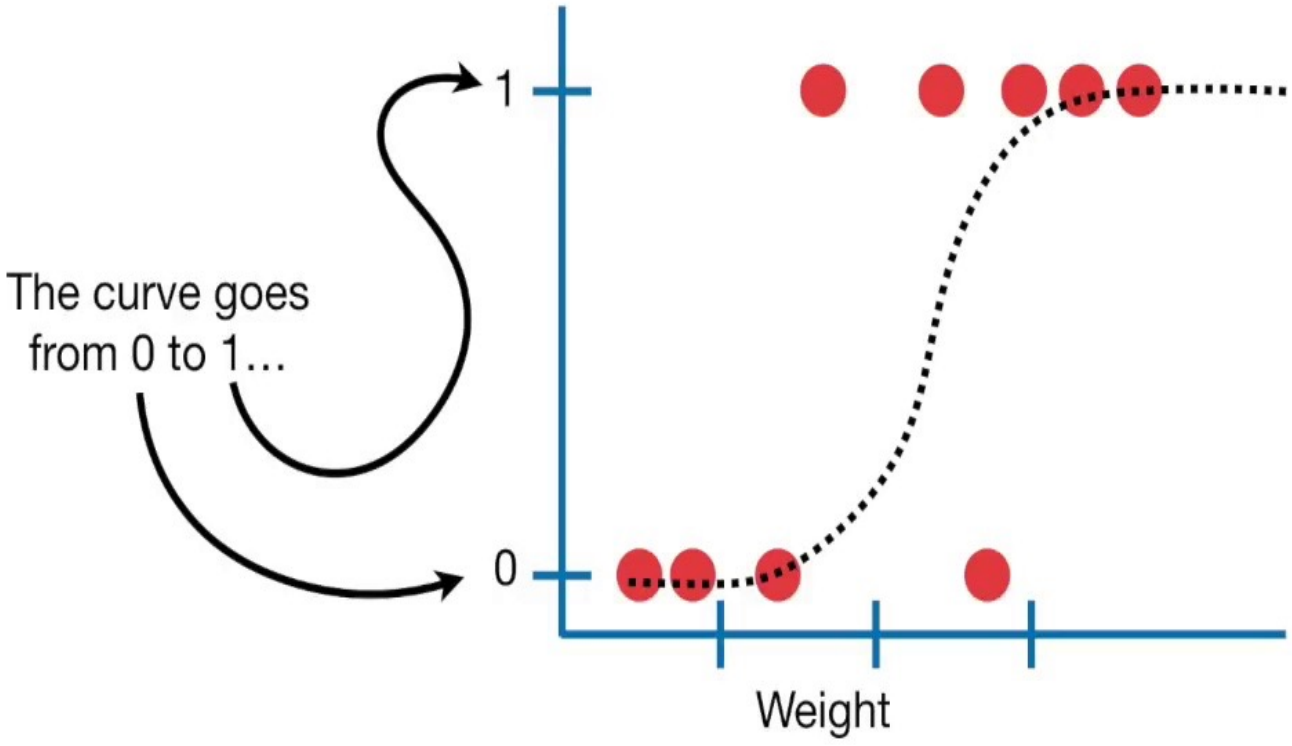


# Logistic Regression

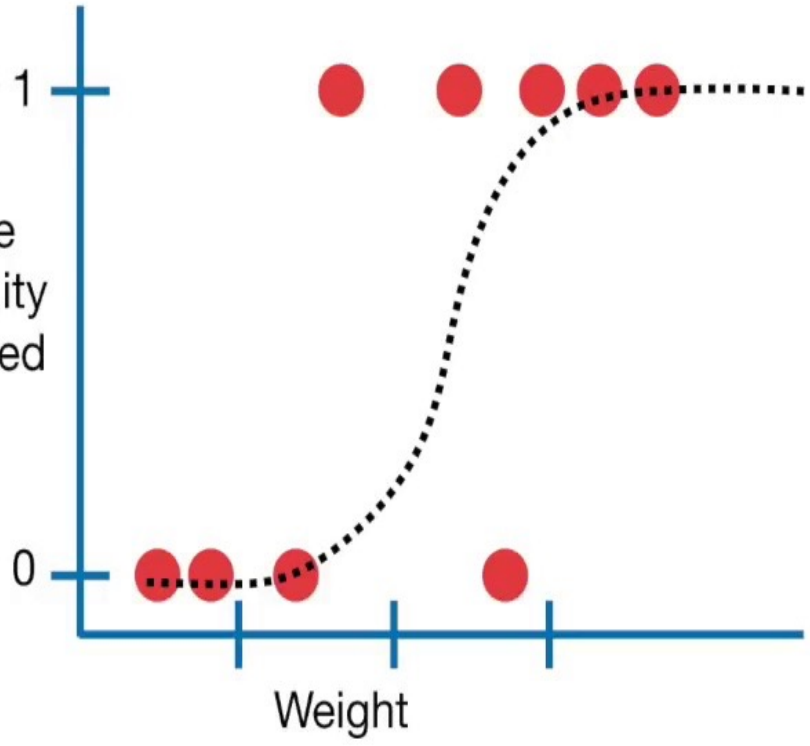


# Logistic Regression

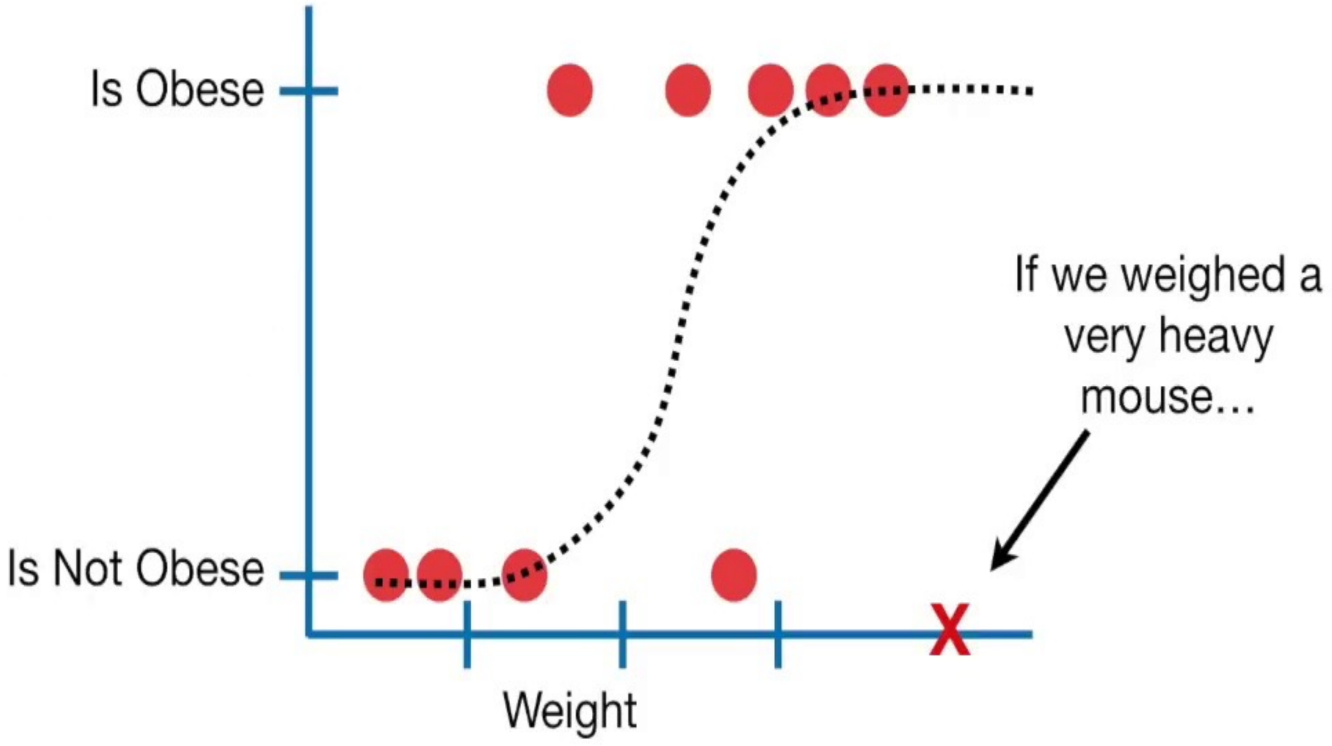


# Logistic Regression

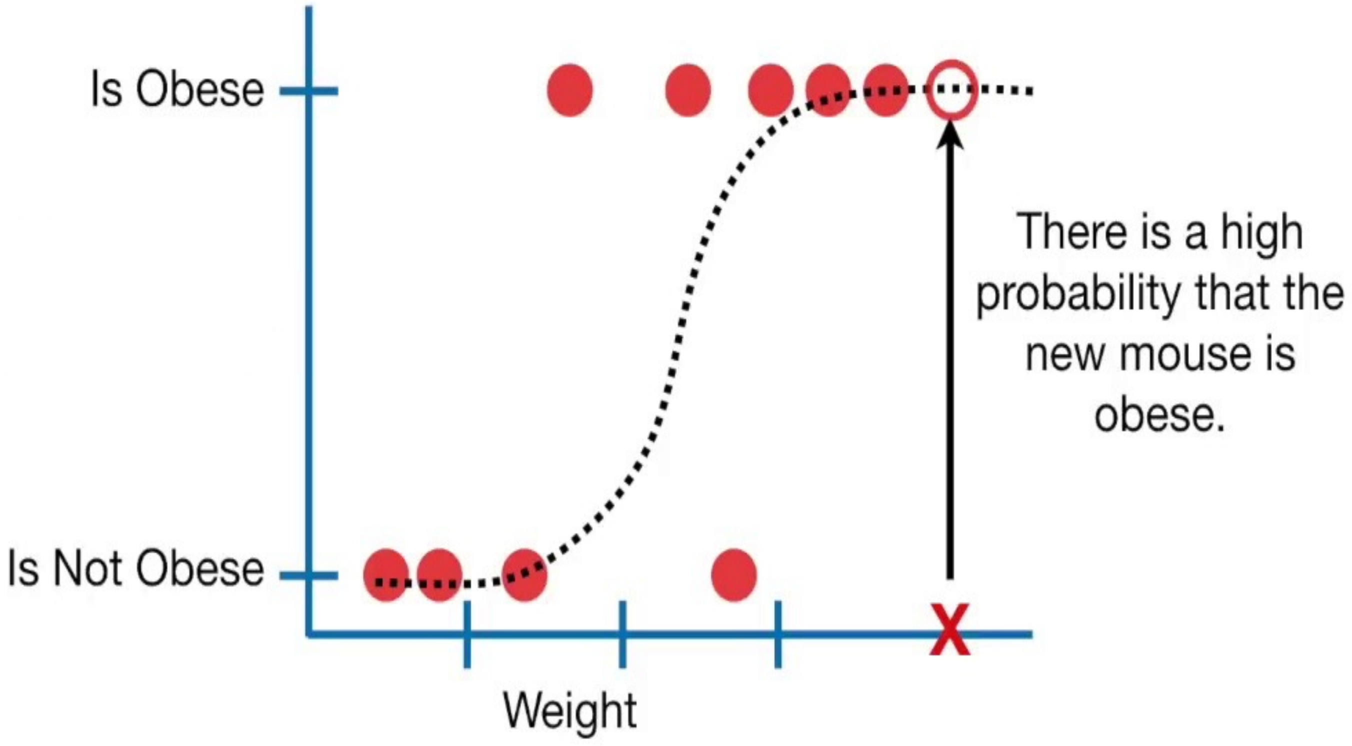
...and that means that the curve tells you the probability that a mouse is ***obese*** based on its ***weight***.



# Logistic Regression



# Logistic Regression



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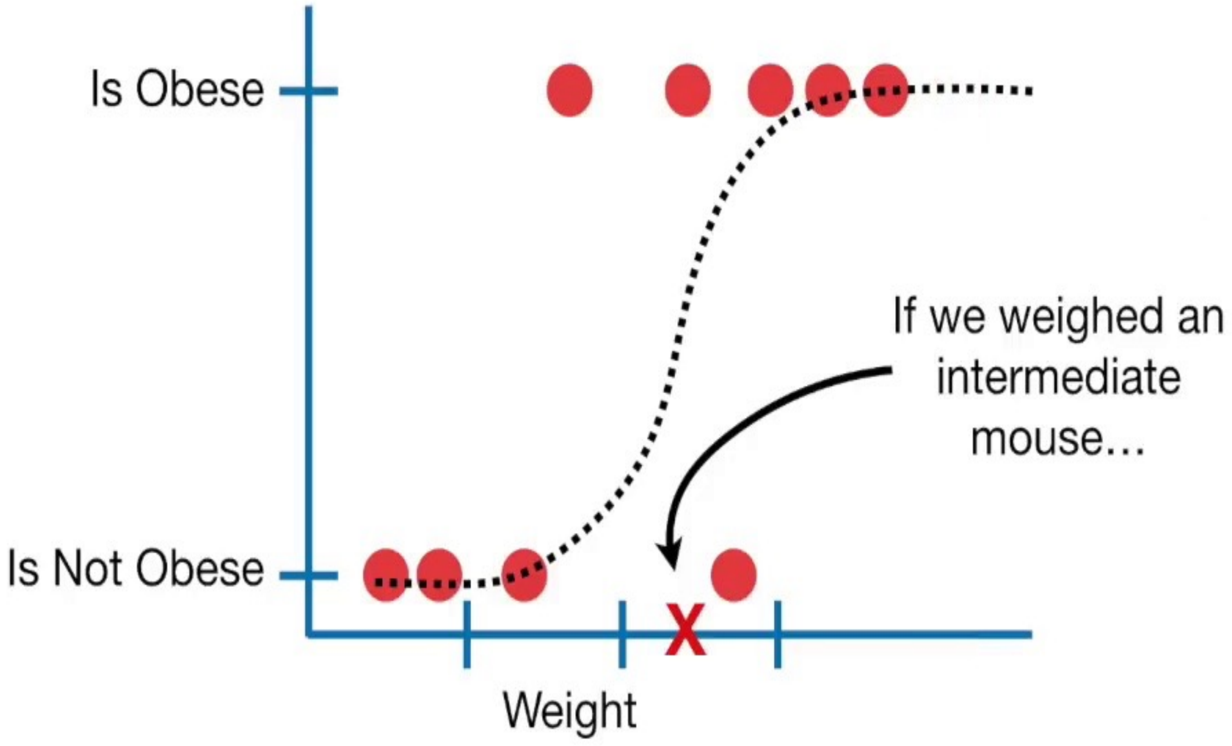
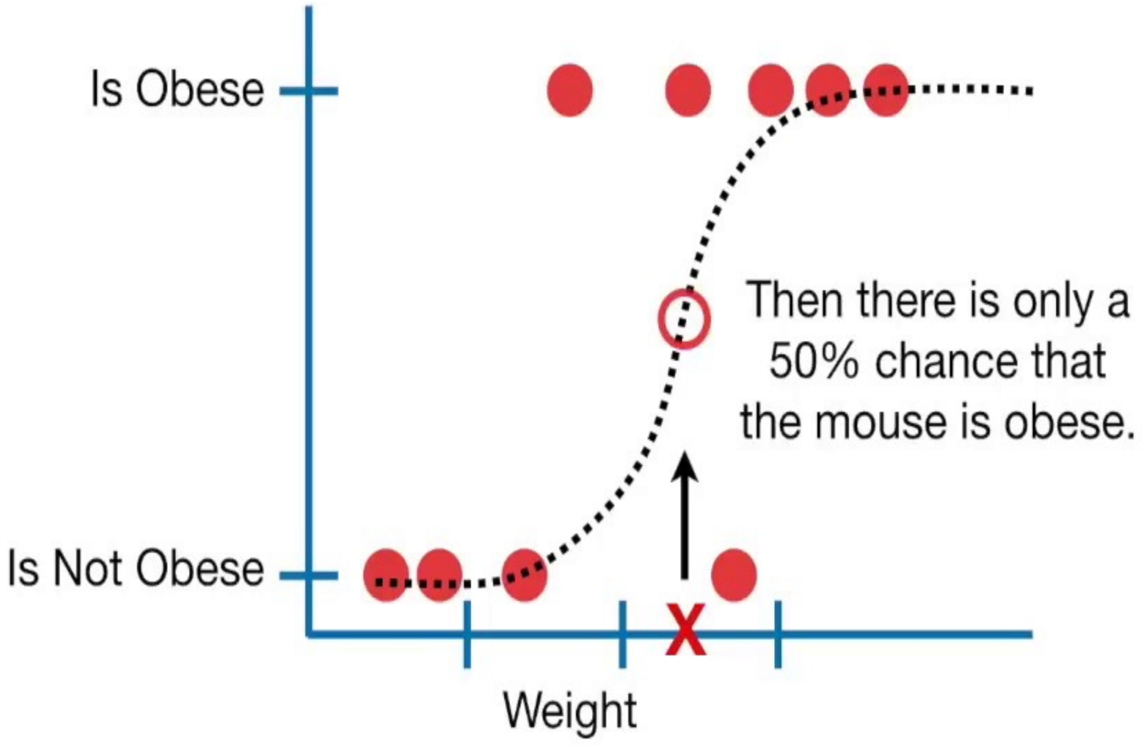
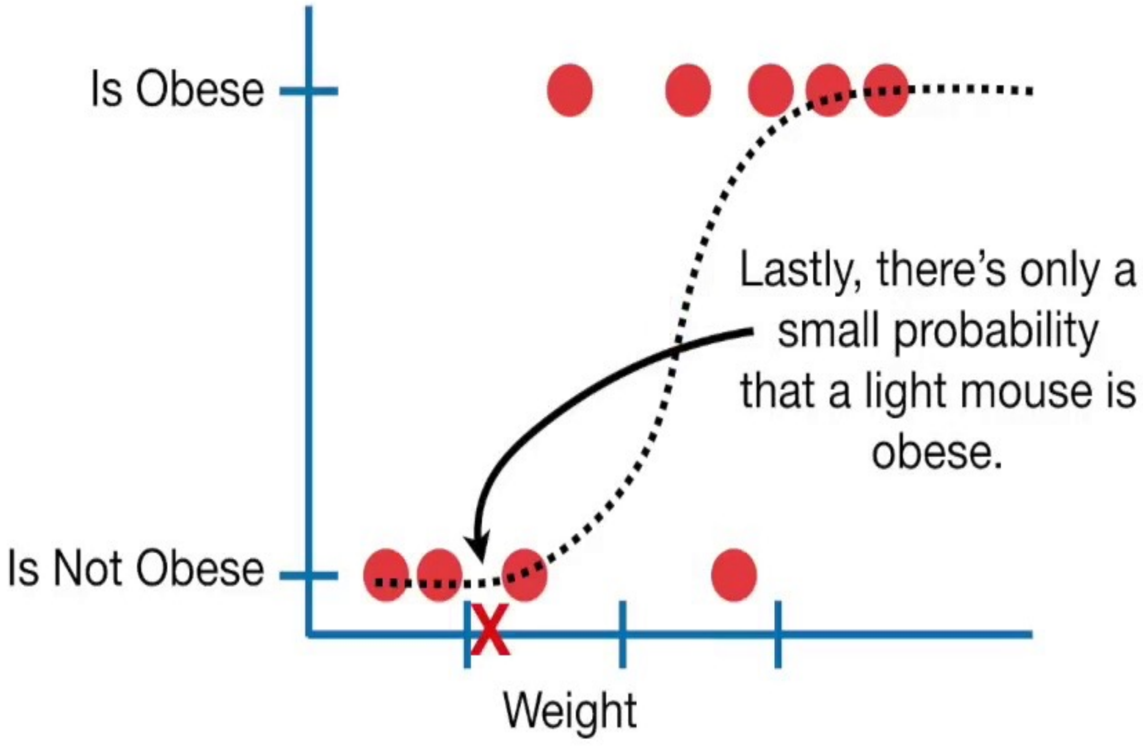


