COUNTING # KNAPSACK (a.k.a 0-1 version ①; no values v_i; [SUBSET SUM])

Exact counting: take DP1 in $O(nW)$ time and modify using the following rule:

$$T_{ij} = \begin{cases} 1 & \text{if } i=0 \text{ or } j=0 \\ T_{i-1,j} + T_{i-1,j-w_{i-1}} & \text{o.w.} \end{cases}$$

$T_{i-1,j-w_{i-1}} = 0$ if $j < w_{i-1}$ ← i-th element

T_{nW} = # feasible subsets $\subseteq [n]$ of total weight $\leq W$

UNIFORM SAMPLING of the above subsets w.r.t DP1:

Return one feasible set S with uniform probability $\frac{1}{T_{n,W}}$ using T_{ij}

- start with $i = n-1$ and $j = W$
- backward choose element $i \in [n]$ with probability $\frac{T_{i,j-w_i}}{T_{i+1,j}}$ ← how many subsets $\subseteq [n+1]$ containing w_i have total weight $\leq j$ ← how many subsets $\subseteq [n+1]$ have total weight $\leq j$

[note: it's the same for $A \subseteq B$: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$]

[see the Python code]

CLAIM Each feasible S is sampled with $Pr = \frac{1}{T_{n,W}}$

proof sketch

- define: $p_i = \begin{cases} f_i & \text{if } i \in S \\ 1 - f_i & \text{o.w.} \end{cases}$
- observation $1 - f_i = \frac{T_{i,j}}{T_{i+1,j}}$, $p_0 = \frac{1}{2}$ as $T_{0,j} = 1 \ \forall j$
- $Pr(S \text{ is chosen}) = \prod_{i=0}^{n-1} p_i = \frac{T_{0,j}}{T_{n,W}} = \frac{1}{T_{n,W}}$ (for a value j)

(by induction on $i = n-1, n-2, \ldots, 0$:

$p_i p_{i+1} \cdots p_{n-1} = \prod_{g=i}^{n-1} p_g = \begin{cases} \frac{T_{i,j-w_i}}{T_{n,W}} & \text{if } i \in S \\ T_{i,j} / T_{n,W} & \text{o.w.} \end{cases}$ for some j

QED

APPROXIMATE COUNTING

as W can be very large!

IDEA: Scale weights by a factor $k = \frac{W}{n^2}$

$$\tilde{w}_i = \left\lfloor \frac{w_i}{k} \right\rfloor = \left\lfloor \frac{n^2 w_i}{W} \right\rfloor \quad \text{and use} \quad \tilde{W} = n^2$$

obs This time we scale w_i and change W

① Original instance: v_i, w_i, W (not really needed), F = set of all feasible sets in ①

② Scaled instance: $v_i, \tilde{w}_i, \tilde{W}$, $\tilde{F}$ = set of all feasible sets in ②

Here $O(n\tilde{W}) = O(n^3)$ to find $|\tilde{F}| = T_{n,\tilde{W}}$

What about $|F|$? ← OUR GOAL

obs $F \subseteq \tilde{F}$ and thus $|F| \leq |\tilde{F}|$

proof: $S \in F \Rightarrow S \in \tilde{F}$.

$$\sum_{i \in S} w_i \leq W \qquad \sum_{i \in S} \tilde{w}_i \leq \tilde{W}$$

Thus $\sum_{i \in S} \tilde{w}_i \leq 1/k \cdot \sum_{i \in S} w_i \leq \frac{n^2}{W} \cdot W = n^2 = \tilde{W}$

($\leq$ by def. of $\tilde{w}_i$; $\leq W$ as $S \in F$)

QED

We show that $|\tilde{F}|$ is an approximation for $|F|$

$$\frac{|\tilde{F}|}{n+1} \leq |F| \leq |\tilde{F}|$$

($\leq$: show how... ; $\leq$: seen before)

w.l.o.g. assume $0 \leq w_0 \leq w_1 \leq \cdots \leq w_{n-1} \leq W$

CLAIM (A) Let largest r : $w_r \leq \frac{W}{n}$ (obs: it exists as average value $\frac{w_0 + \cdots + w_{n-1}}{n} \leq \frac{W}{n}$)

Then, any $T \subseteq [r] \Rightarrow T \in F$

$$\Rightarrow \sum_{i \in T} w_i \leq \sum_{i \in T} \frac{W}{n} \overset{|T| \leq n}{\leq} W$$

COROLLARY (A) $S \in \tilde{F}/F \Rightarrow S \not\subseteq [r]$

IDEA to prove $\frac{|\tilde{F}|}{n+1} \leq |F|$:

find a mapping $f: \tilde{F} \to F$ such that for each $S' \in F$ there exists at most $n+1$ sets $S \in \tilde{F}$ with $f(S) = S'$.

$\Rightarrow |\tilde{F}| \leq (n+1)|F|$ as wanted

Given $S \in \widetilde{F}$, define

$$f(S) = \begin{cases} S & \text{if } S \in F \\ S \setminus \{\ell\} & \text{o.w., where } w_\ell \text{ is the largest for } \ell \in S \end{cases}$$

OBS. (B) $\ell > r$ and thus $w_\ell > \frac{W}{n}$ (by COROLLARY (A))

CLAIM (C) $f(S) \in F$

$S \in F$ is trivial. Thus, let $S \in \widetilde{F}/F$.

