APPROX
Knapsack $\geq (1-\varepsilon)\cdot OPT$ in $O(n^3/\varepsilon)$ time

(i.e. $(1-\varepsilon)^{-1}$-approximation so the smaller ε, the better is)

We saw 2 DP exact solutions

DP1: $O(nW)$ time

DP2: $O(n^2 V_{max})$ time, where $V_{max} = \max_i V_i$

IDEA Reduce by a factor K all values $\tilde{V}_i := \left\lfloor \frac{V_i}{K} \right\rfloor$

(we will fix $K = \varepsilon \frac{V_{max}}{n}$)

$\Rightarrow \tilde{V}_{max} = \left\lfloor \frac{V_{max}}{K} \right\rfloor$ and use DP2 with a choice of K so that is polynomial time in n and $1/\varepsilon$

HENCE

① Original instance $\underbrace{V_i, w_i, W}_{n \text{ items}} \rightarrow$ optimal solution $S^* \subseteq [n]$

② Scaled instance for K $\underbrace{\tilde{V}_i, w_i, W}_{n \text{ item}} \rightarrow$ optimal solution $\tilde{S} \subseteq [n]$

fact A feasible solution for ② is also feasible in ①

(w_i, W did not change!)

CLAIM Choosing $K = \varepsilon \frac{V_{max}}{n}$, then

$$\sum_{i \in \tilde{S}} V_i \geq (1-\varepsilon) \cdot \underbrace{\sum_{i \in S^*} V_i}_{OPT}$$

Since $\tilde{V}_i = \left\lfloor \frac{V_i}{k} \right\rfloor$, multiplying both sides by k:

(a) $V_i \geq k \tilde{V}_i$

(b) $k\tilde{V}_i \geq V_i - k$

Moreover, since S is optimal in (2), S^* cannot be better in (2)

(c) $\sum_{i \in S} \tilde{V}_i \geq \sum_{i \in S^*} \tilde{V}_i$

Now consider the sums on the V_i's in (1)

$$\sum_{i \in S} V_i \overset{(a)}{\geq} \sum_{i \in S} k\tilde{V}_i \overset{(c)}{\geq} \sum_{i \in S^*} k\tilde{V}_i \overset{(b)}{\geq} \sum_{i \in S^*} (V_i - k)$$

since $|S^*| \leq n$, the sum can be written as $\left(\sum_{i \in S^*} V_i\right) - kn$

Thus,
$$\sum_{i \in S} V_i \geq \underbrace{\sum_{i \in S^*} V_i}_{\overset{\text{OPT}}{=}} - kn$$

We obtain our claim by choosing k s.t. $\boxed{-kn \geq -\varepsilon \cdot \text{OPT}}$

$kn = \varepsilon V_{max} \leq \varepsilon \cdot \text{OPT}$ gives the answer, i.e. $k = \varepsilon \frac{V_{max}}{n}$

Since $\tilde{V}_{max} = \left\lfloor \frac{V_{max}}{k} \right\rfloor \leq \frac{V_{max}}{k} = \underbrace{\frac{n}{\varepsilon V_{max}}}_{= \frac{1}{k}} \cdot V_{max} = \frac{n}{\varepsilon}$

running time is

$$O\left(n^2 \tilde{V}_{max}\right) = O\left(\frac{n^3}{\varepsilon}\right) \qquad \text{QED}$$