# Logistic Regression

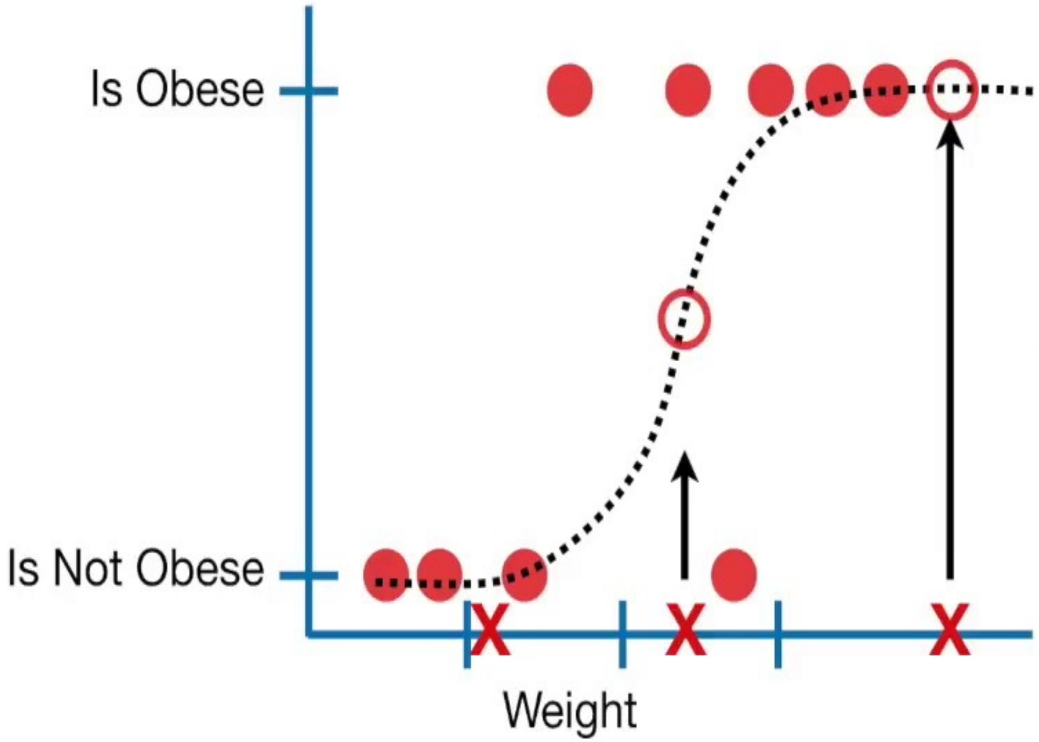


# Logistic Regression

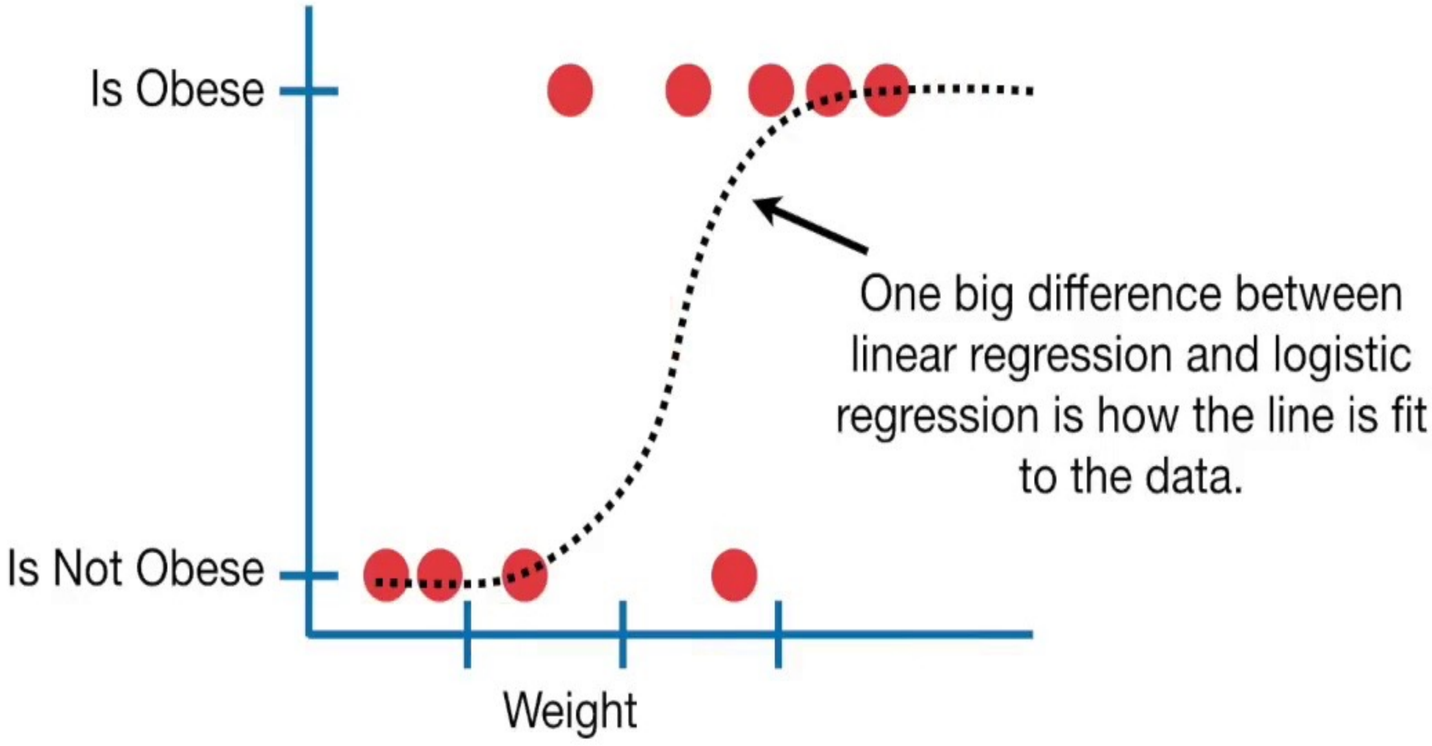


# Logistic Regression

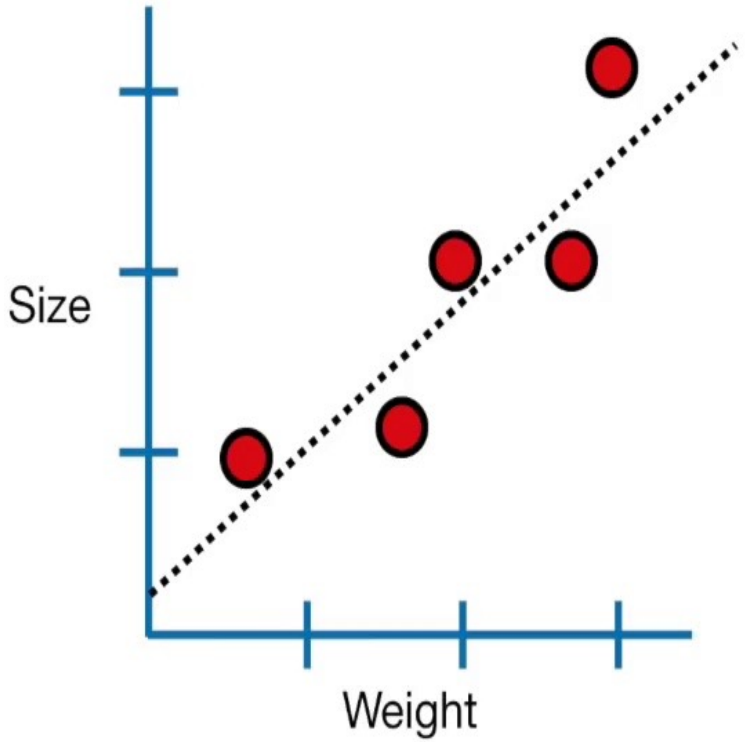
Although logistic regression tells the probability that a mouse is obese or not, it's usually used for classification.



# Linear vs Logistic

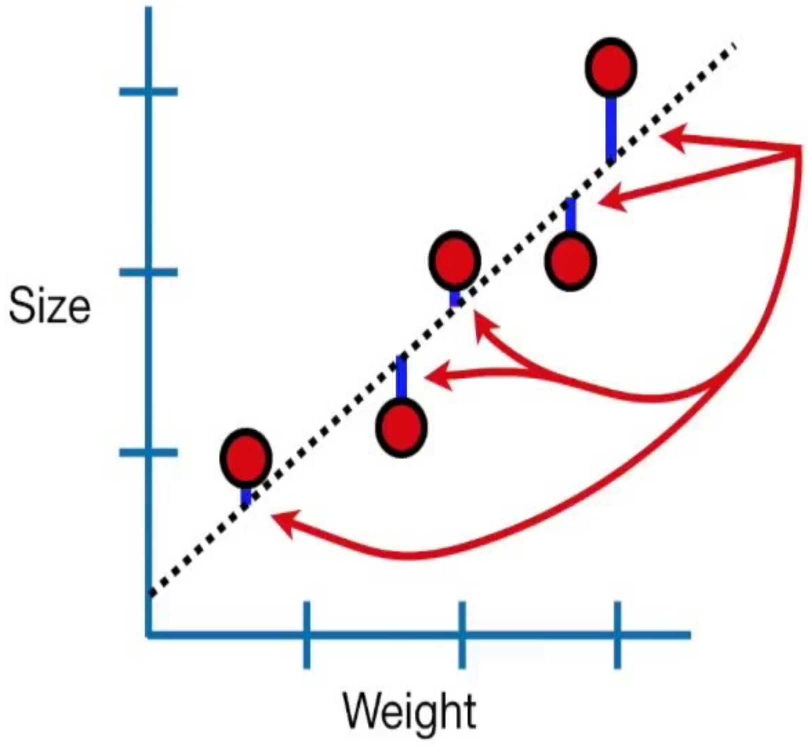


# Linear vs Logistic




With linear regression, we fit the line using "least squares".

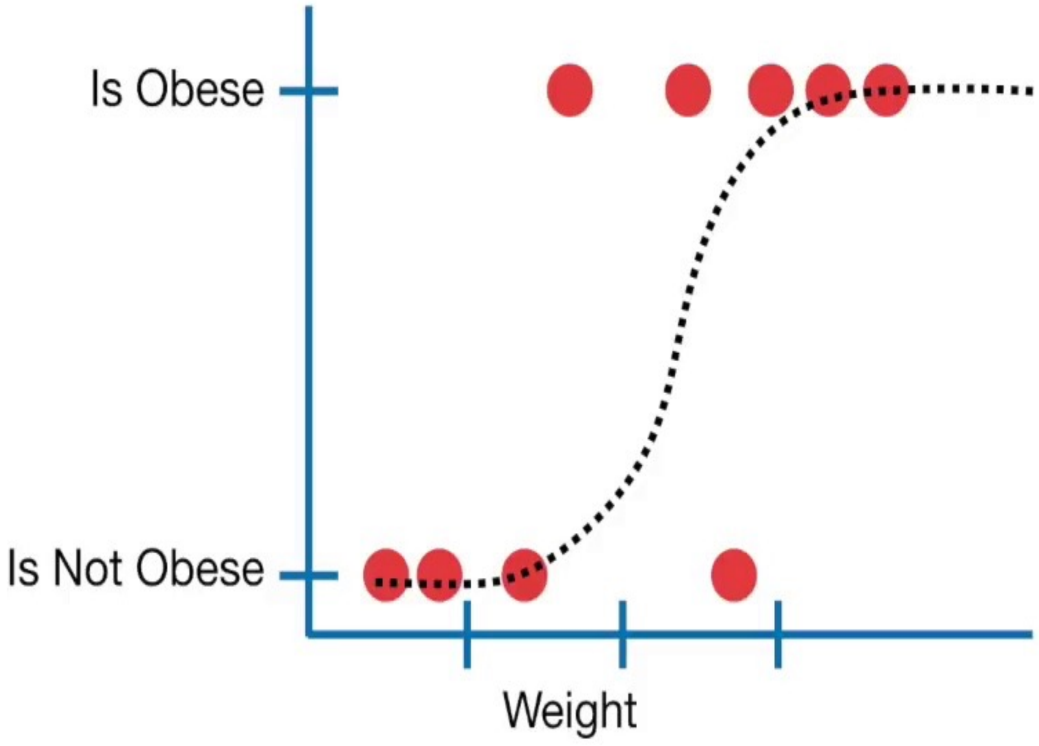
# Linear vs Logistic




In other words, we find the line that minimizes the sum of the squares of these residuals.

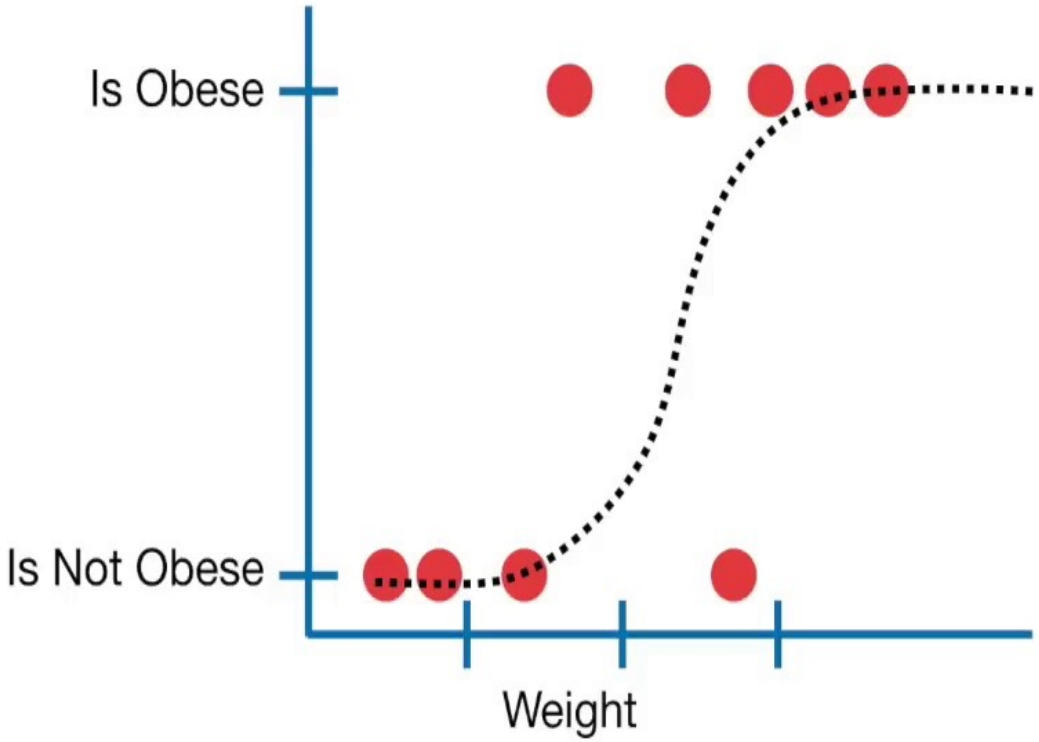
# Linear vs Logistic

Logistic regression doesn't have the same concept of a "residual", so it can't use least squares

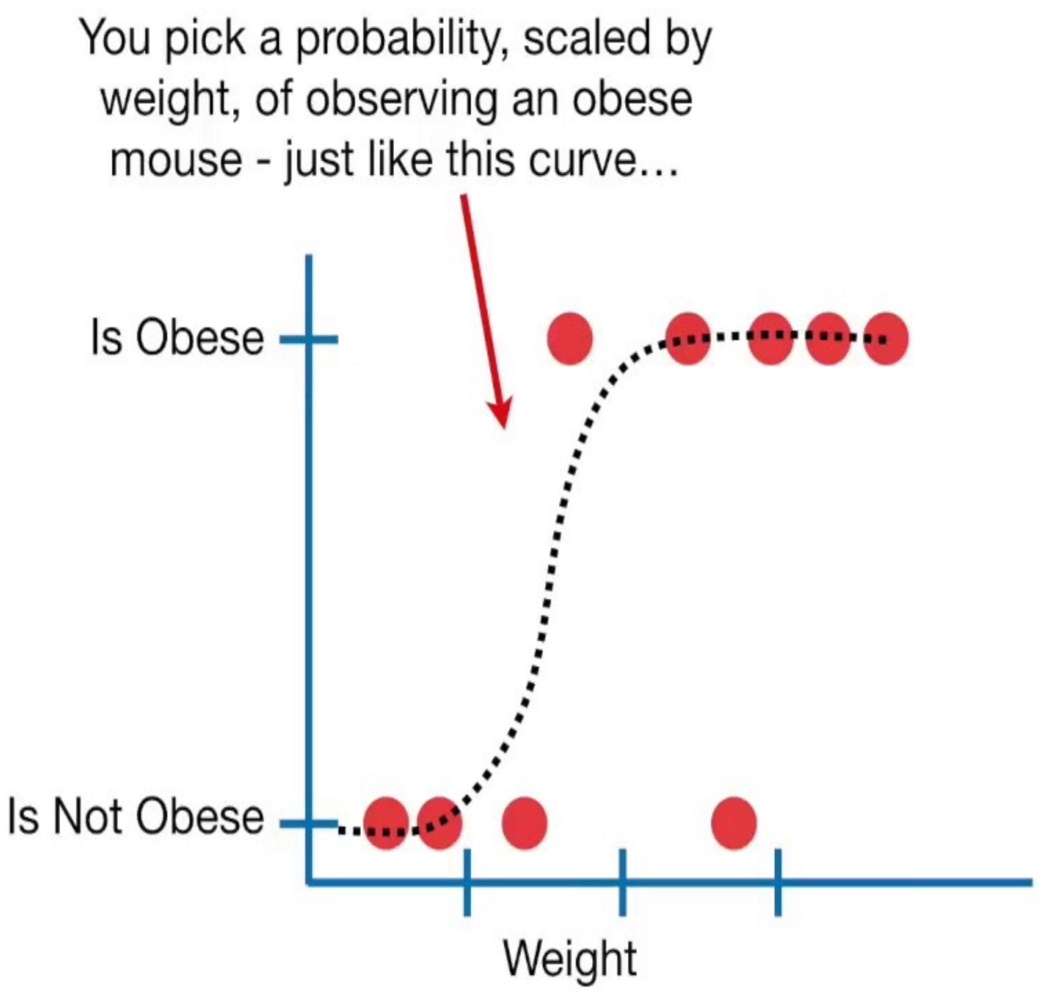


# Linear vs Logistic

Instead it uses something called
“maximum likelihood”.



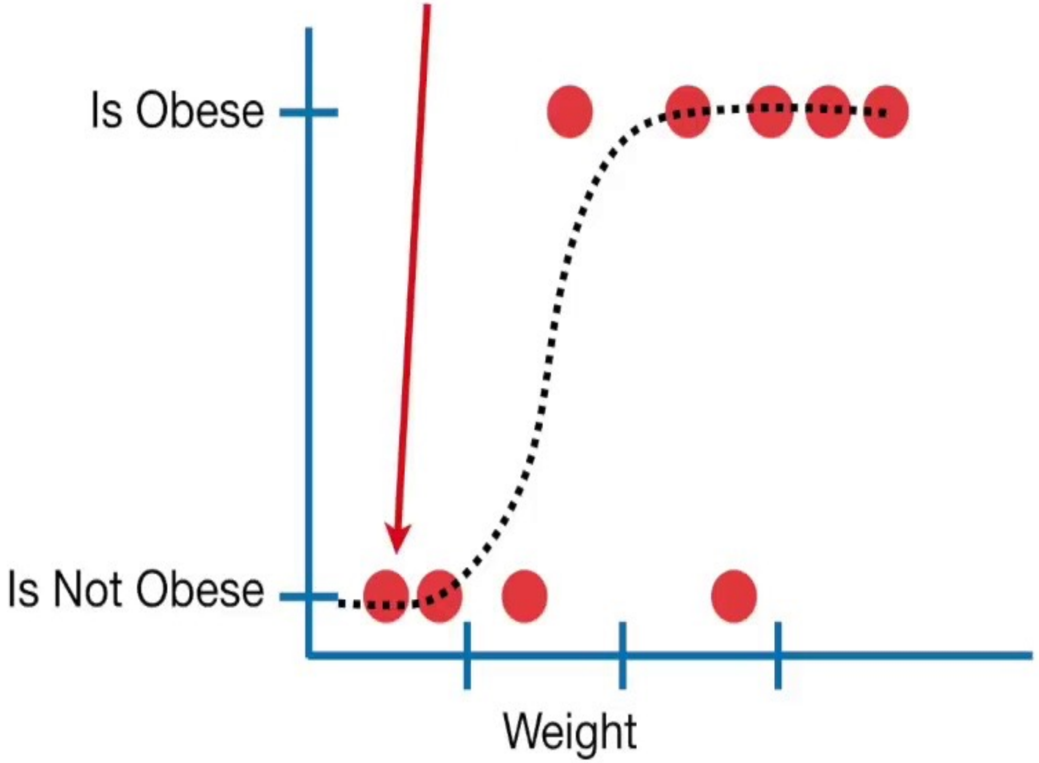
# MLE



You pick a probability, scaled by weight, of observing an obese mouse - just like this curve...
Is Obese
Is Not Obese
Weight

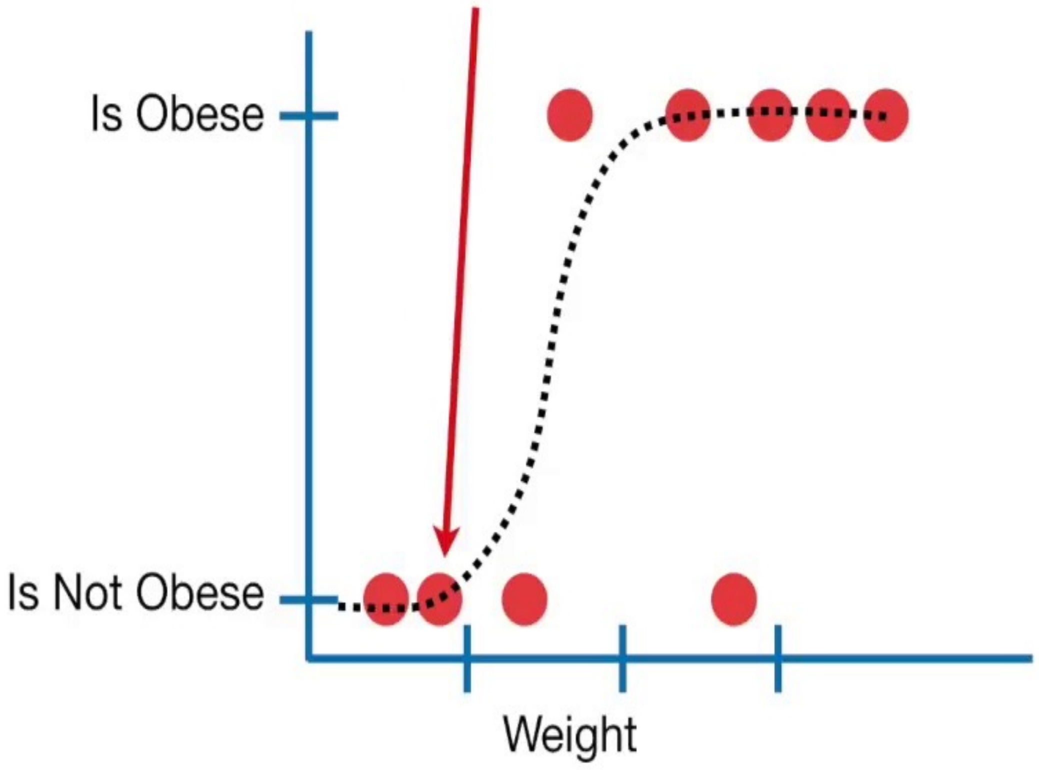
# MLE

...and you use that to calculate the likelihood of observing a non-obese mouse that weighs this much...