Def $0 \le \delta_i < 1$ rounding error $-\tilde{w}_i + \frac{w_i}{k} = \delta_i$

$$w_i' = k\left(\tilde{w}_i + \delta_i\right) = \frac{W}{n^2}\left(\tilde{w}_i + \delta_i\right)$$

Then

$$\sum_{i \in f(S)} w_i' = \frac{W}{n^2} \cdot \sum_{i \in f(S)} \left(\tilde{w}_i + \delta_i\right)$$

$$\overset{f(S) = S \setminus \{\ell\}}{=} \frac{W}{n^2}\left[\underbrace{\left(\sum_{i \in S} \tilde{w}_i\right)}_{\le n^2 \text{ as } S \in F'} - \tilde{w}_\ell + \underbrace{\left(\sum_{i \in S} \delta_i\right)}_{< n} - \delta_\ell\right]$$

$$\le W + \underbrace{\frac{W}{n} - w_\ell}_{< 0 \text{ by OBS (B)}}$$

$$\le W$$

QED

How many sets $S \in \widetilde{F}$ maps to $S' \in F$?

As $f(S) = S \setminus \{e\} \in F$:

- 1 choice if $S \in F$
- $\leq n$ choices if $S \in \widetilde{F}/F$ for e

Hence $\leq (n+1)$ - QED

As a result, we can estimate $|F|$ using $|\widetilde{F}|$ in $O(n^3)$ time: $\frac{|\widetilde{F}|}{n+1} \leq |F| \leq |\widetilde{F}|$

Can we do better? YES use SAMPLING

FPRAS $\equiv$ FPTAS + randomized (ε, δ)

ε-approximation polynomial time in $n, 1/\varepsilon, \lg 1/\delta$
with probability $\geq 1-\delta$
(CM sketch is a FPRAS)

Remark. In 2011, Gopalan et al. published a FPTAS alg. and there are recent deterministic solutions based on a more sophisticated dynamic programming.

FPRAS for #KNAPSACK

IDEA Estimate $p = \frac{|F|}{|\tilde{F}|}$ using sampling $\left(\frac{1}{n+1} \le p \le 1\right)$

[a] Repeat for $t = 1, 2, \ldots, N$: ← to be fixed

[1] random sample a feasible set $S \in \tilde{F}$ (we saw this before)

[2] $X_t = \begin{cases} 1 & \text{if } S \in F \\ 0 & \text{o.w.} \end{cases}$ (thus $Pr(X_t = 1) = p$)

[b] Let $X = \sum_{t=1}^{N} X_t$ (thus $E[X] = Np$)

[c] Return $Z = \frac{|\tilde{F}| \cdot X}{N}$

FACT $\mu = E[Z] = \frac{|\tilde{F}| \, E[X]}{N} = |\tilde{F}| p = |F|$

thus Z is an unbiased estimator

Running time is $O\left(\underbrace{n^3}_{\text{by DP1}} + N \underbrace{n}_{\text{one random sample}}\right) = O\left(n\left(n^2 + N\right)\right)$

Which choice for N?

Use Chernoff-Hoeffding bounds to make

(*) $\Pr[|Z-\mu| \geq \varepsilon\mu] \leq \delta$

General technique useful for sampling N times.

We saw these two bounds:

(1) $\Pr[X > \mu' + \lambda] \leq e^{-\frac{\lambda^2}{2\mu'+\lambda}}$, $\Pr[X < \mu' - \lambda] \leq e^{-\frac{\lambda^2}{3\mu'}}$

where $\mu' = E[X] = Np$; also, we set $\lambda = \varepsilon\mu' < \mu'$

(2) We also note: $Z = \frac{|\tilde{F}|}{N} \cdot X$ and $\mu = E[Z] = \frac{|\tilde{F}|}{N}\mu'$

(1)+(2) $\Rightarrow$ $\Pr[Z > (1+\varepsilon)\mu \text{ or } Z < (1-\varepsilon)\mu] =$

$\Pr[X > (1+\varepsilon)\mu' \text{ or } X < (1-\varepsilon)\mu'] \leq$

$e^{-\frac{\lambda^2}{2\mu'+\lambda}} + e^{-\frac{\lambda^2}{3\mu'}} \leq 2e^{-\frac{\lambda^2}{3\mu'}} = 2e^{-\frac{\varepsilon^2 Np}{3}}$

($\lambda < \mu'$ here; $\lambda = \varepsilon\mu'$, $\mu' = Np$)

To get (*) we set $2e^{-\frac{\varepsilon^2 Np}{3}} \leq \delta$

This gives $\frac{\varepsilon^2 Np}{3} \geq \ln 2/\delta \Leftrightarrow N \geq \frac{3\ln 2/\delta}{\varepsilon^2 p}$

As $p \geq \frac{1}{n+1}$, it suffices to choose $N = \frac{3(n+1)\ln 2/\delta}{\varepsilon^2}$

to guarantee (*).