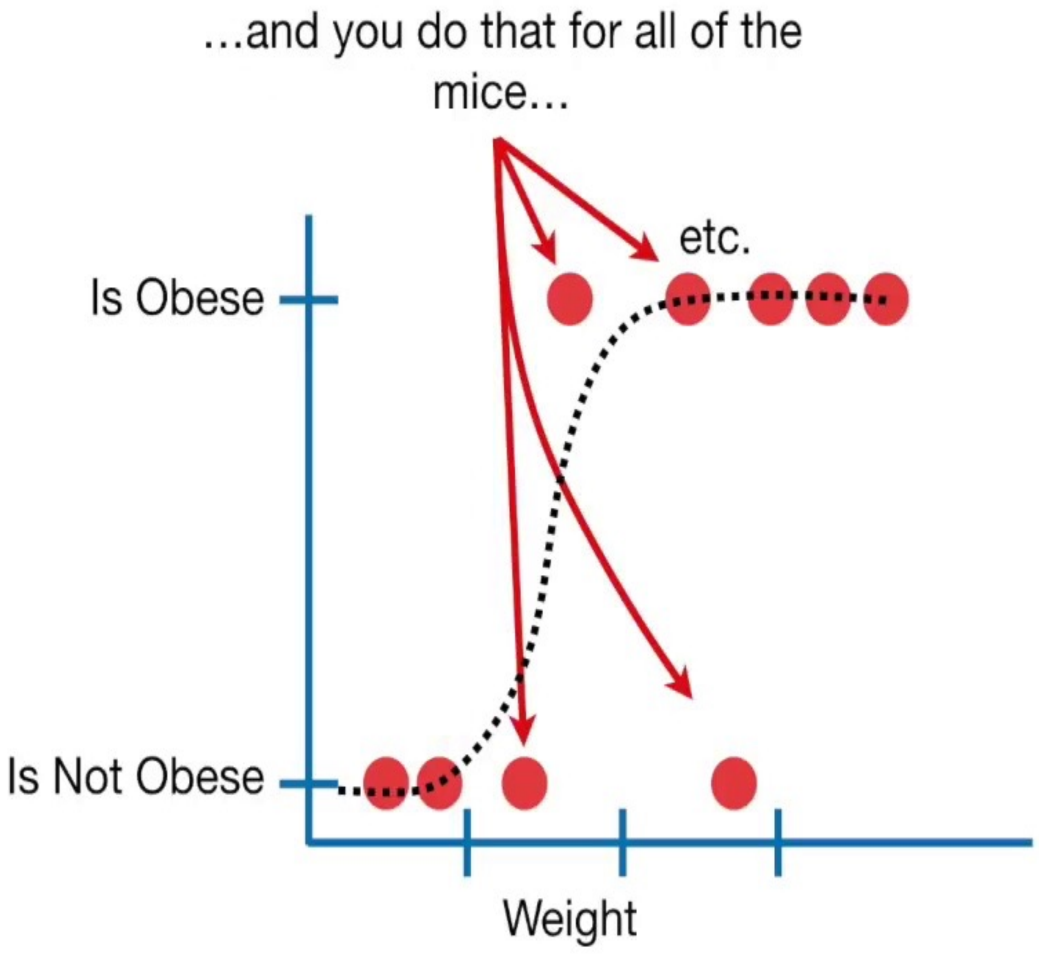


# MLE

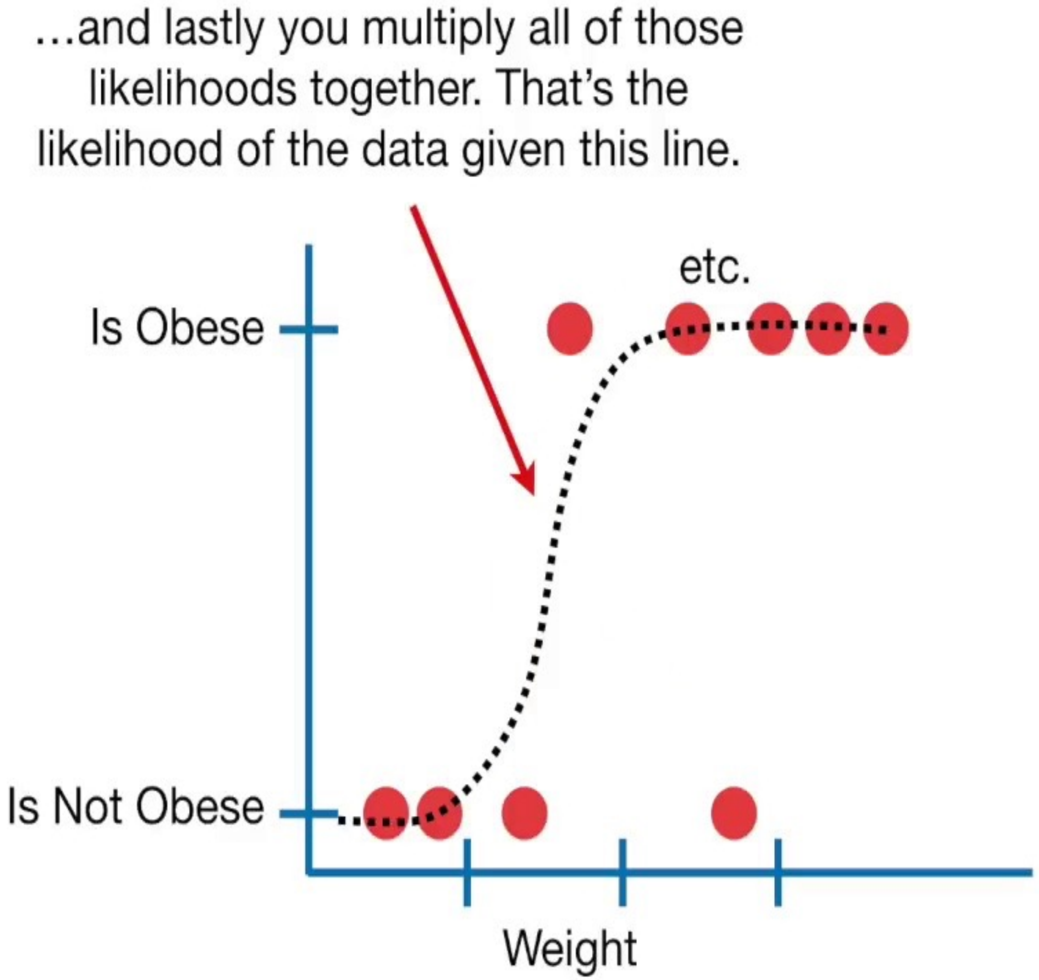
...and then you calculate the likelihood of observing this mouse...



# MLE

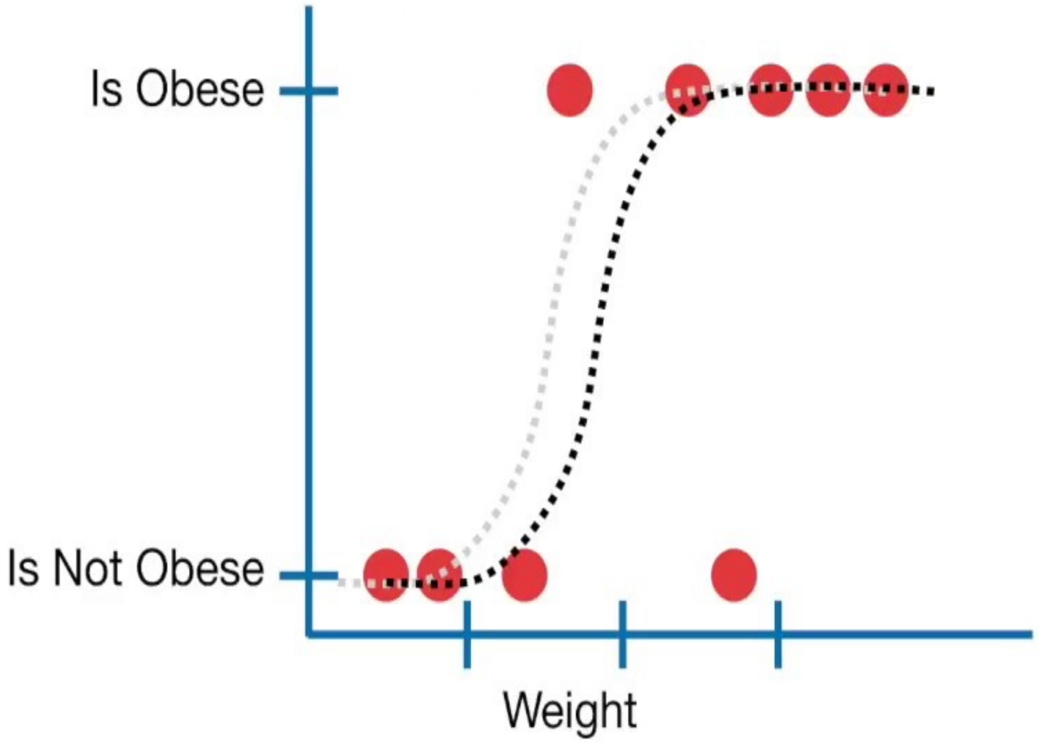

...and you do that for all of the mice...
etc.
Is Obese
Is Not Obese
Weight

# MLE

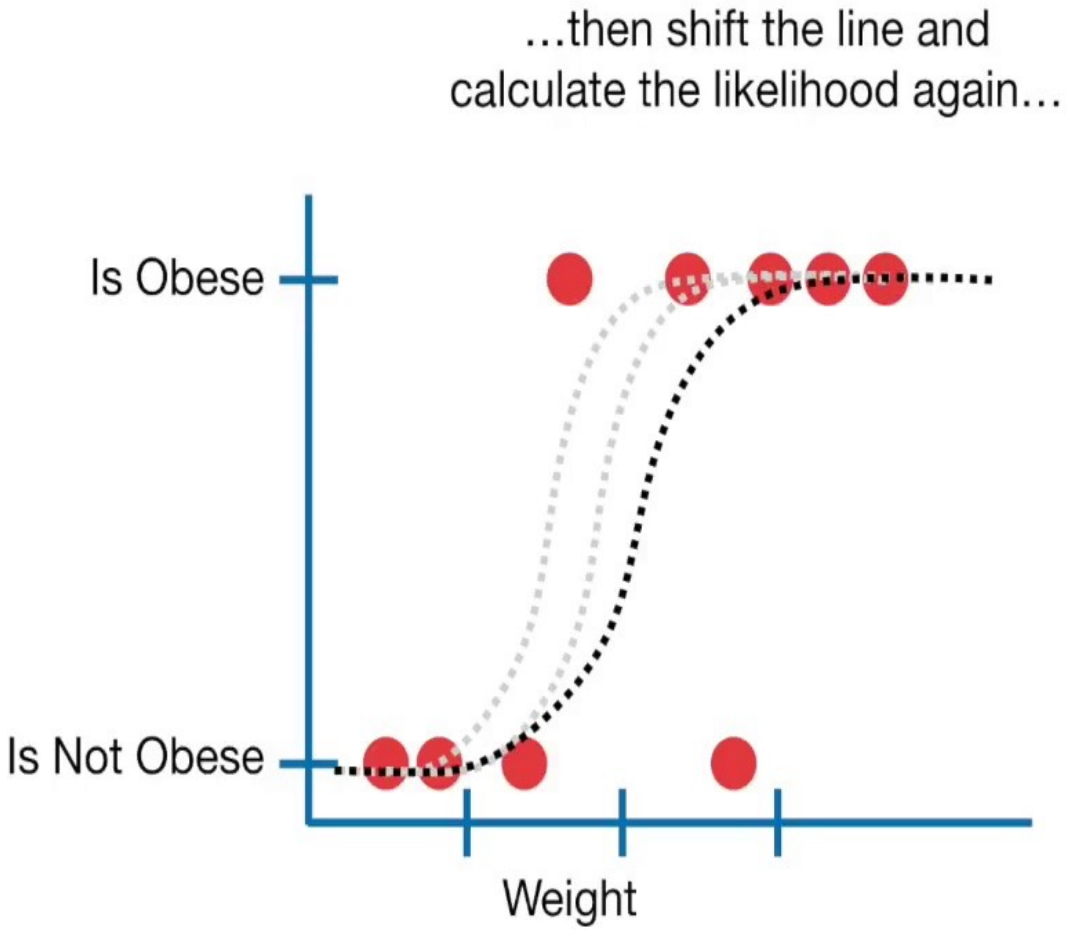

...and lastly you multiply all of those likelihoods together. That's the likelihood of the data given this line.
etc.
Is Obese
Is Not Obese
Weight

# MLE

Then you shift the line and calculate a new likelihood of the data...

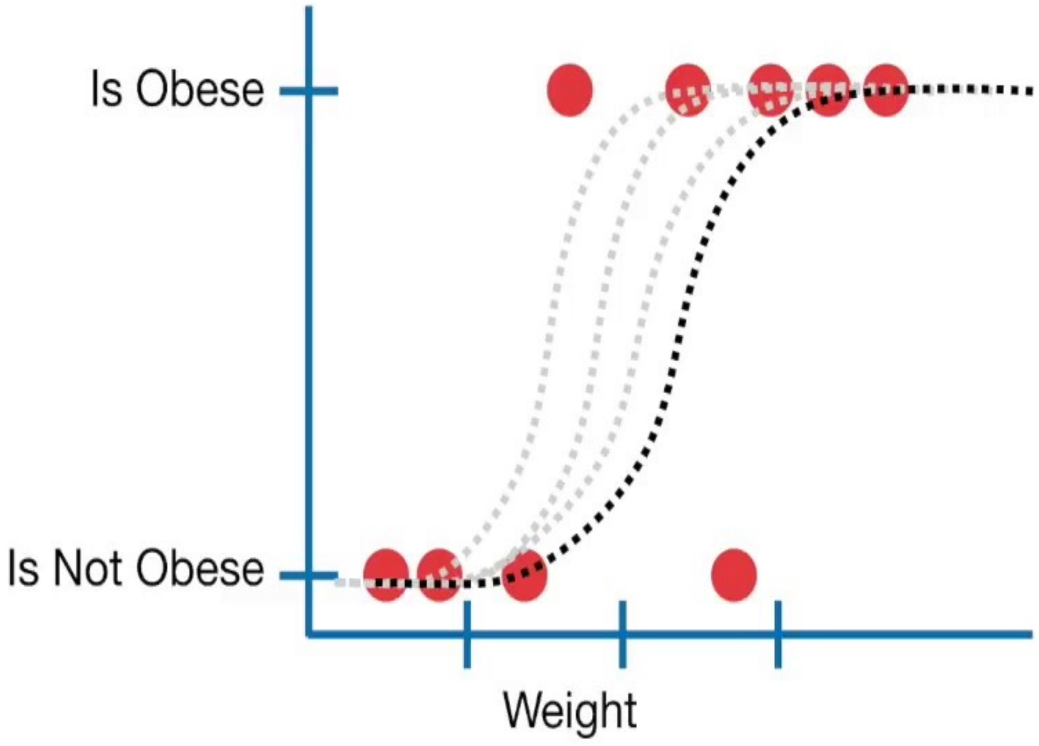


# MLE

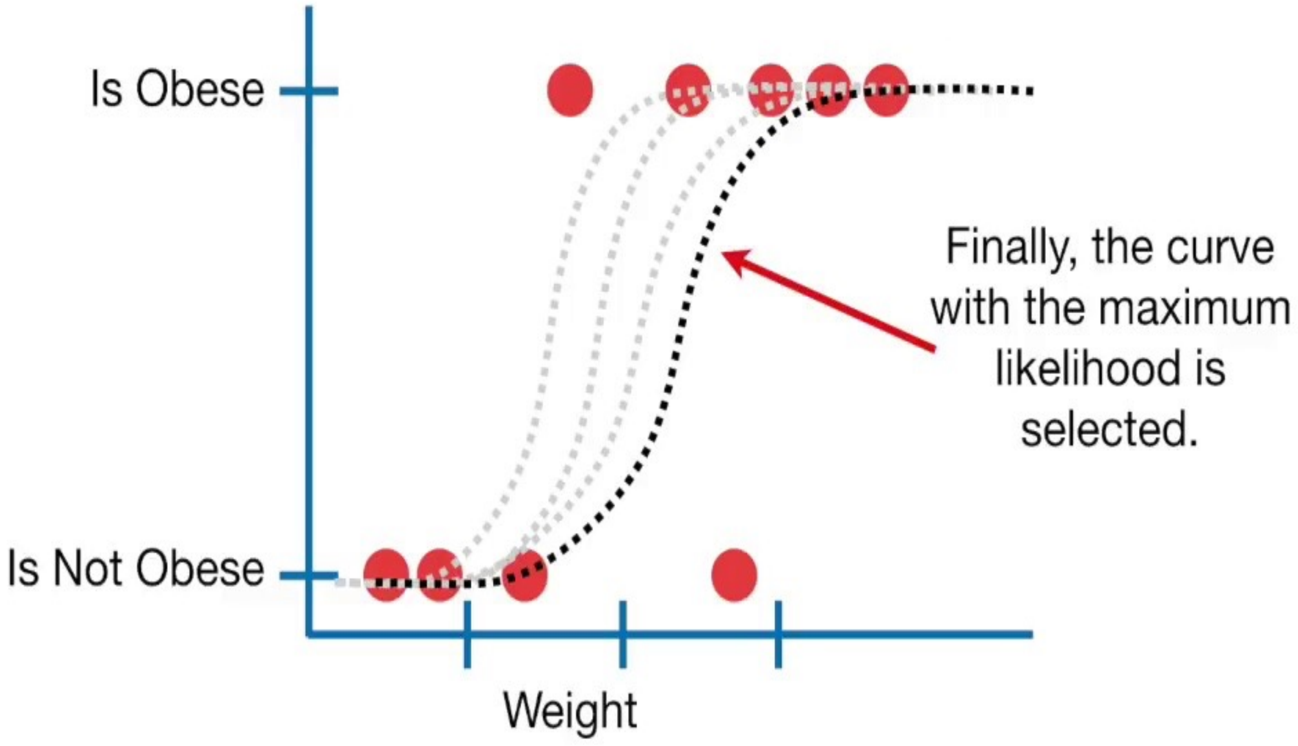

...then shift the line and
calculate the likelihood again...
Is Obese
Is Not Obese
Weight

# MLE

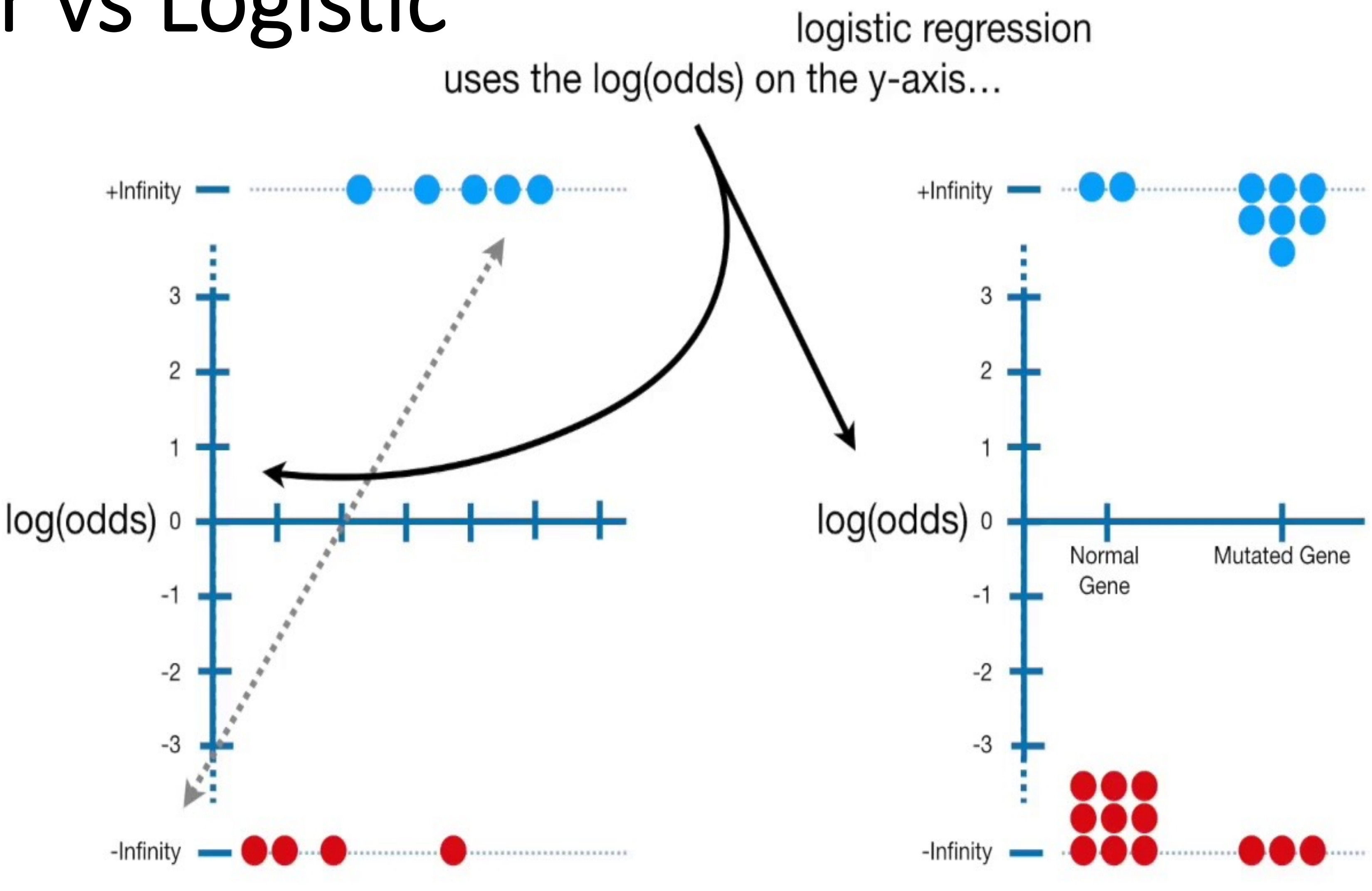
…and again…



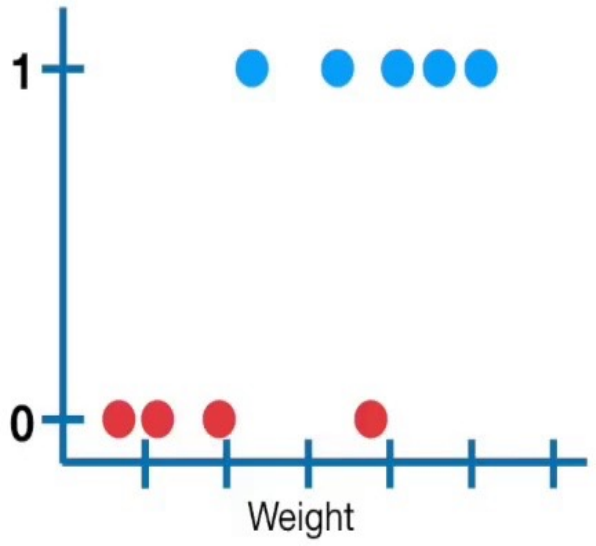
# MLE


Is Obese
Is Not Obese
Weight
Finally, the curve with the maximum likelihood is selected.

# Linear vs Logistic

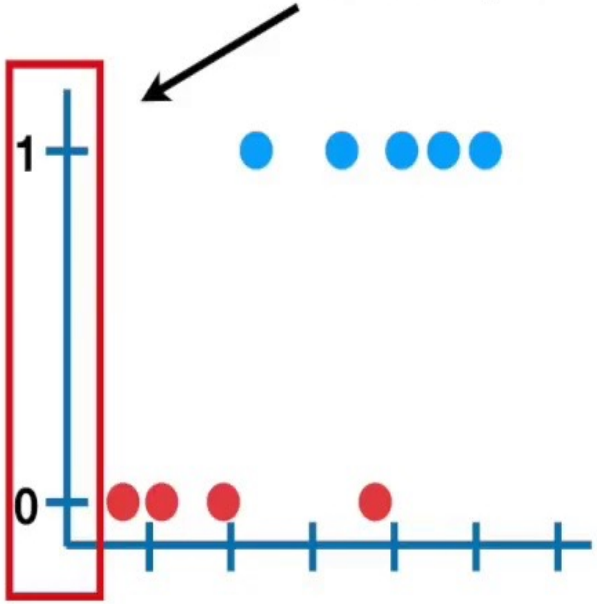


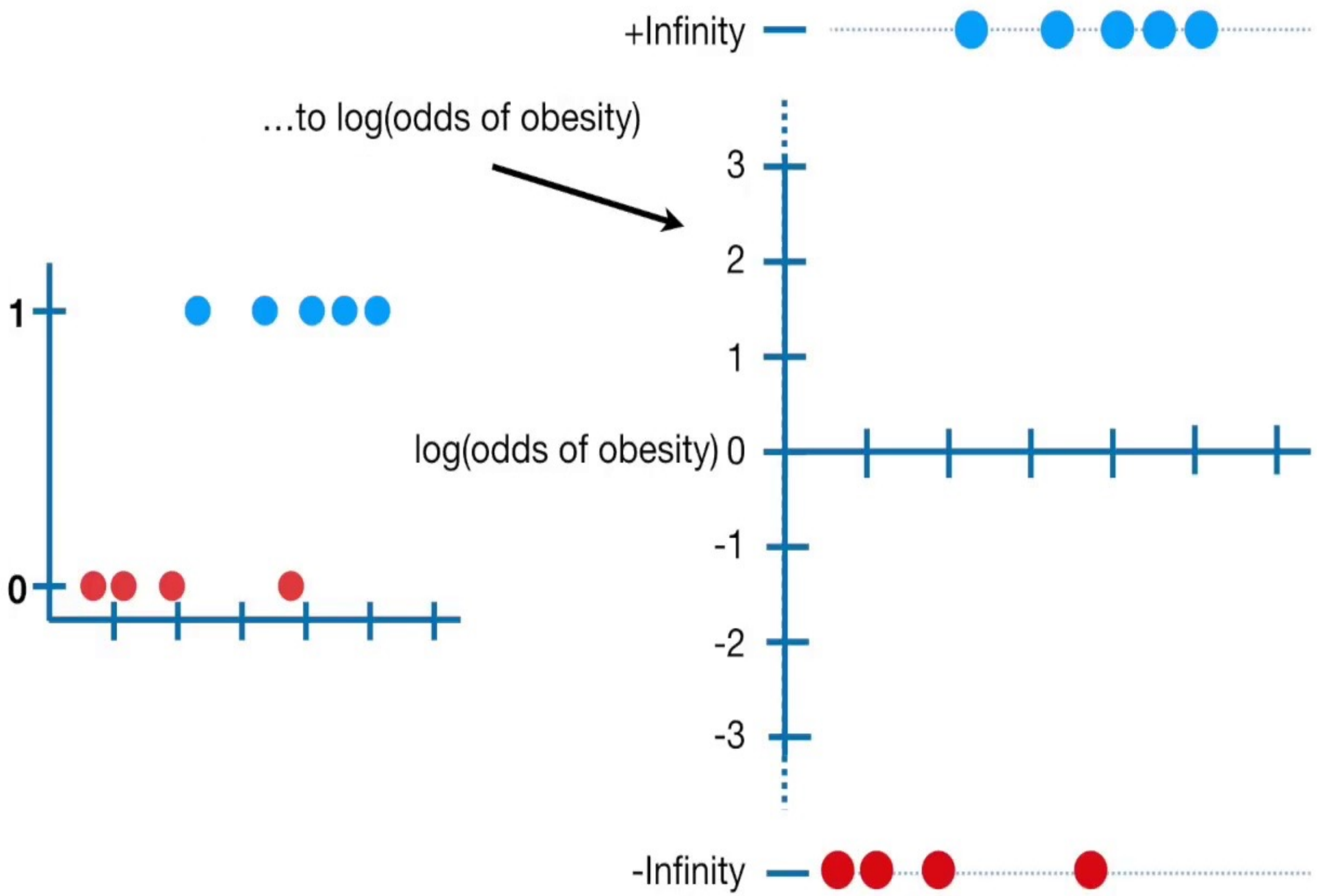
# Fit a Line with Logistic Regression



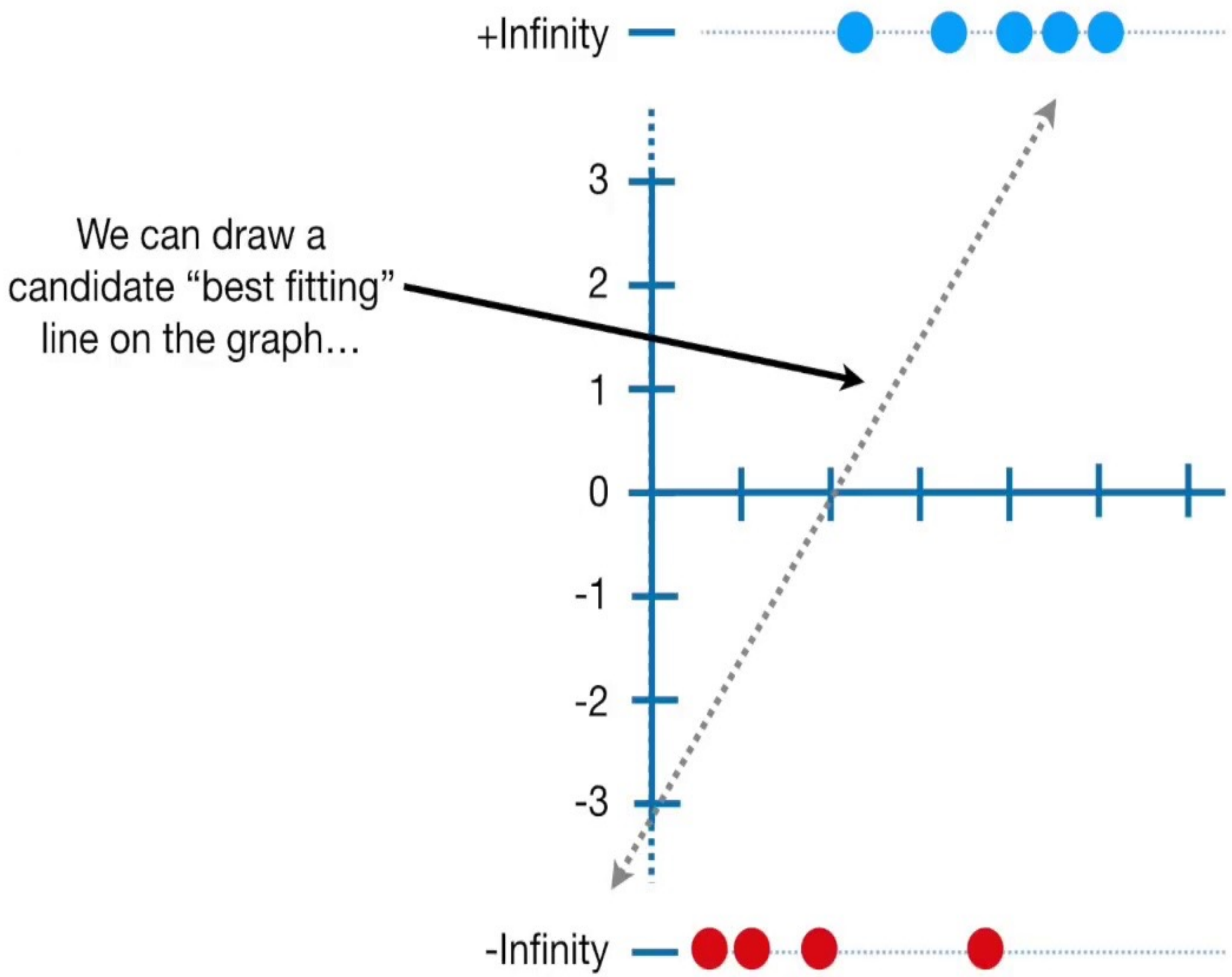
# Fit a Line with Logistic Regression

As we know, in logistic regression, we transform the y-axis from the probability of obesity…



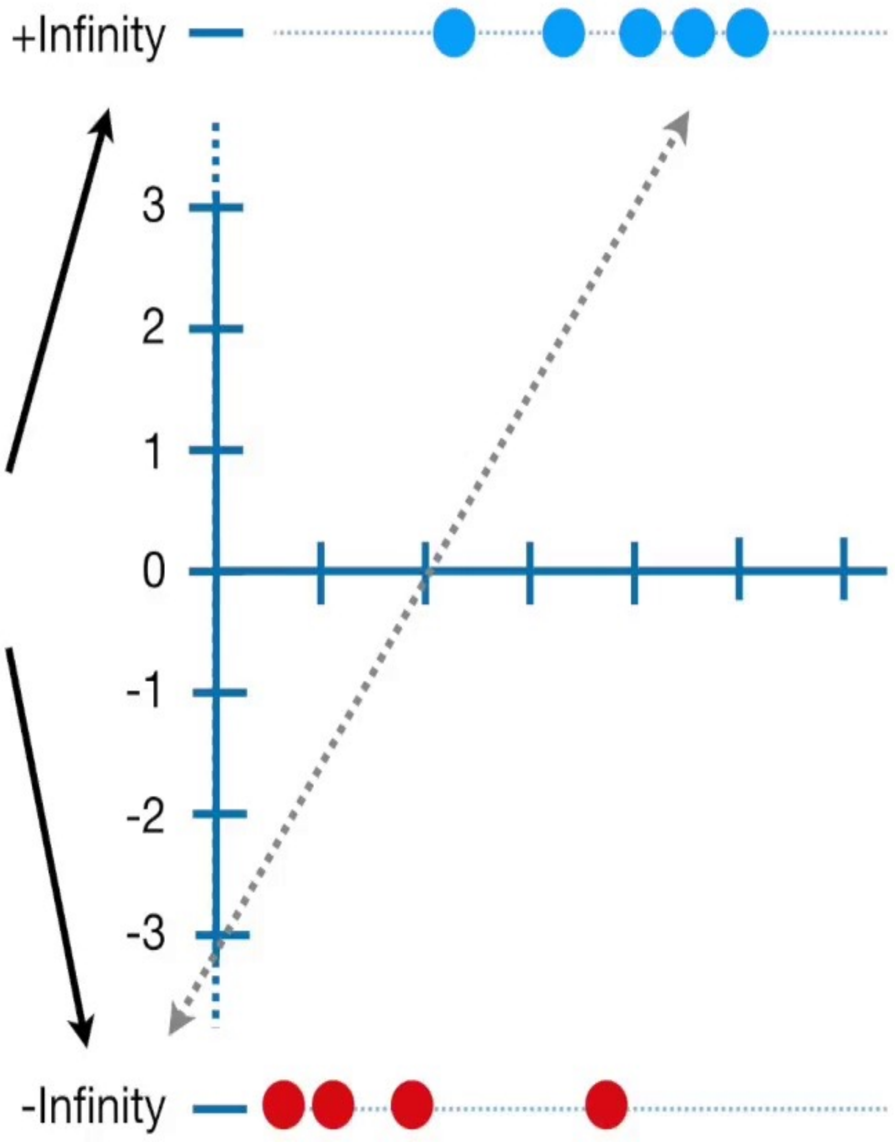


+Infinity
...to log(odds of obesity)
3
2
1
log(odds of obesity) 0
-1
-2
-3
-Infinity
1
0




$$\beta_0 + \beta_1 X$$

The only problem is that the transformation pushes the raw data to positive and negative infinity…




$$\beta_0 + \beta_1 X$$

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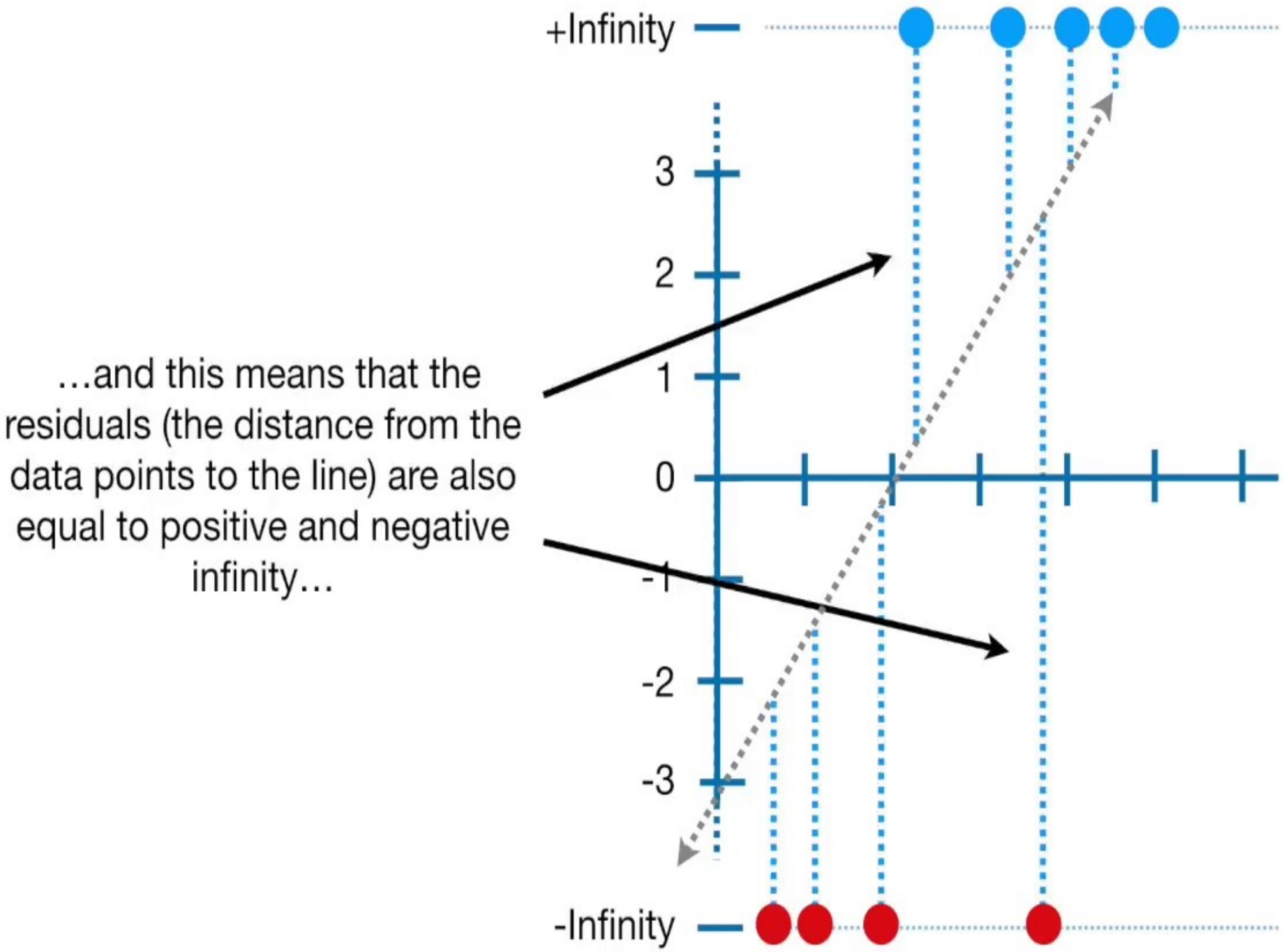
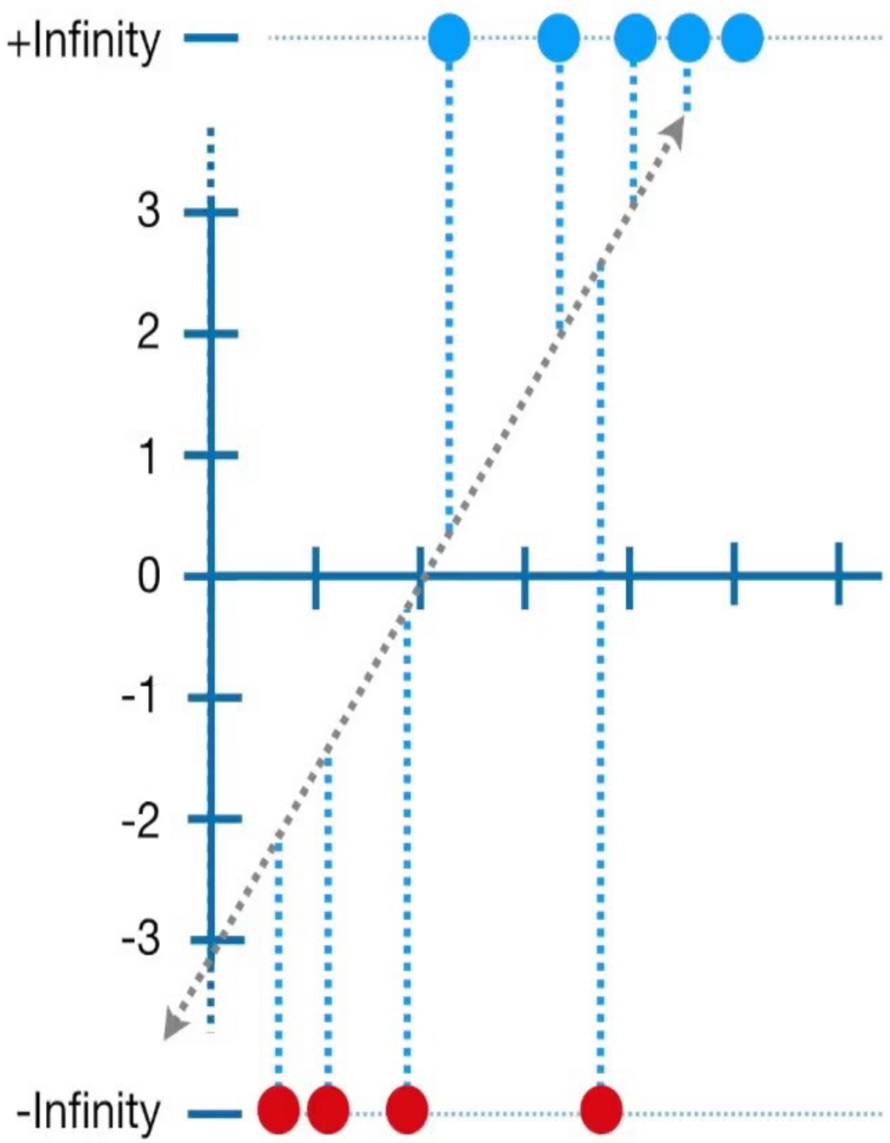
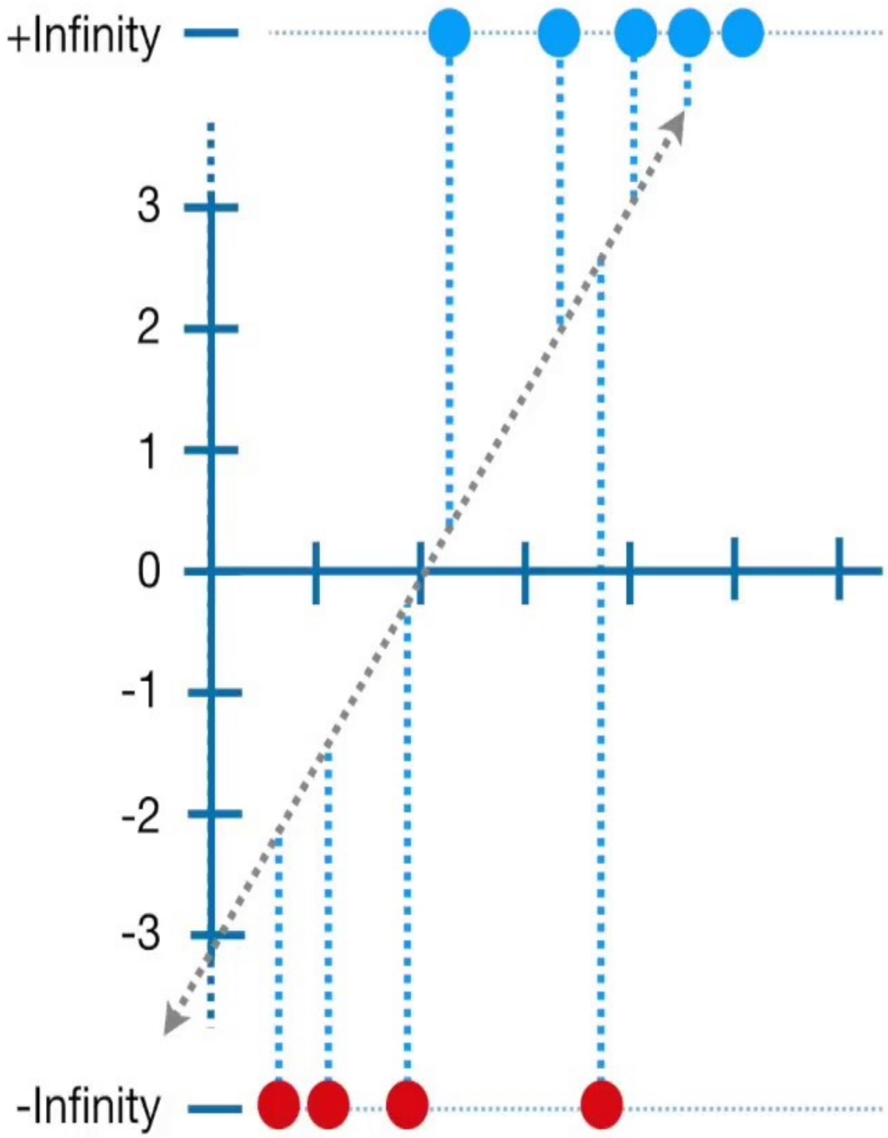


…and this means we can't use least-squares to find the best fitting line.




$$\beta_0 + \beta_1 X$$

Instead, we use maximum likelihood...

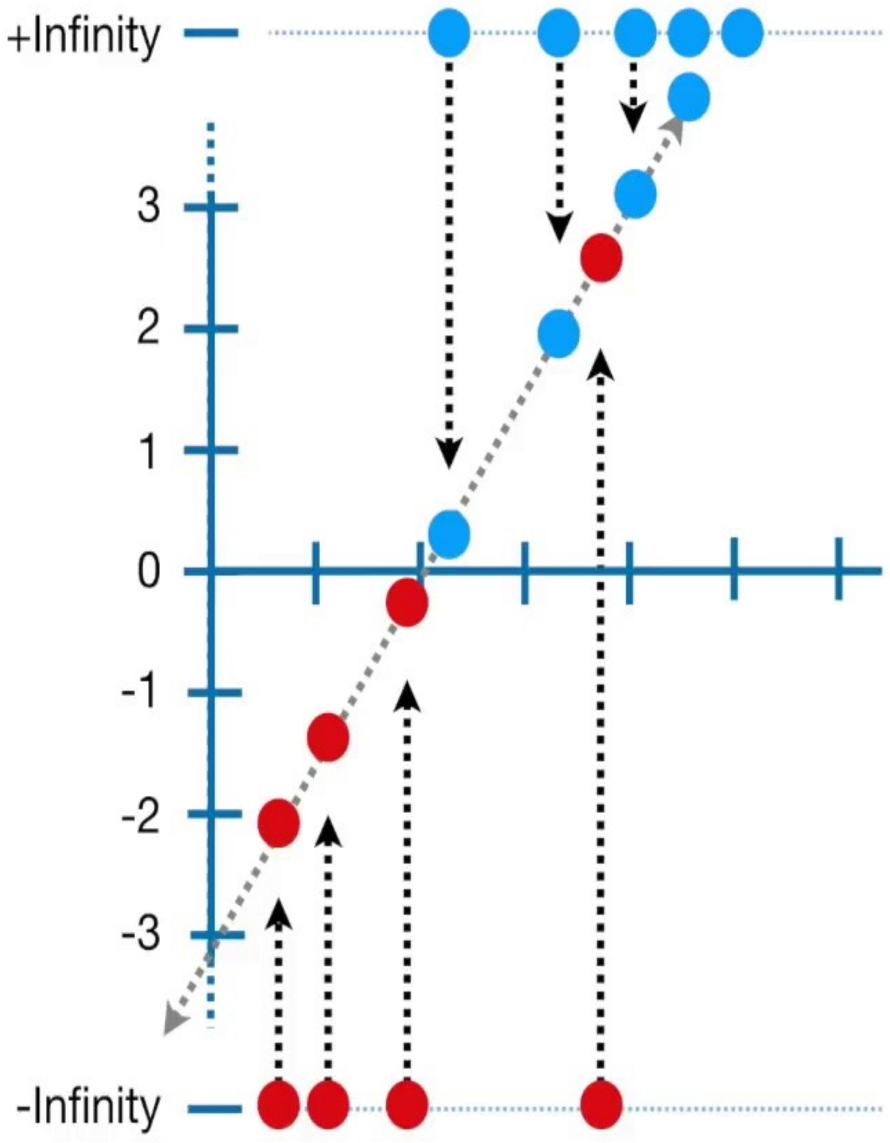


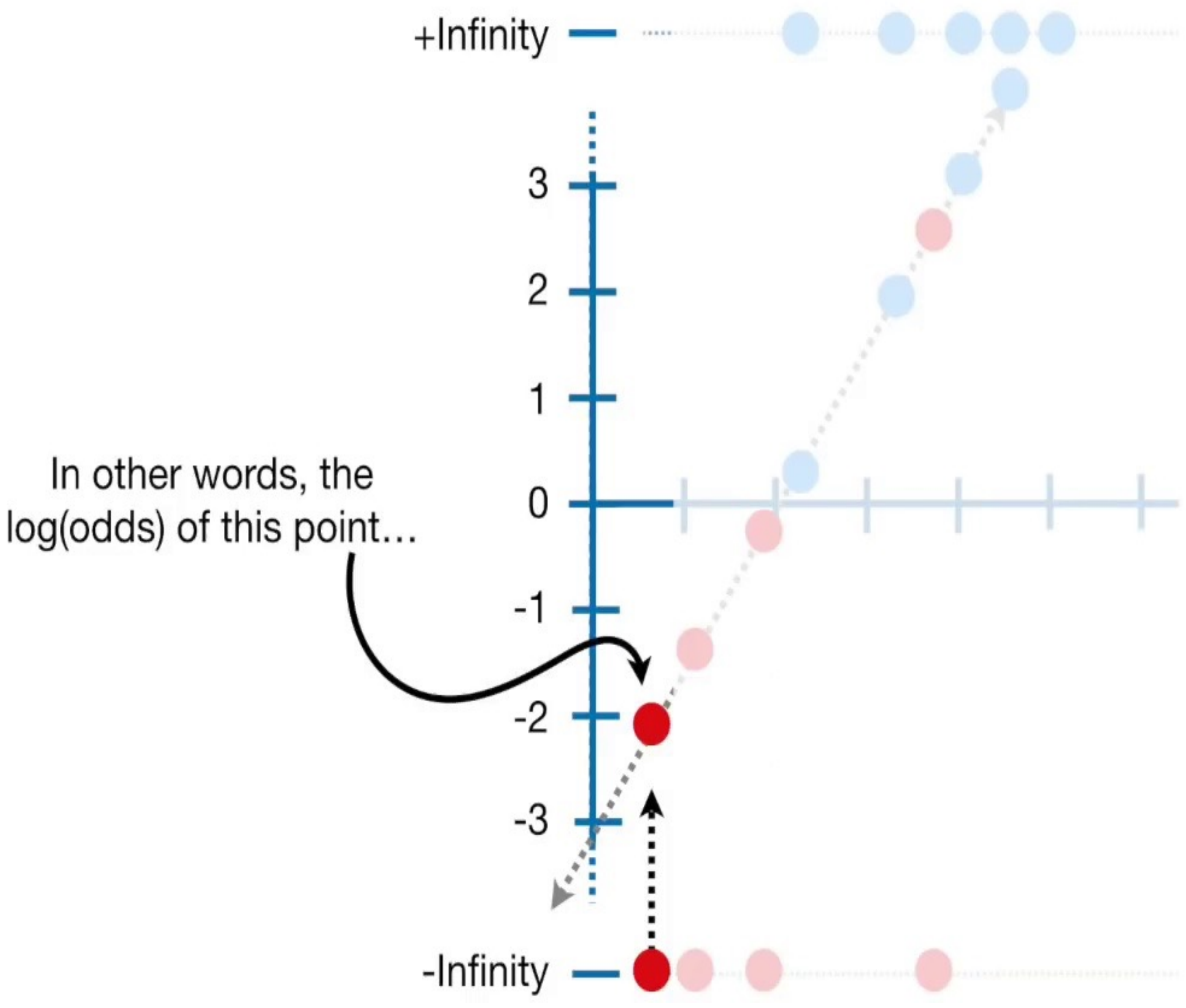

$$\beta_0 + \beta_1 X$$

$$\beta_0 + \beta_1 X$$

The first thing we do is project the original data points onto the candidate line.

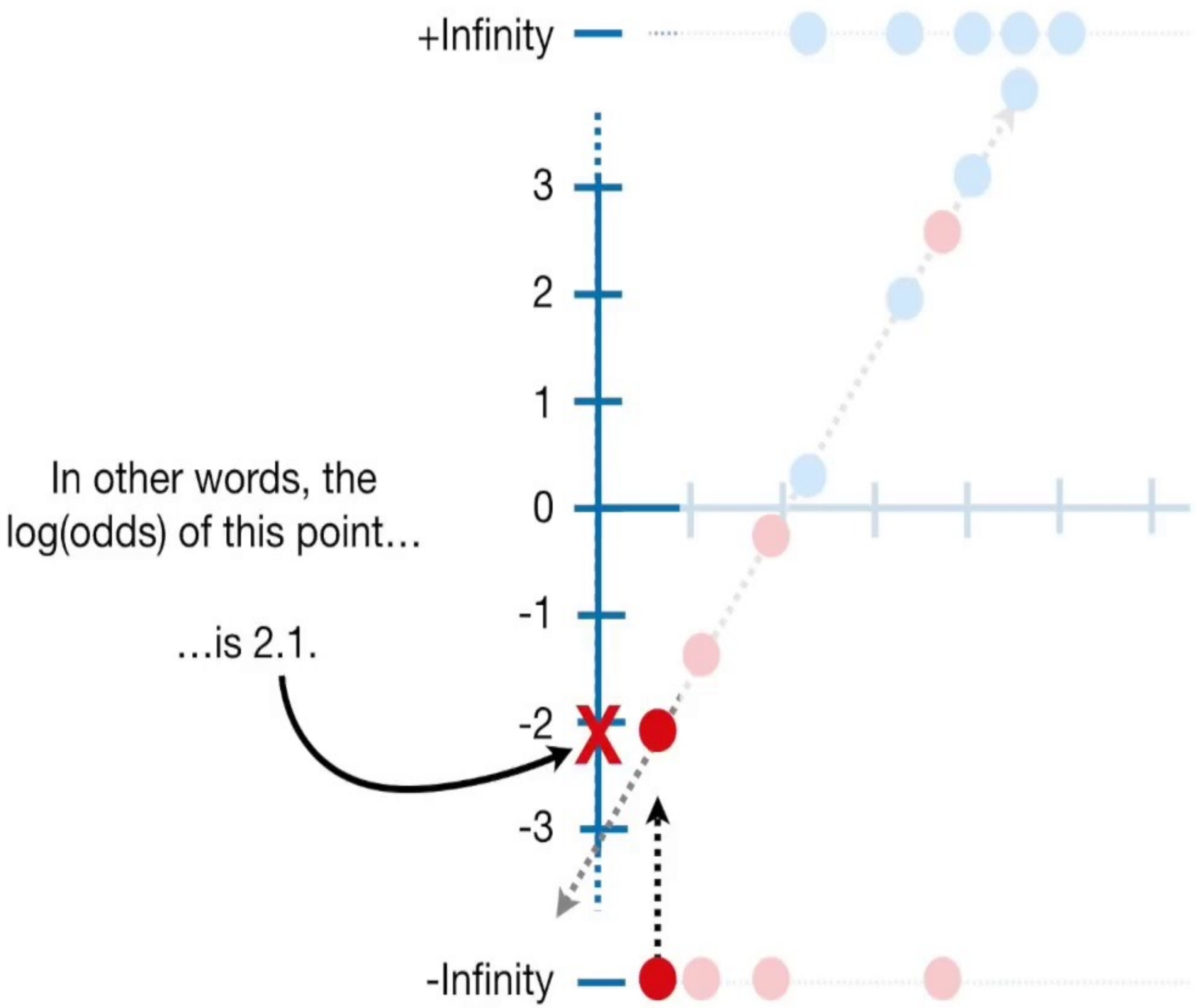
This gives each sample a candidate log(odds) value.

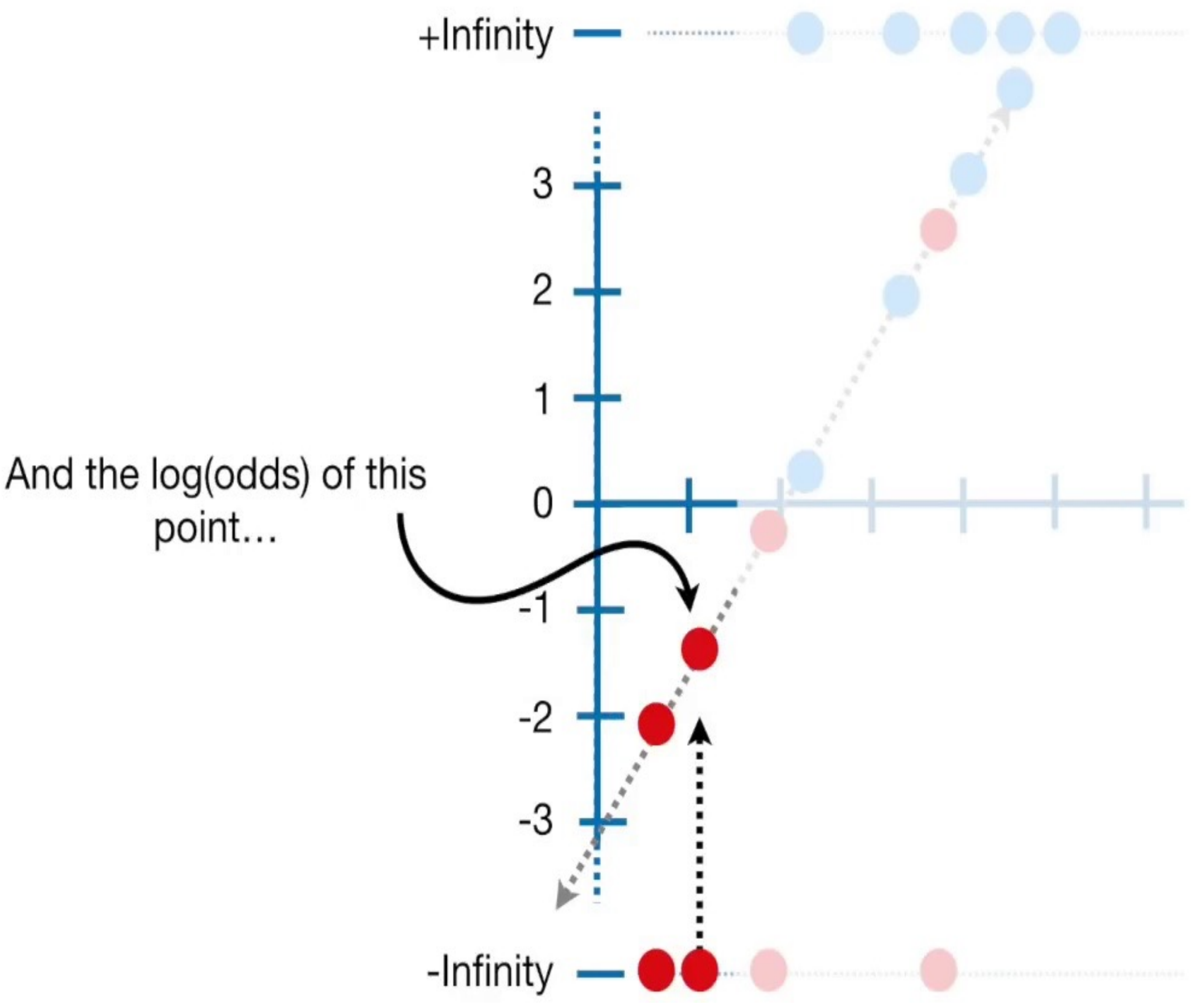





$$\beta_0 + \beta_1 X$$

$$\beta_0 + \beta_1 X$$






$$\beta_0 + \beta_1 X$$

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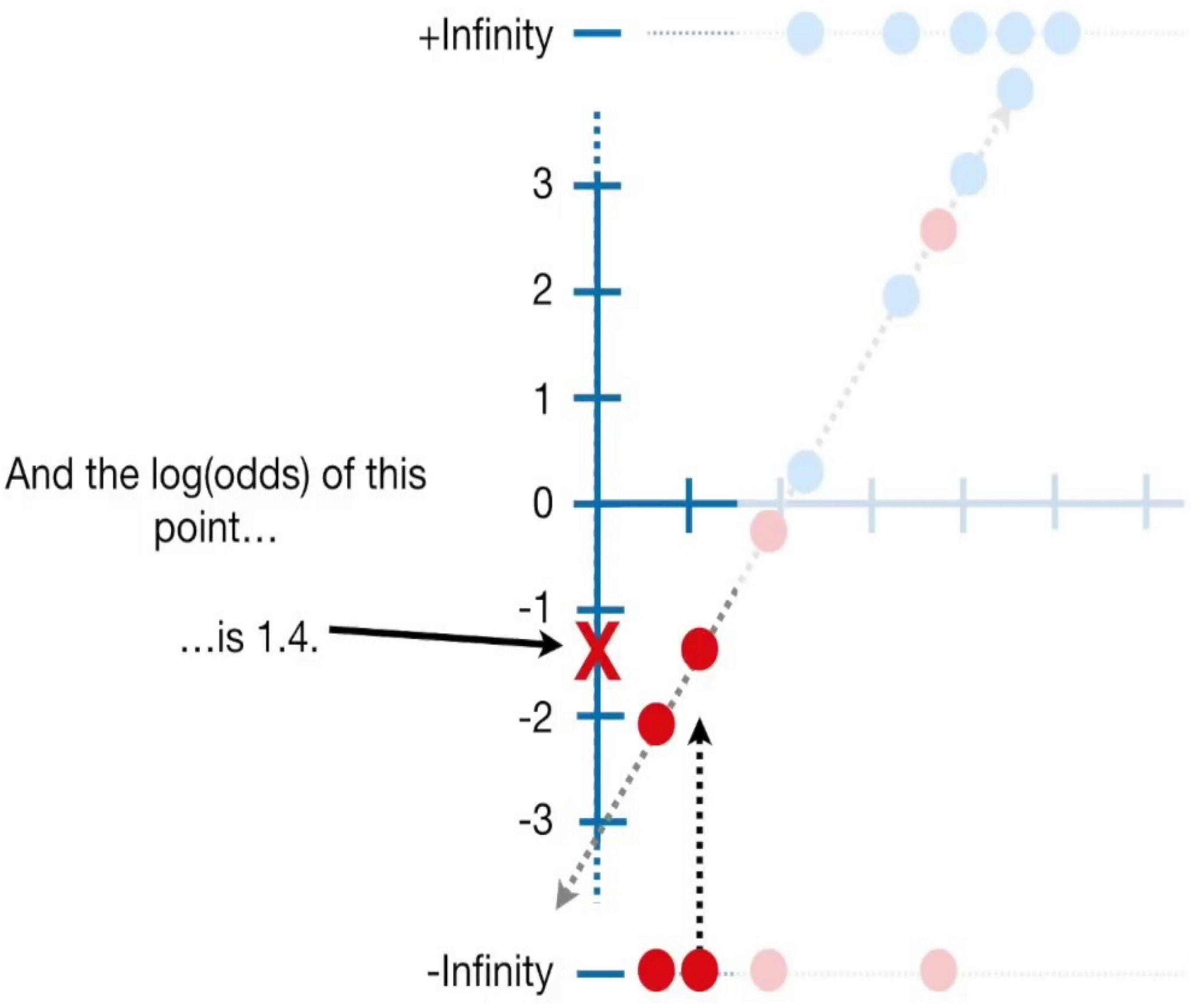
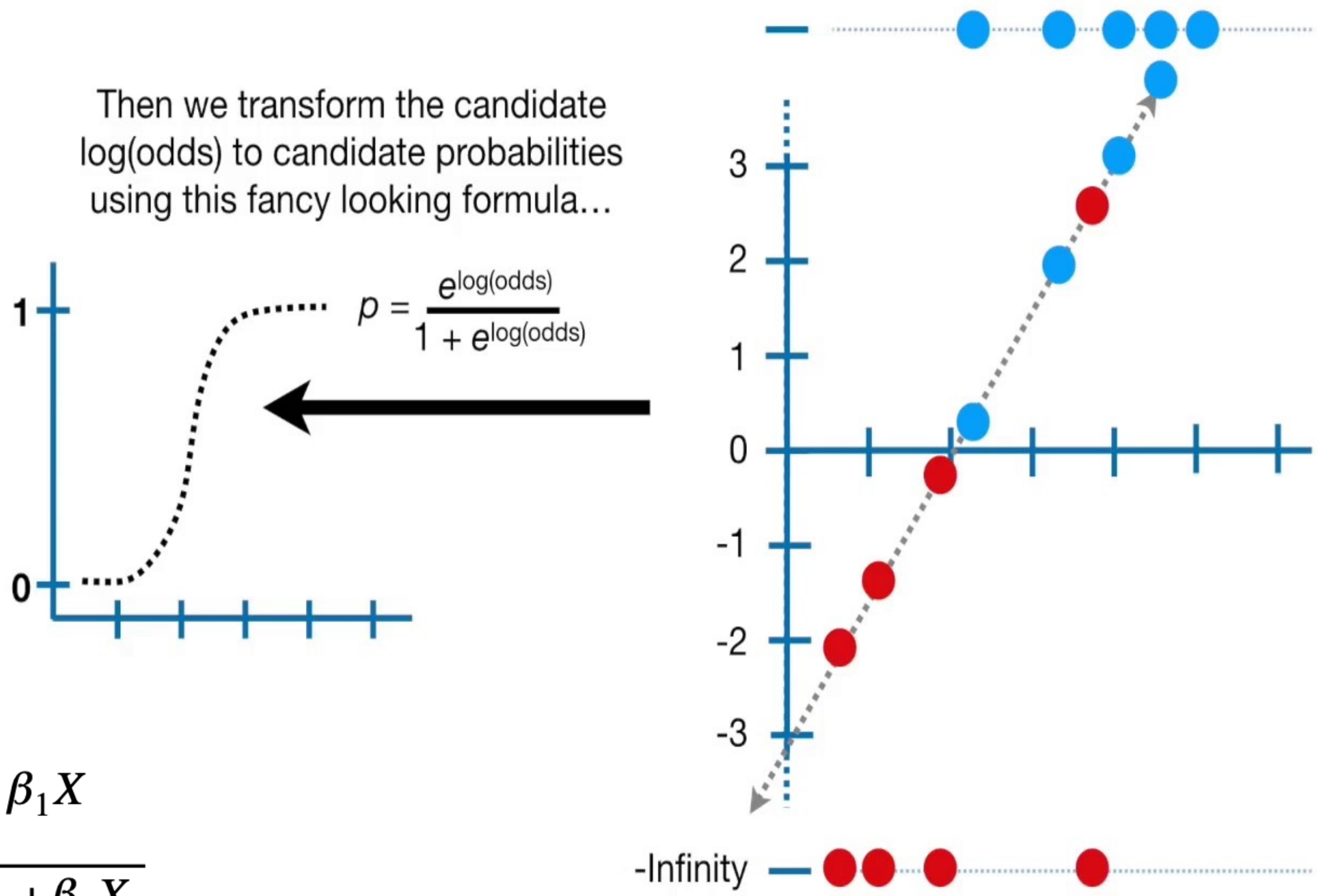
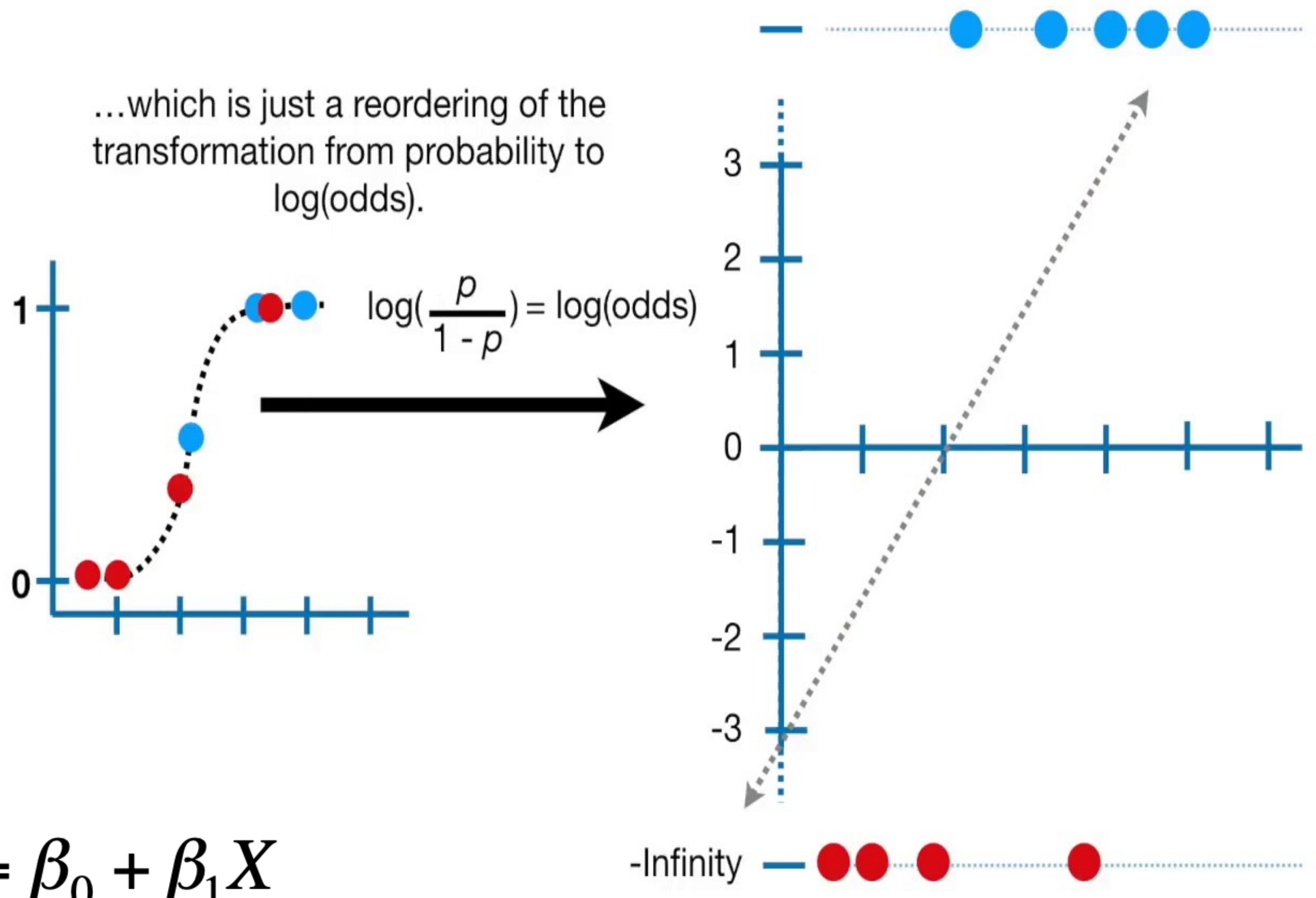





$$\beta_0 + \beta_1 X$$

$$p = \frac{e^{\beta_0 + \beta_1 X}}{1 + e^{\beta_0 + \beta_1 X}}$$

$\beta_0 + \beta_1 X$



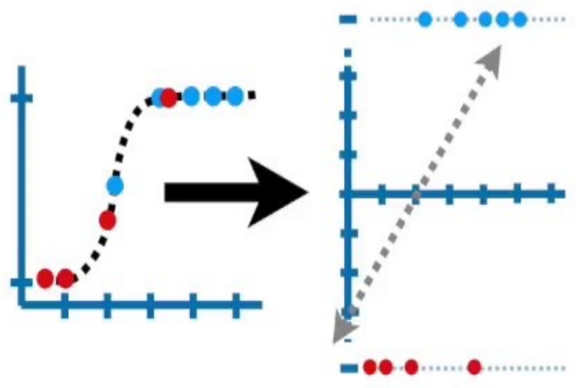

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$

# Logistic Regression Equation

$$\log\left(\frac{p}{1-p}\right) = \log(\text{odds})$$

Exponentiate both sides...

$$\frac{p}{1-p} = e^{\log(\text{odds})}$$



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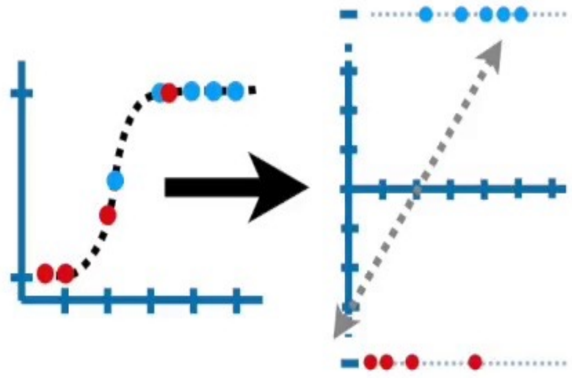
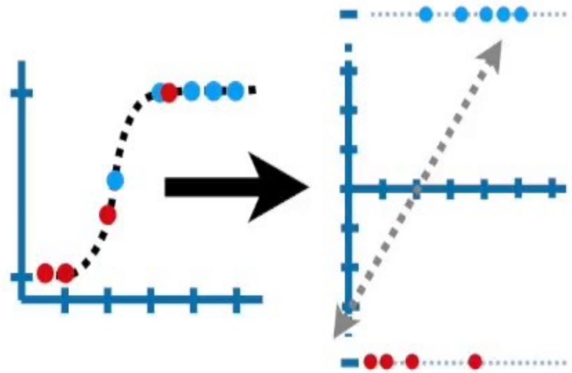


Exponentiate both sides... $$\frac{p}{1-p} = e^{\log(\text{odds})}$$

Multiply both sides by (1 - $p$)... $$p = (1-p)e^{\log(\text{odds})}$$

# Logistic Regression Equation

$$\log\left(\frac{p}{1-p}\right) = \log(\text{odds})$$



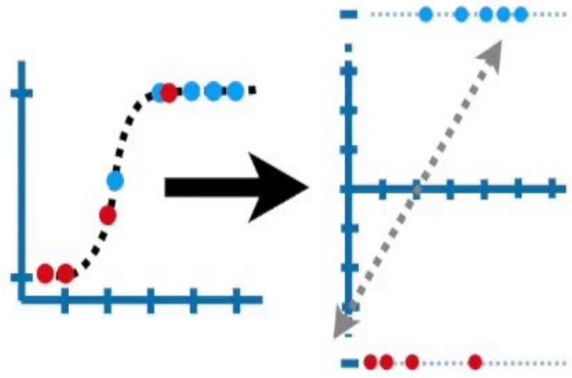
Exponentiate both sides... $$\frac{p}{1-p} = e^{\log(\text{odds})}$$

Multiply both sides by $(1 - p)$... $$p = (1 - p)e^{\log(\text{odds})}$$

Multiply $(1 - p)$ and $e^{\log(\text{odds})}$... $$p = e^{\log(\text{odds})} - pe^{\log(\text{odds})}$$

# Logistic Regression Equation

$$\log\left(\frac{p}{1-p}\right) = \log(\text{odds})$$



Exponentiate both sides... $$\frac{p}{1-p} = e^{\log(\text{odds})}$$

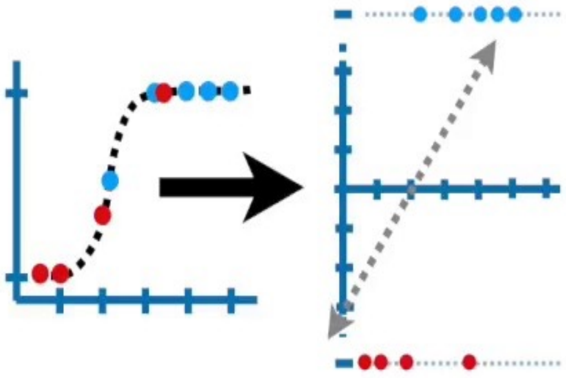
Multiply both sides by $(1 - p)$... $$p = (1-p)e^{\log(\text{odds})}$$

Multiply $(1 - p)$ and $e^{\log(\text{odds})}$... $$p = e^{\log(\text{odds})} - pe^{\log(\text{odds})}$$

Add $pe^{\log(\text{odds})}$ to both sides... $$p + pe^{\log(\text{odds})} = e^{\log(\text{odds})}$$

# Logistic Regression Equation

$$\log\left(\frac{p}{1-p}\right) = \log(\text{odds})$$



Exponentiate both sides... $\frac{p}{1-p} = e^{\log(\text{odds})}$

Multiply both sides by $(1 - p)$... $p = (1 - p)e^{\log(\text{odds})}$

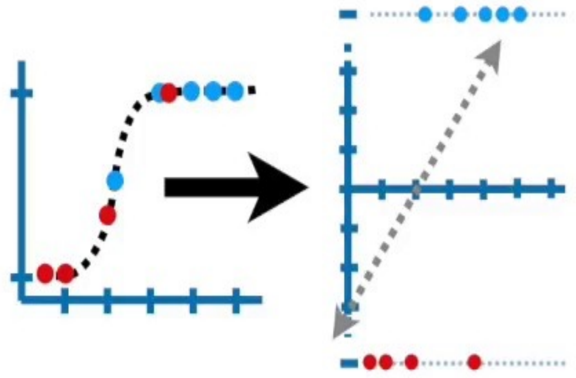
Multiply $(1 - p)$ and $e^{\log(\text{odds})}$... $p = e^{\log(\text{odds})} - pe^{\log(\text{odds})}$

Add $pe^{\log(\text{odds})}$ to both sides... $p + pe^{\log(\text{odds})} = e^{\log(\text{odds})}$

Pull $p$ out... $p(1 + e^{\log(\text{odds})}) = e^{\log(\text{odds})}$

# Logistic Regression Equation

$$\log\left(\frac{p}{1-p}\right) = \log(\text{odds})$$



Exponentiate both sides... $\frac{p}{1-p} = e^{\log(\text{odds})}$

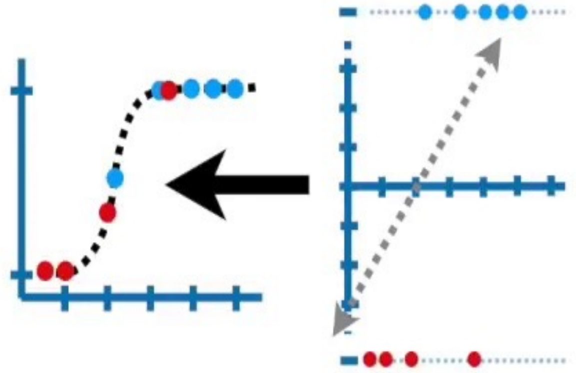
Multiply both sides by $(1 - p)$... $p = (1 - p)e^{\log(\text{odds})}$

Multiply $(1 - p)$ and $e^{\log(\text{odds})}$... $p = e^{\log(\text{odds})} - pe^{\log(\text{odds})}$

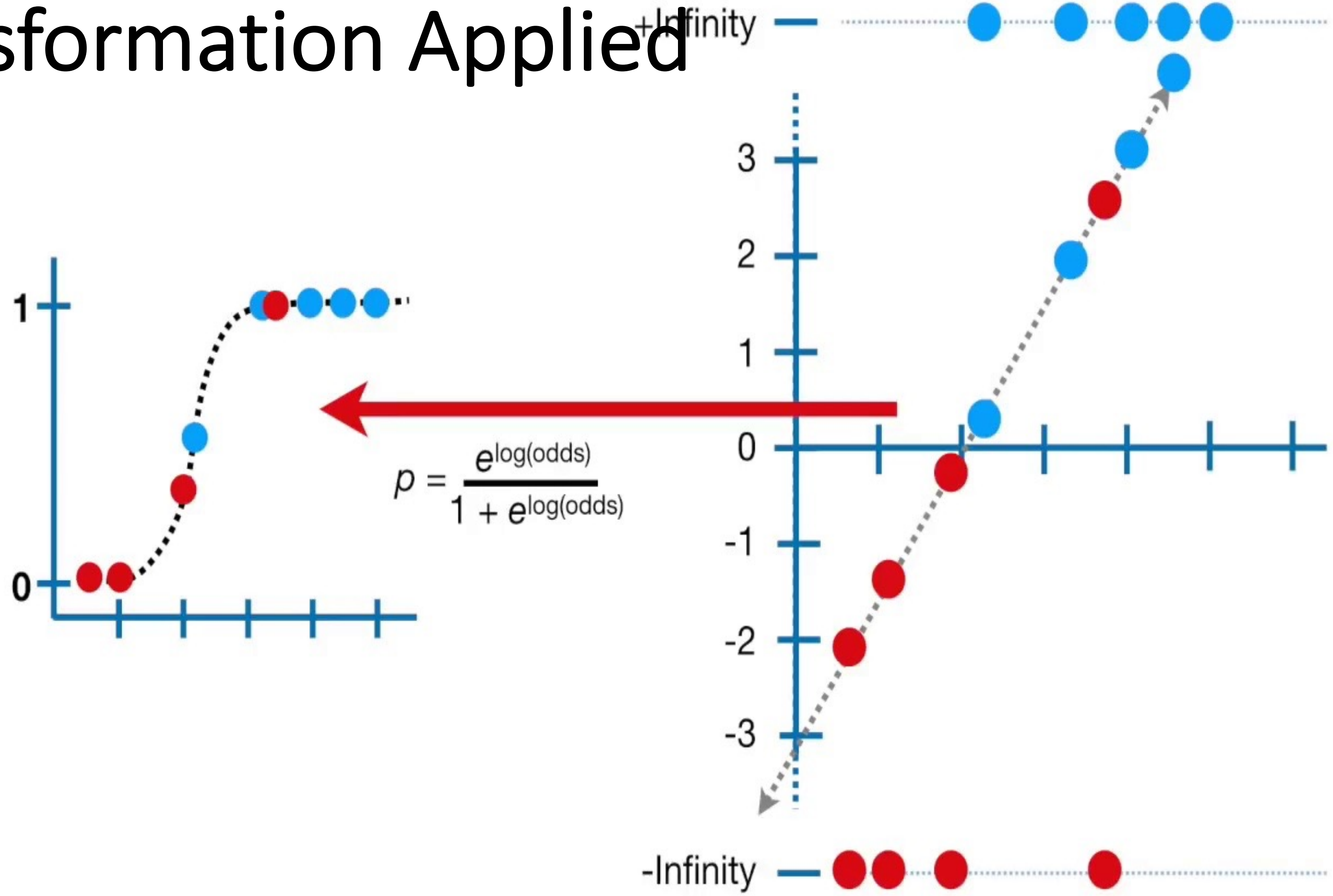
Add $pe^{\log(\text{odds})}$ to both sides... $p + pe^{\log(\text{odds})} = e^{\log(\text{odds})}$

Pull $p$ out... $p(1 + e^{\log(\text{odds})}) = e^{\log(\text{odds})}$

Divide both sides by $(1 + e^{\log(\text{odds})})$... $p = \frac{e^{\log(\text{odds})}}{1 + e^{\log(\text{odds})}}$

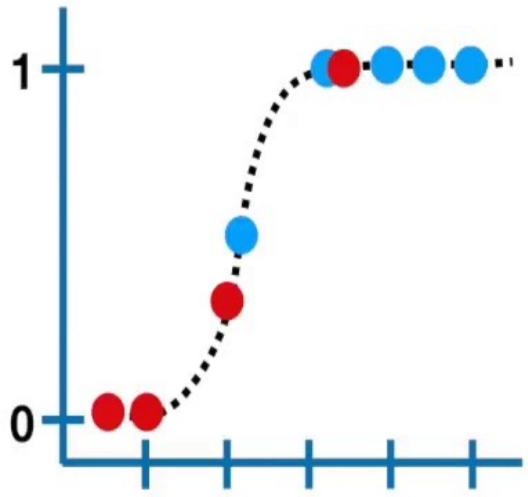


# Transformation Applied

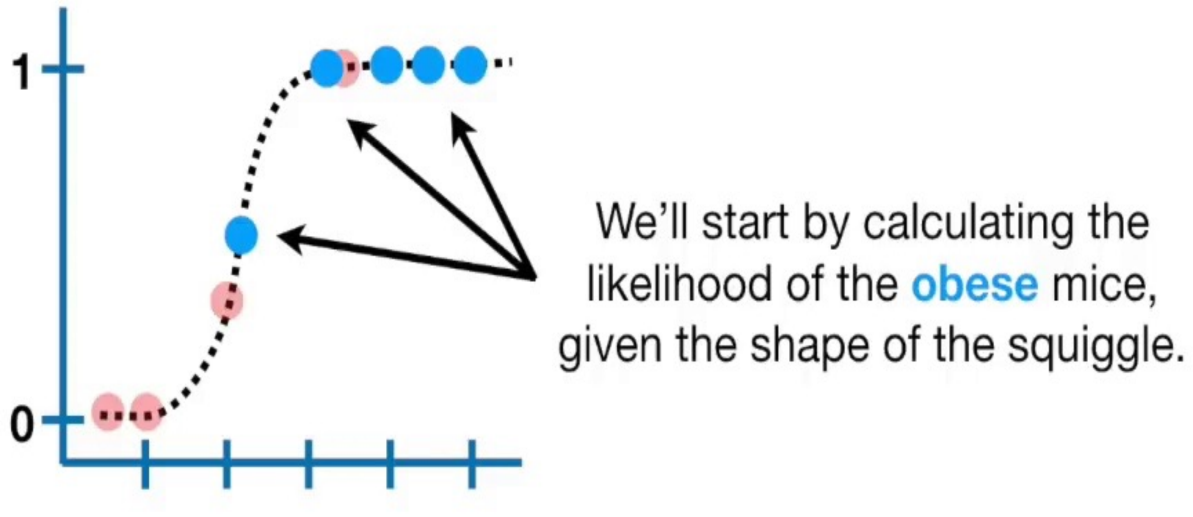


# Find Best Line

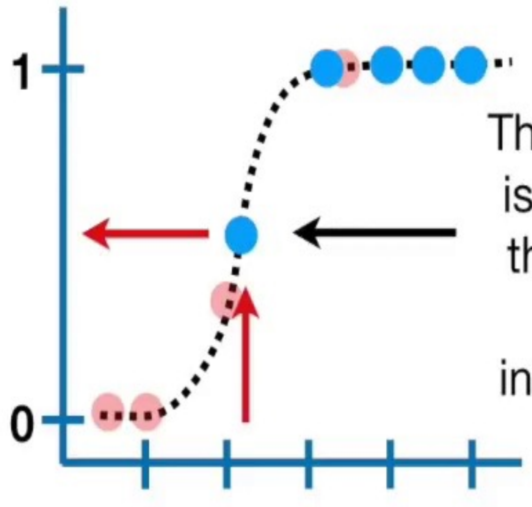
Now we use the observed status (**obese** or **not obese**) to calculate their likelihood given the shape of the squiggly line.



# Find Best Line

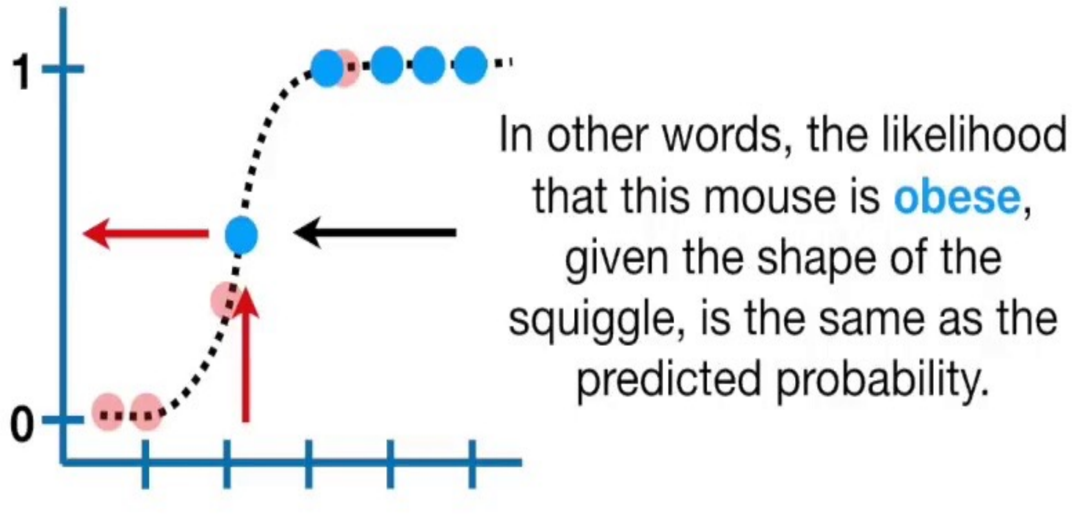


# Find Best Line

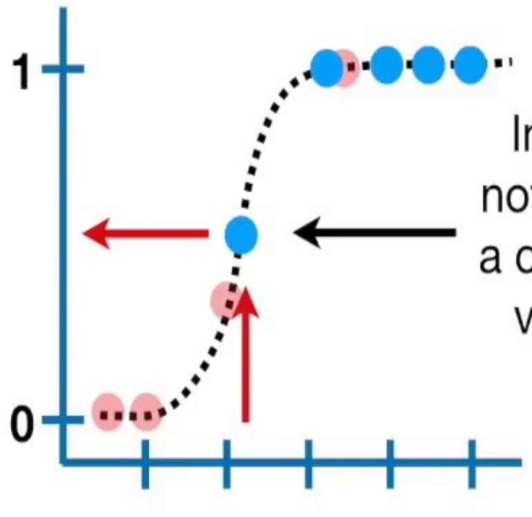



The likelihood that this mouse is **obese**, given the shape of the squiggle, is the value on the y-axis where point intersects the squiggle, **0.49**.

# Find Best Line

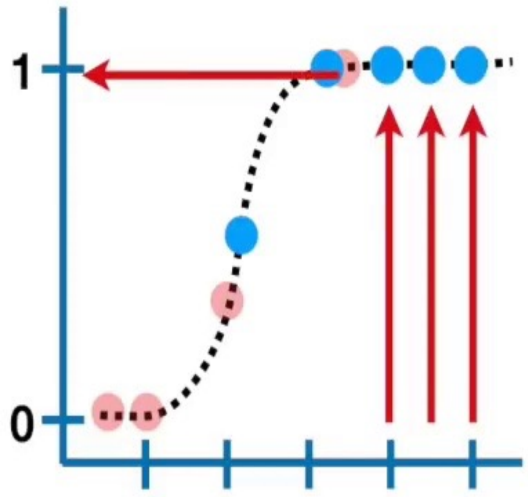


# Find Best Line




In this case, the probability is not calculated as the area under a curve, but instead is the y-axis value, and that's why it is the same as the likelihood.

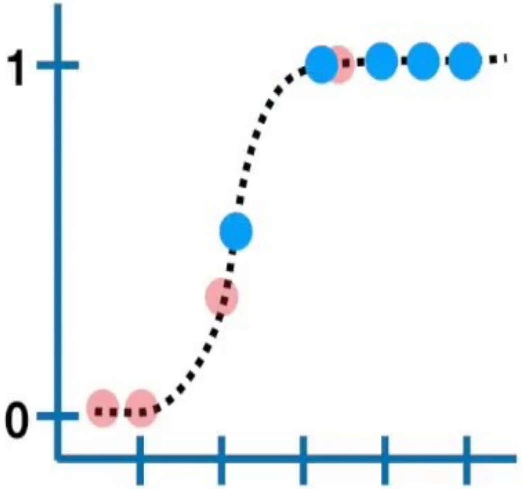
# Find Best Line



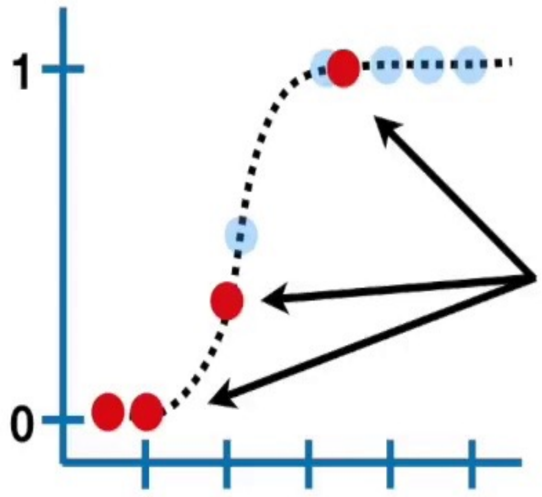

The likelihoods that these mice are obese are 0.91, 0.91 and 0.92

likelihood of data given the squiggle = 0.49 × 0.9 × 0.91 × 0.91 × 0.92 × …

The likelihood for all of the obese mice is just the product of the individual likelihoods.

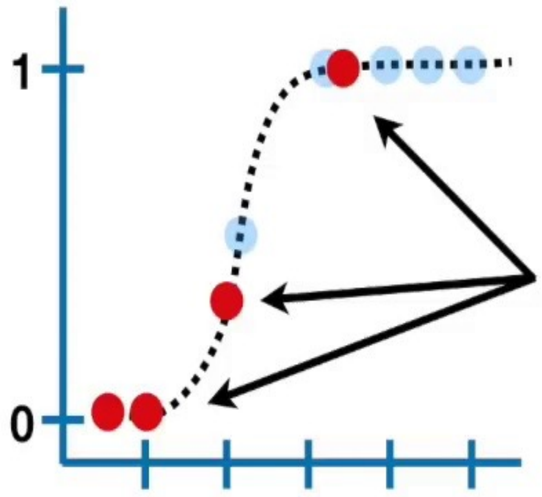


likelihood of data given the squiggle = 0.49 x 0.9 x 0.91 x 0.91 x 0.92 x …



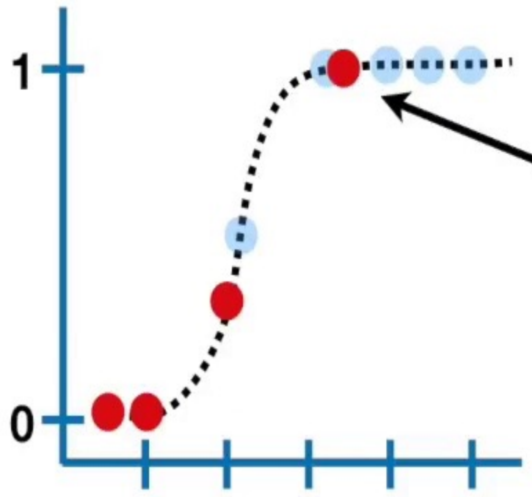

**NOTE:** The lower the probability of being obese, the higher the probability of not being obese.

likelihood of data given the squiggle = 0.49 x 0.9 x 0.91 x 0.91 x 0.92 x …



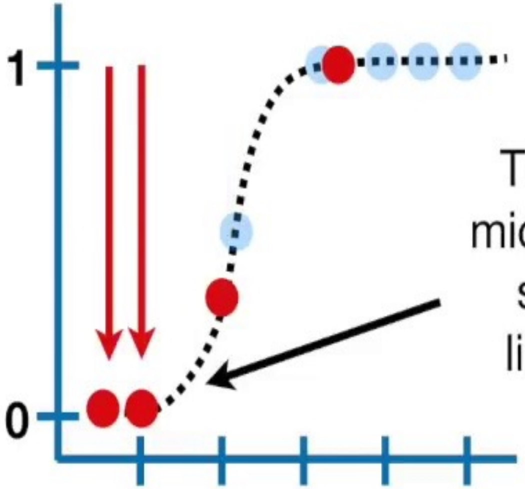

Thus, for these mice, the likelihood = (1 - probability the mouse is obese)

likelihood of data given the squiggle = 0.49 x 0.9 x 0.91 x 0.91 x 0.92 x …

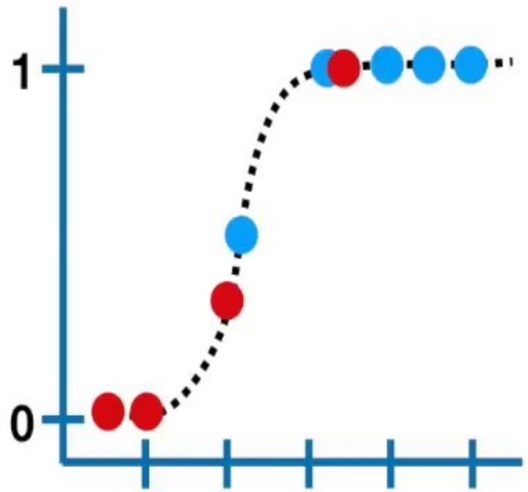



The probability that this mouse is **obese** is 0.9, so the probability and likelihood that it is **not obese** is (**1 - 0.9**)

likelihood of data given the squiggle = 0.49 x 0.9 x 0.91 x 0.91 x 0.92 x …

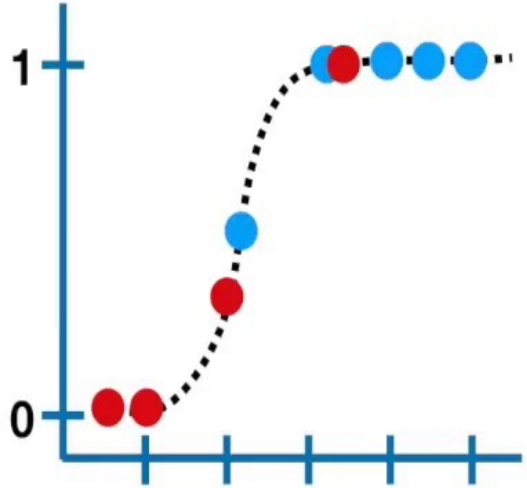



The probabilities that these mice are **obese** are both 0.01, so the probability and the likelihood that they are **not obese** is (**1 - 0.01**) likelihood of data given the squiggle = $0.49 \times 0.9 \times 0.91 \times 0.91 \times 0.92 \times (1 - 0.9) \times (1 - 0.3) \times (1 - 0.01) \times (1 - 0.01)$



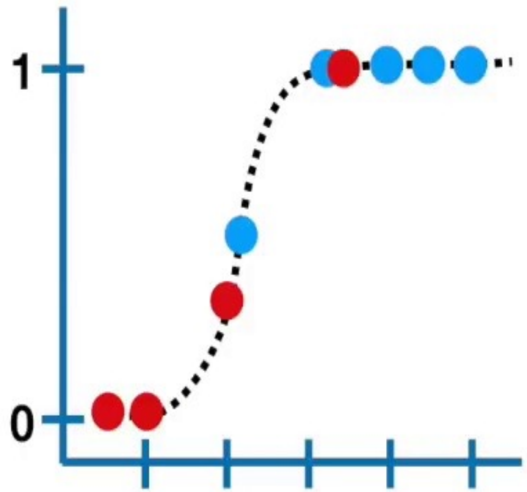

**NOTE:** Although it is possible to calculate the likelihood as the product of the individual likelihoods, statisticians prefer to calculate the **log of the likelihood** instead.

$$\log(\text{likelihood of data given the squiggle}) = \log(0.49) + \log(0.9) + \log(0.91) + \log(0.91) + \log(0.92) + \log(1 - 0.9) + \log(1 - 0.3) + \log(1 - 0.01) + \log(1 - 0.01)$$



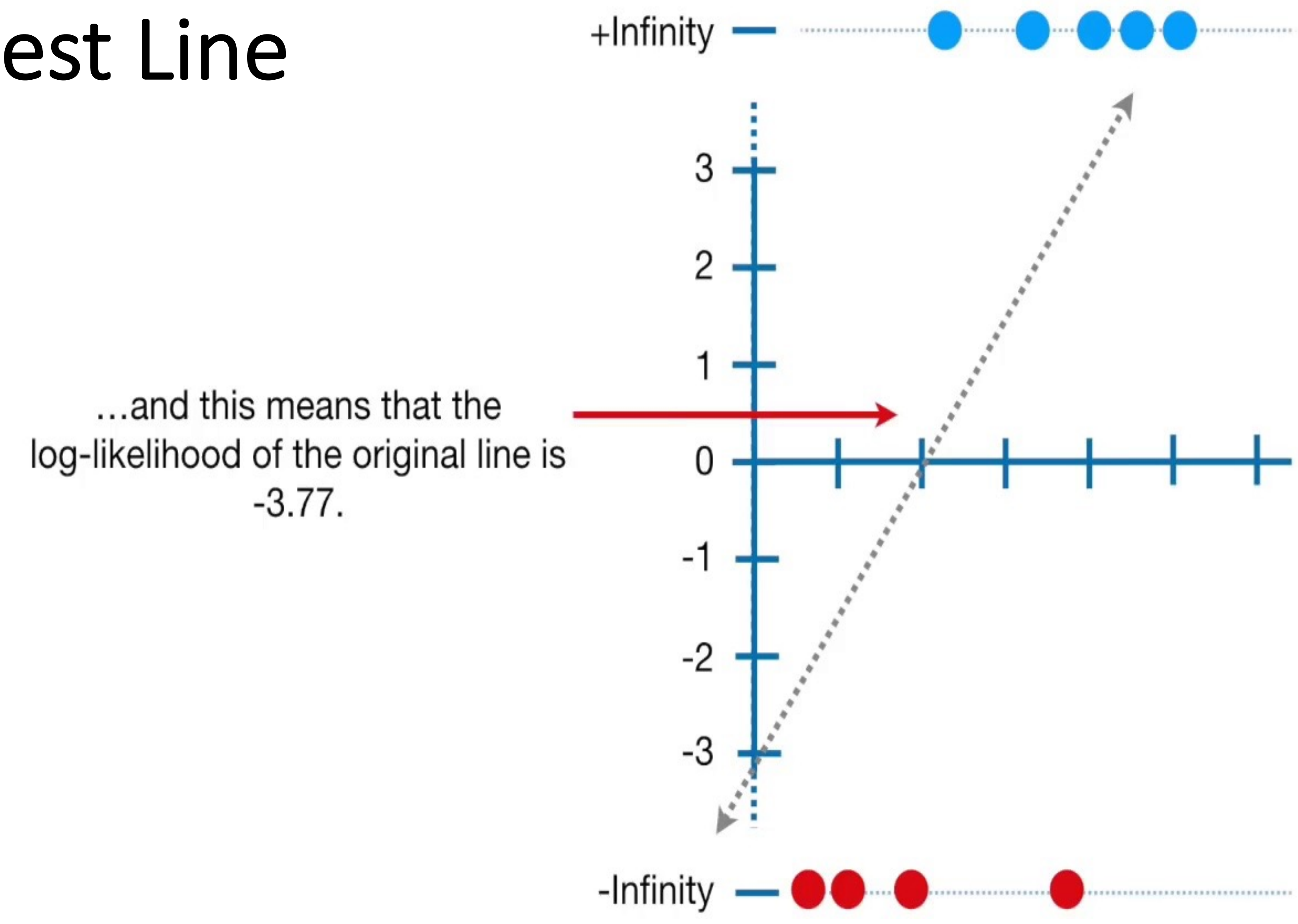

With the log of the likelihood, or "log-likelihood" to those in the know, we **add the logs of the individual likelihoods** instead of multiplying the individual likelihoods…

log(likelihood of data given the squiggle) = -3.77



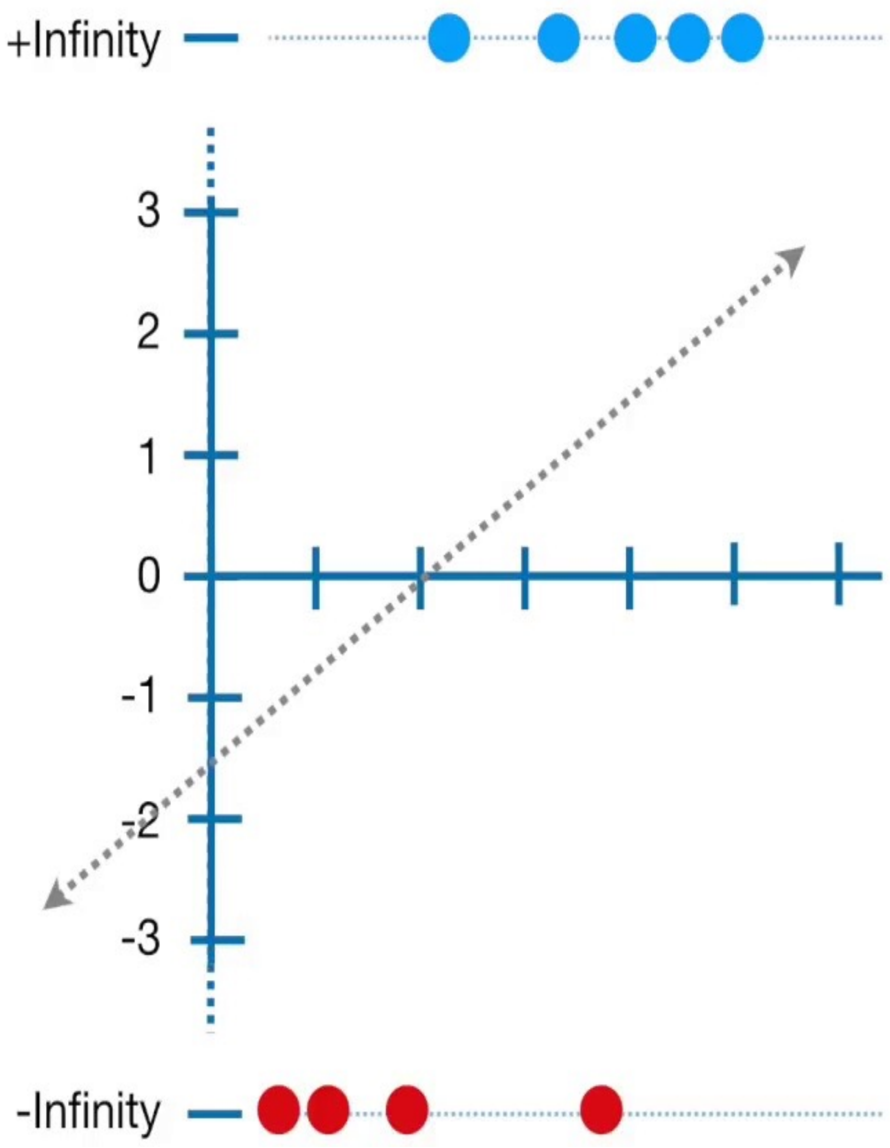

Thus, the log-likelihood of the data given the squiggle is -3.77…

# Find Best Line



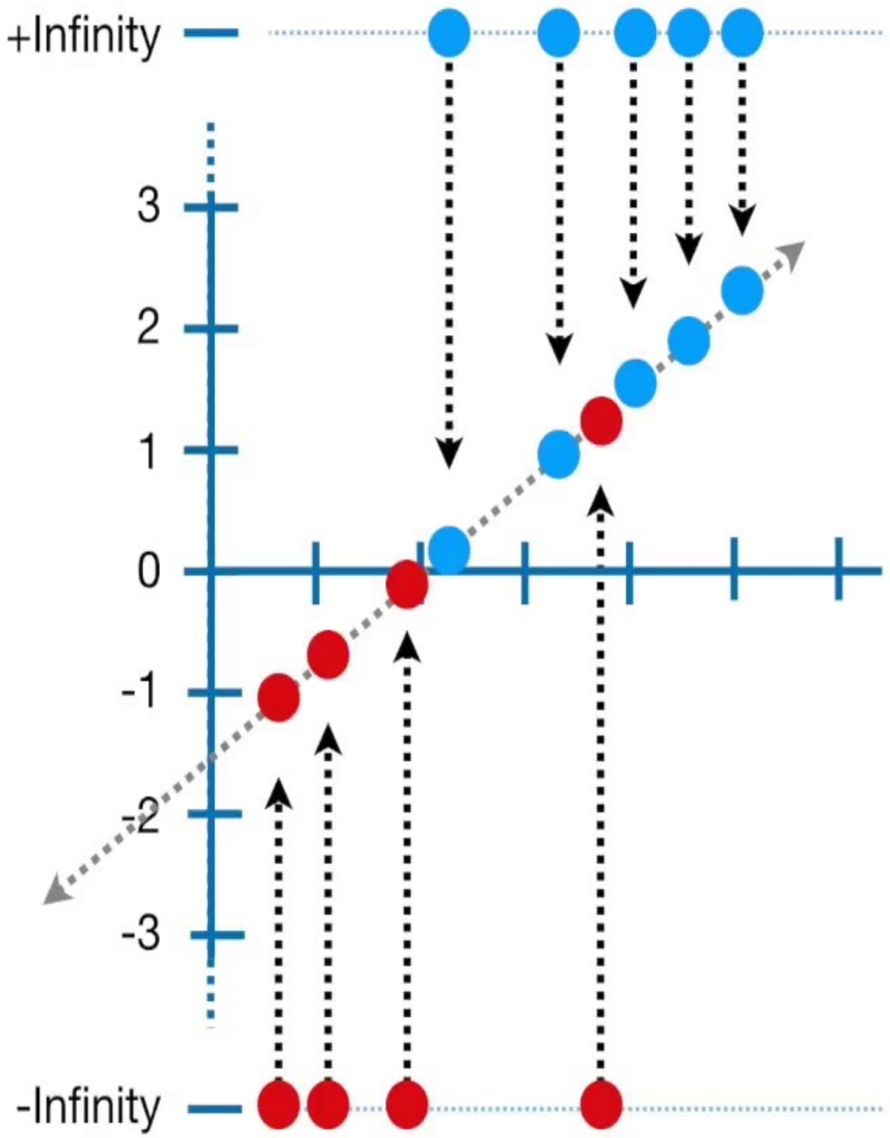
# Find Best Line

Now we rotate the line...

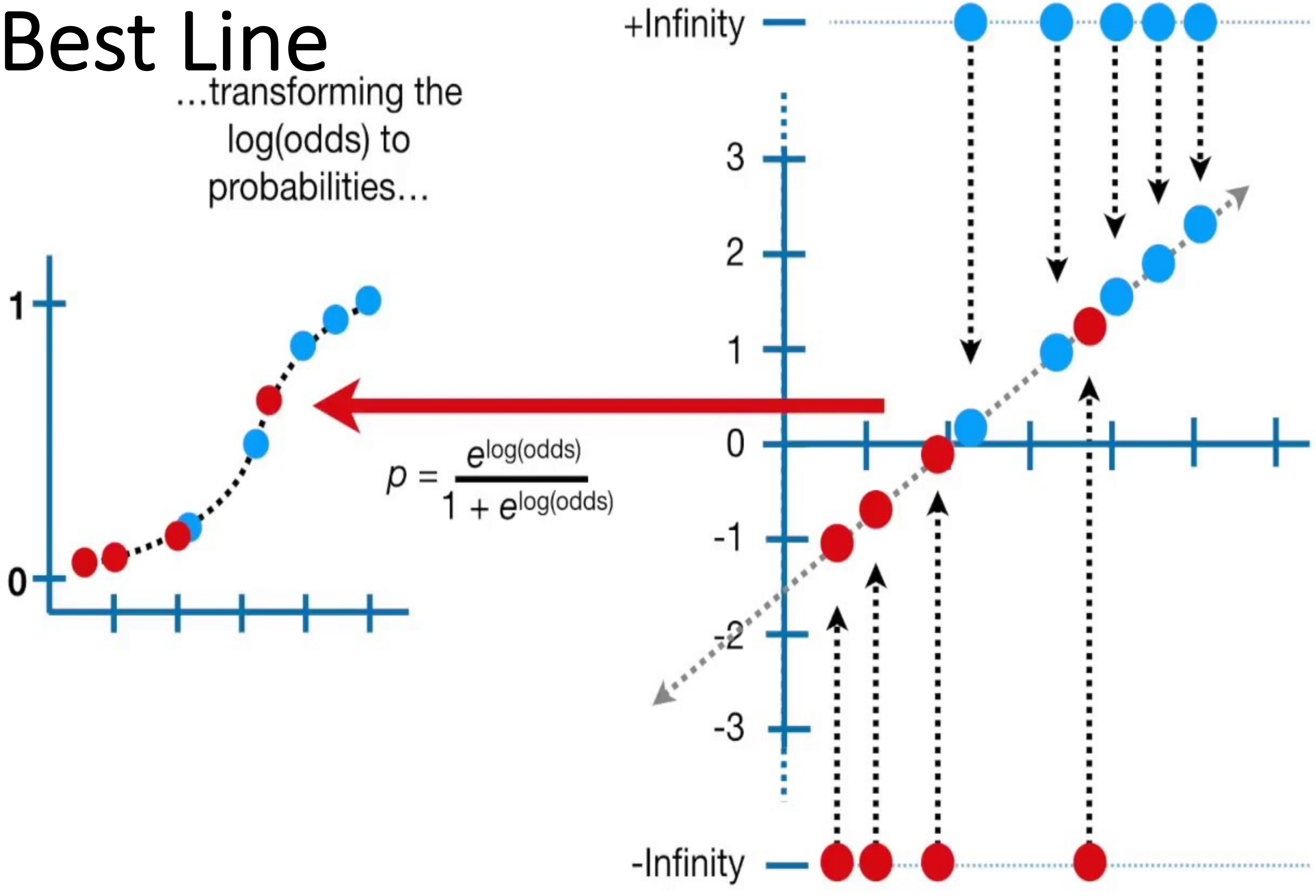


# Find Best Line

...and calculate its log-likelihood
by projecting the data onto it...

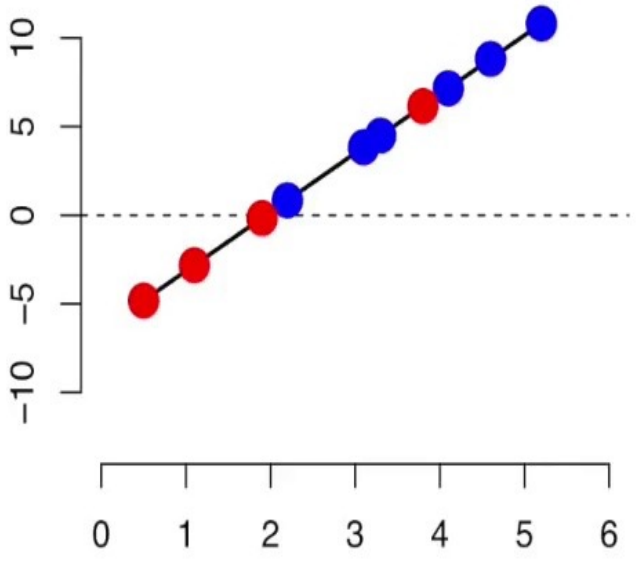


# Find Best Line



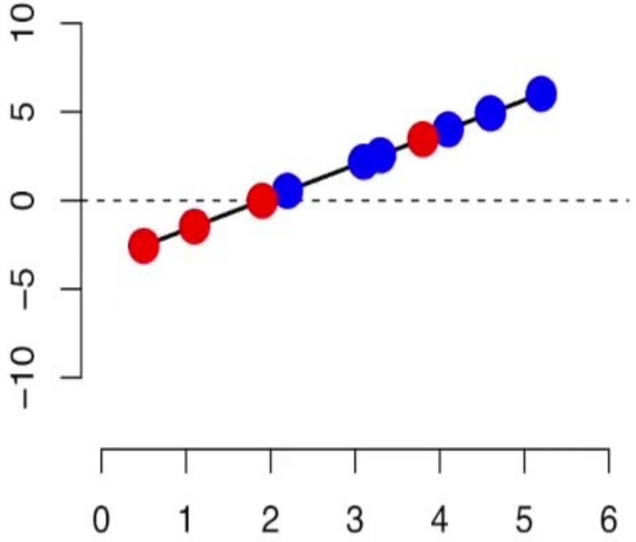
# Find Best Line

...and we just keep rotating the log(odds) line and projecting the data onto it...



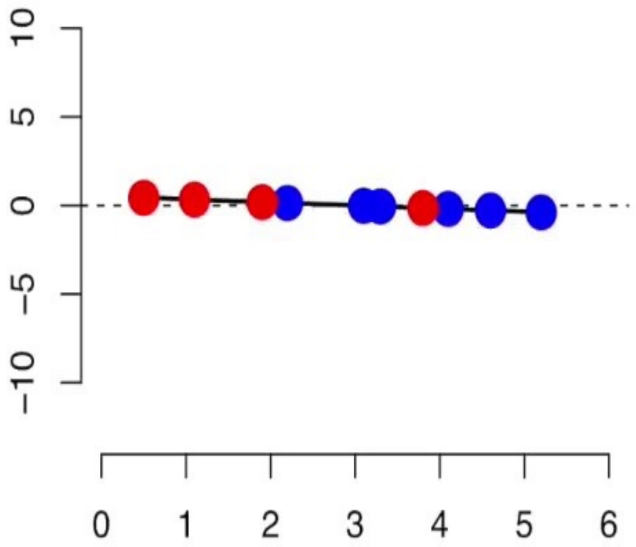
# Find Best Line

...and we just keep rotating
the log(odds) line and
projecting the data onto it...



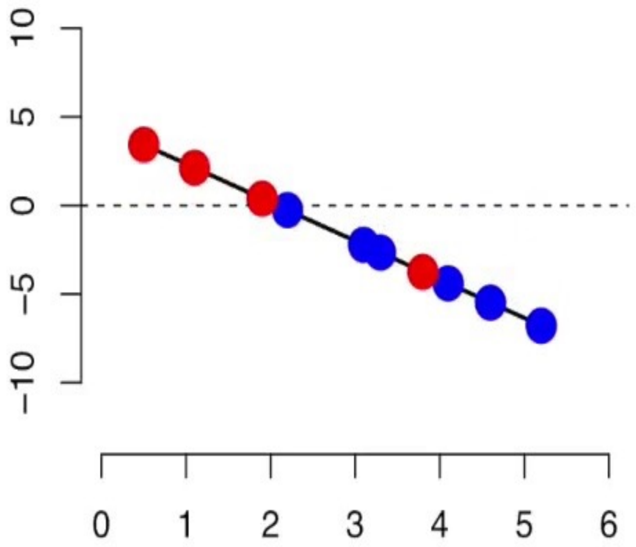
# Find Best Line

...and we just keep rotating
the log(odds) line and
projecting the data onto it...

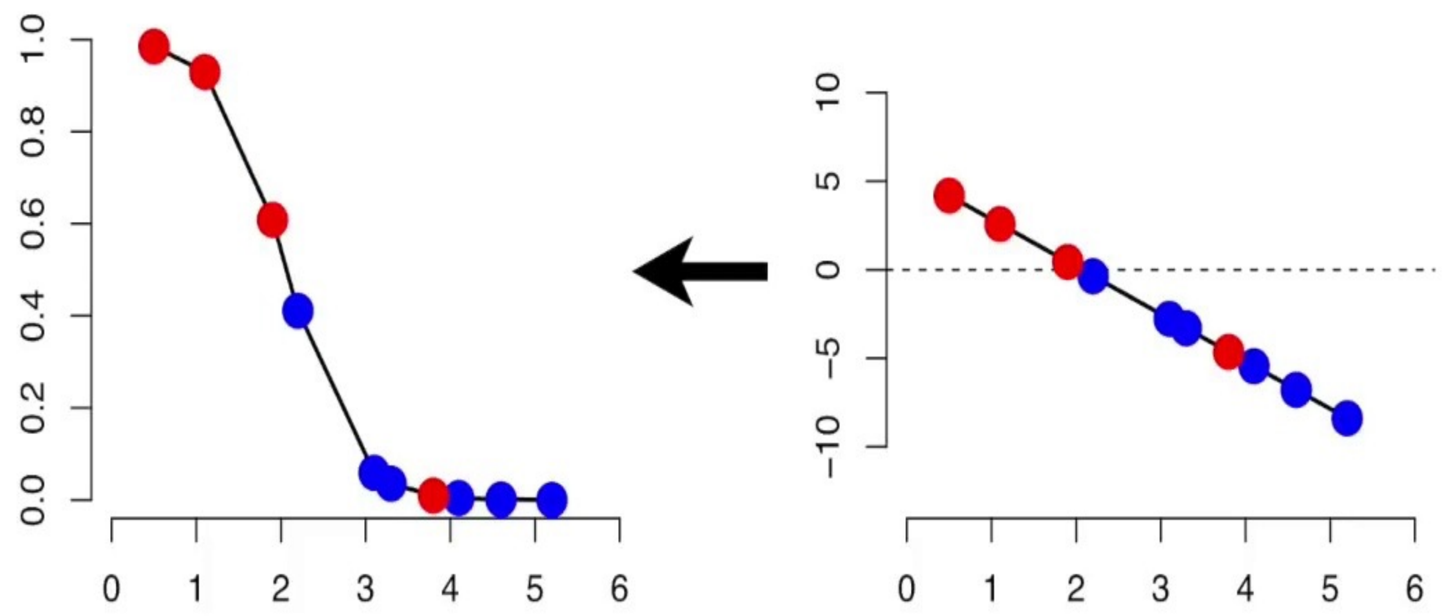


# Find Best Line

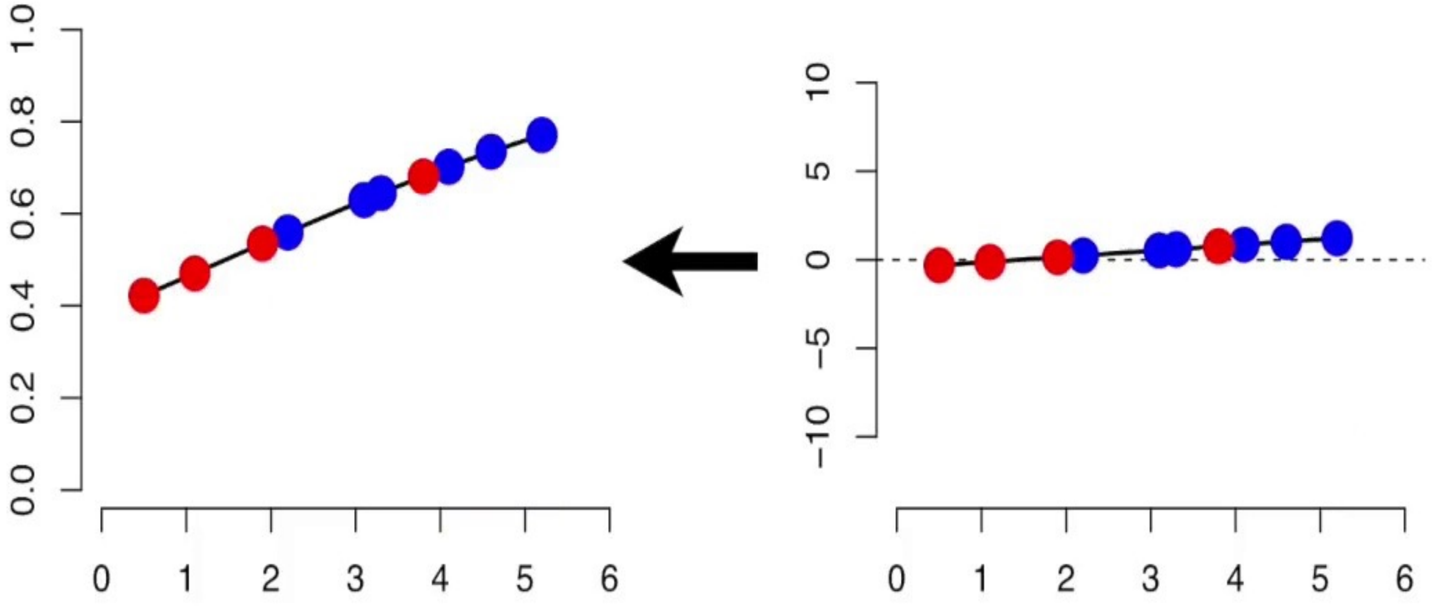
...and we just keep rotating the log(odds) line and projecting the data onto it...



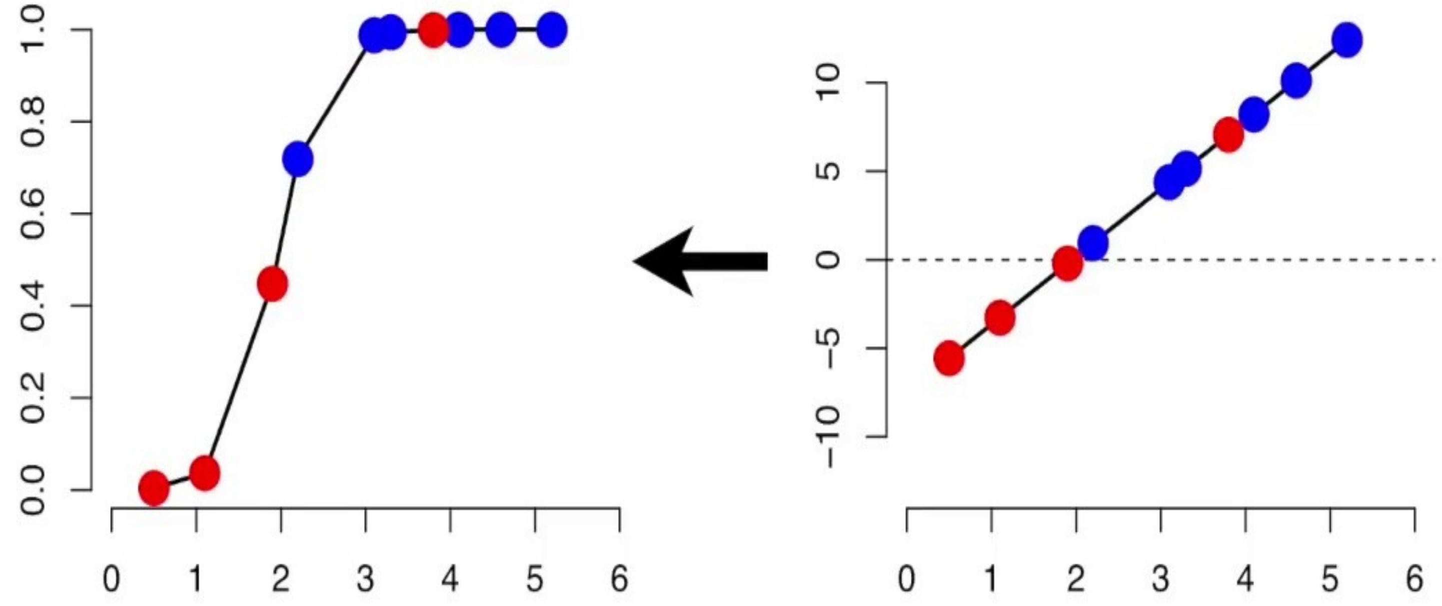
...and transforming it to probabilities and calculating the log-likelihood.



...and transforming it to probabilities and calculating the log-likelihood.

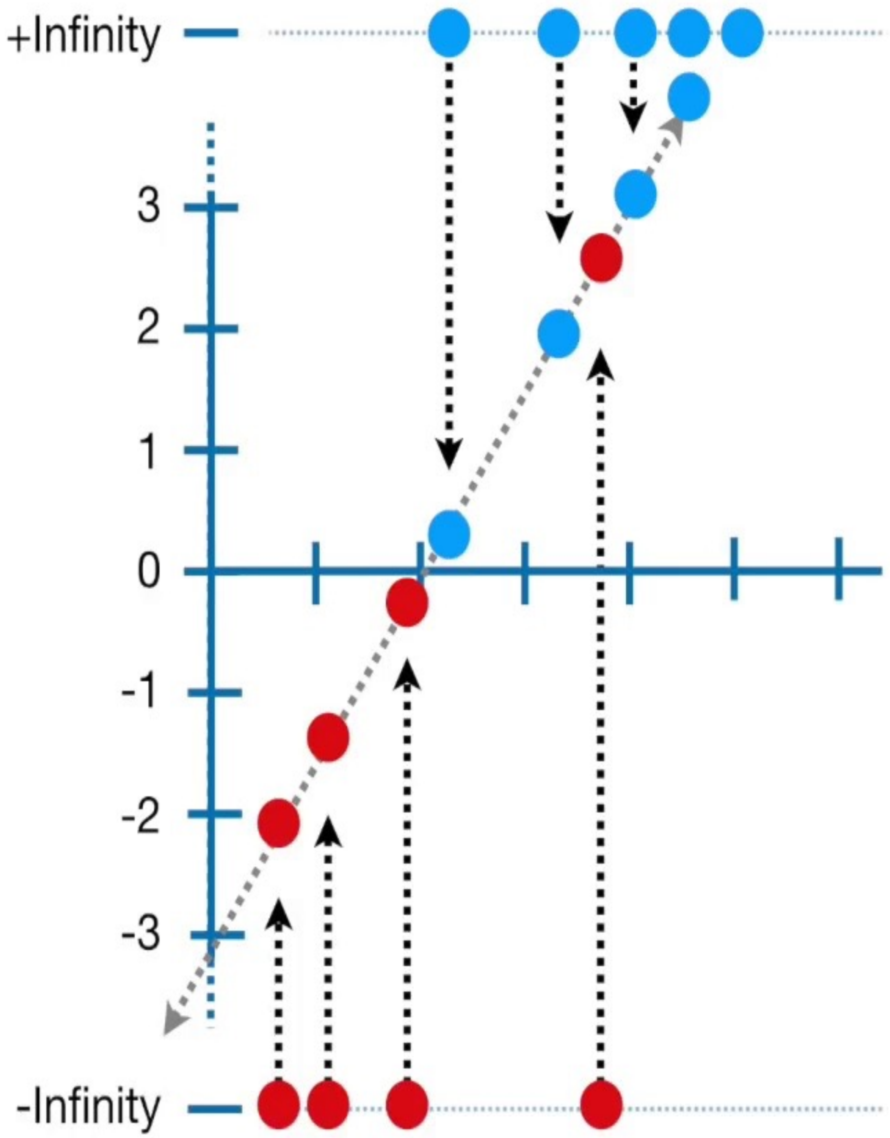


...and transforming it to probabilities and calculating the log-likelihood.

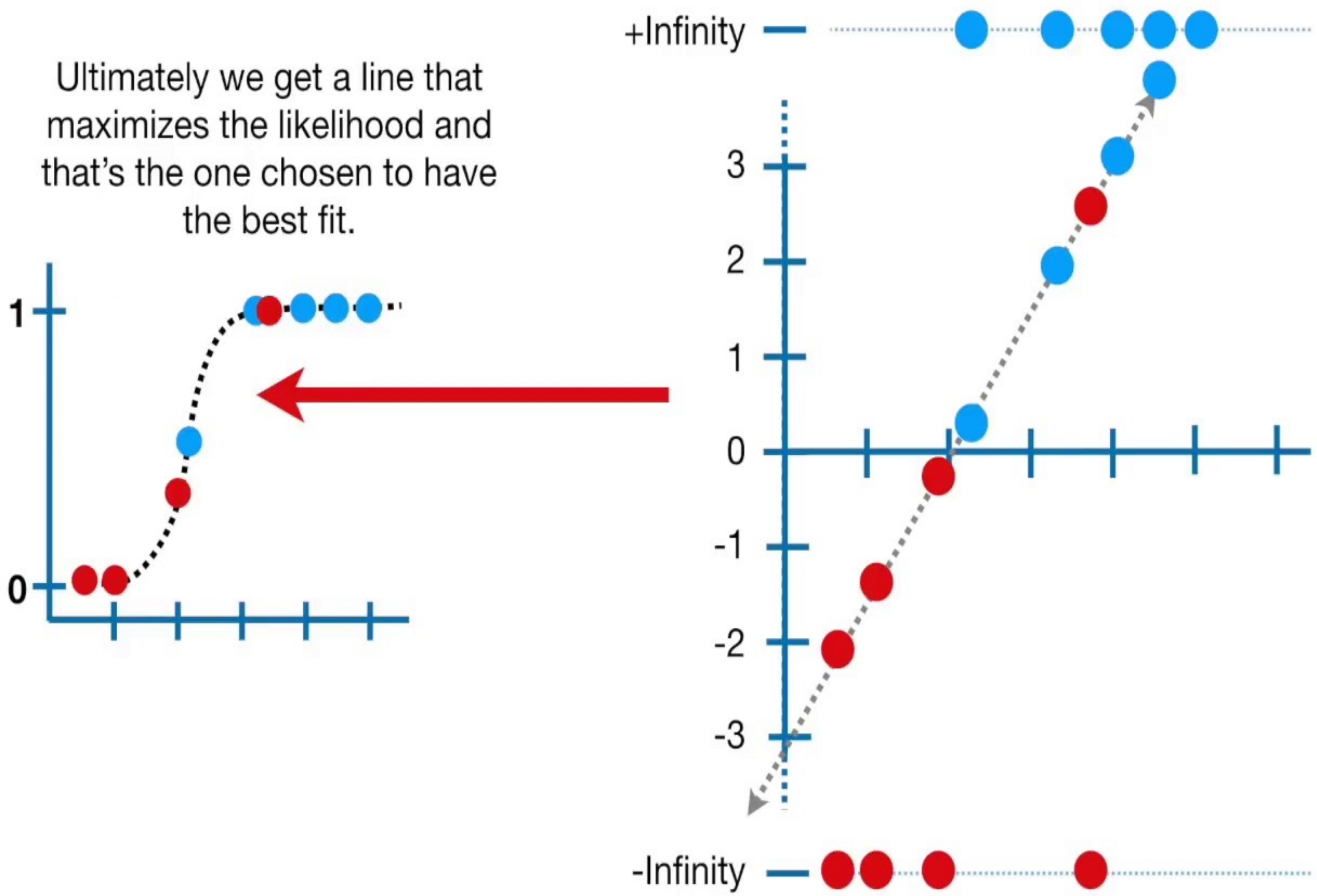


# Find Best Line

**NOTE:** The algorithm that finds the line with the maximum likelihood is pretty smart - each time it rotates the line, it does so in a way that increases the log-likelihood. Thus, the algorithm can find the optimal fit after a few rotations.



Ultimately we get a line that maximizes the likelihood and that's the one chosen to have the best fit.

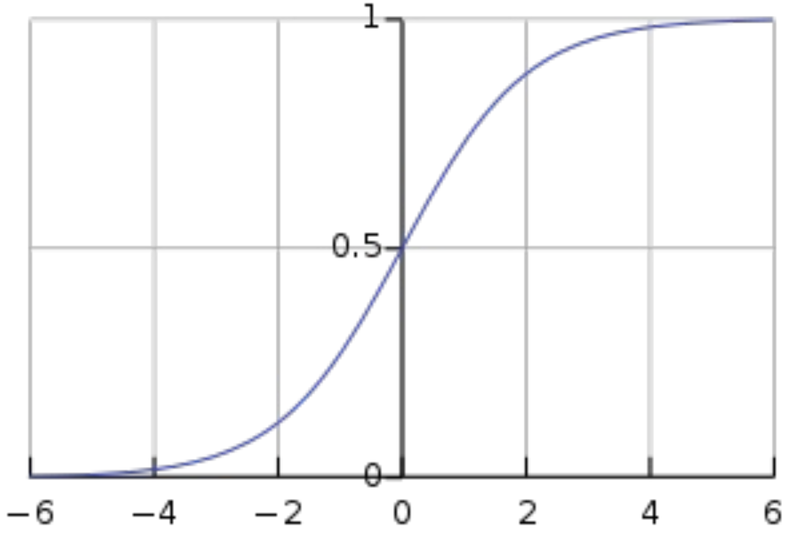


# Recovering Probabilities from Log Odds

$$\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$

$$\Leftrightarrow \frac{p}{1-p} = e^{\beta_0 + \beta_1 X}$$

$$\Leftrightarrow p = \frac{e^{\beta_0 + \beta_1 X}}{1+e^{\beta_0 + \beta_1 X}} = \frac{1}{1+e^{-(\beta_0 + \beta_1 X)}}$$




- which gives *p* as the sigmoid function!

# Logistic Regression

- In Logistic Regression we seek a model

$$Y = logit(p) = log(p/(1-p)$$

- That is, the **log odds, i.e., the logit,** is assumed to be linearly related to the independent variable X
- In this way it is possible to solve an ordinary (linear) regression.

# Interpretation of Beta1

- Let:
  - odds1 = odds for value X (p/(1–p))
  - odds2 = odds for value X + 1 unit
- Then:

$$\frac{odds2}{odds1} = \frac{e^{\beta_0 + \beta_1 (X+1)}}{e^{\beta_0 + \beta_1 X}}$$

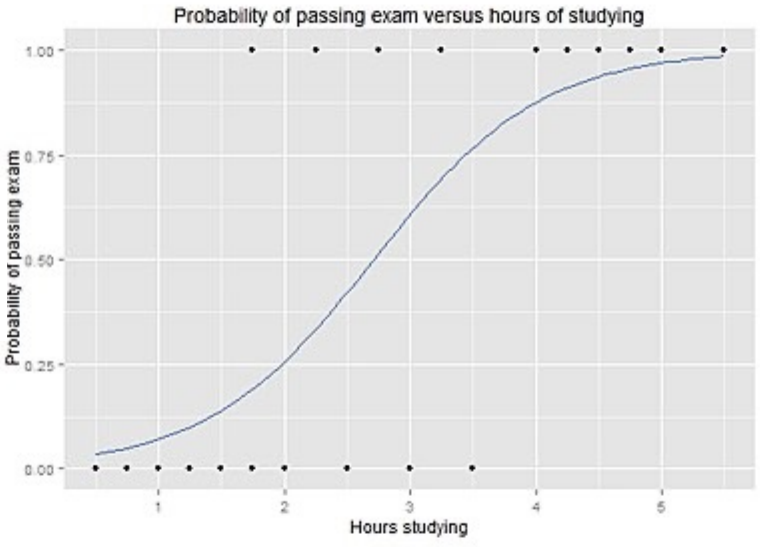
If the odds ratio of two consecutive value is large it means that an increment on X has a large impact in the prediction of Y.

$$= \frac{e^{(\beta_0 + \beta_1 X) + \beta_1}}{e^{\beta_0 + \beta_1 X}} = \frac{e^{(\beta_0 + \beta_1 X)} e^{\beta_1}}{e^{\beta_0 + \beta_1 X}} = e^{\beta_1}$$

- The exponent of the slope describes the proportionate rate at which the predicted odds ratio changes with each successive unit of *X*

# Example

- Hours: 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 1.75, 2.00, 2.25, 2.50, 2.75, 3.00, 3.25, 3.50, 4.00, 4.25, 4.50, 4.75, 5.00, 5.50
- Pass: 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1




Beta0 = −4.0777, Beta1 = 1.5046

$$\text{Log-odds of passing exam} = 1.5046 \cdot \text{Hours} - 4.0777$$

$$\text{Odds of passing exam} = \exp(1.5046 \cdot \text{Hours} - 4.0777)$$

$$\text{Probability of passing exam} = \frac{1}{1 + \exp(-(1.5046 \cdot \text{Hours} - 4.0777))}$$

One additional hour of study is estimated to increase log-odds by 1.5046, so multiplying odds by $e^{1.5046}$ = 4.5.

For example, for a student who studies 2 hours we have an estimated probability of passing the exam of 0.26.

Similarly, for a student who studies 4 hours, the estimated probability of passing the exam is 0.87.

# References

- Regression. Appendix D. Introduction to Data Mining.