COMPUTERS AND INTRACTABILITY

A Guide to the Theory of NP-Completeness

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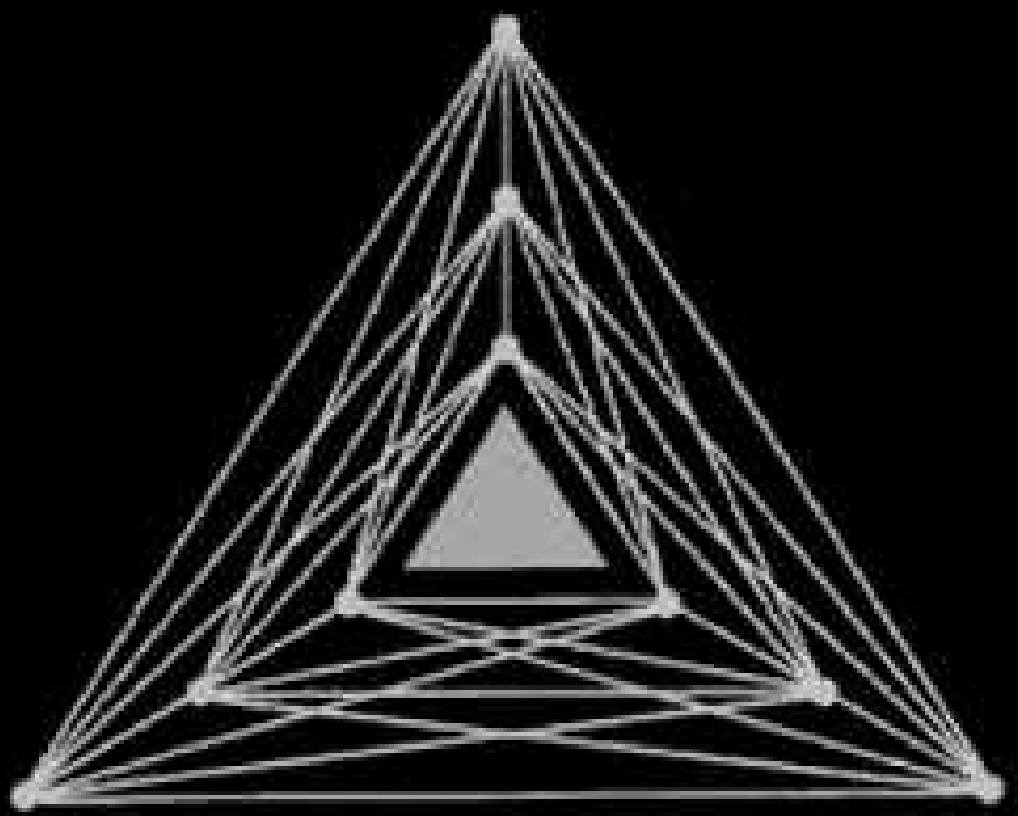

these six are themselves NP-complete. In Section 3.2 we describe three general approaches for transforming one problem to another, and we demonstrate their use by proving a wide variety of problems NP-complete. A concluding section contains some suggested exercises.

3.1 Six Basic NP-Complete Problems

When seasoned practitioners are confronted with a problem Π to be proved NP-complete, they have the advantage of having a wealth of experience to draw upon. They may well have proved a similar problem Π' NP-complete in the past or have seen such a proof. This will suggest that they try to prove Π NP-complete by mimicking the NP-completeness proof for Π' or by transforming Π' itself to Π. In many cases this may lead rather easily to an NP-completeness proof for Π.

All too often, however, no known NP-complete problem similar to Π can be found (even using the extensive lists at the end of this book). In such cases the practitioner may have no direct intuition as to which of the hundreds of known NP-complete problems is best suited to serve as the basis for the desired proof. Nevertheless, experience can still narrow the choices down to a core of basic problems that have been useful in the past. Even though in theory *any* known NP-complete problem can serve just as well as any other for proving a new problem NP-complete, in practice certain problems do seem to be much better suited for this task. The following six problems are among those that have been used most frequently, and we suggest that these six can serve as a "basic core" of known NP-complete problems for the beginner.

3-SATISFIABILITY (3SAT)
INSTANCE: Collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses on a finite set U of variables such that $|c_i| = 3$ for $1 \leqslant i \leqslant m$.
QUESTION: Is there a truth assignment for U that satisfies all the clauses in C?

3-DIMENSIONAL MATCHING (3DM)
INSTANCE: A set $M \subseteq W \times X \times Y$, where W, X, and Y are disjoint sets having the same number q of elements.
QUESTION: Does M contain a *matching*, that is, a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

VERTEX COVER (VC)
INSTANCE: A graph $G = (V,E)$ and a positive integer $K \leqslant |V|$.
QUESTION: Is there a *vertex cover* of size K or less for G, that is, a subset $V' \subseteq V$ such that $|V'| \leqslant K$ and, for each edge $\{u,v\} \in E$, at least one of u and v belongs to V'?

CLIQUE
INSTANCE: A graph $G = (V,E)$ and a positive integer $J \leqslant |V|$.
QUESTION: Does G contain a *clique* of size J or more, that is, a subset $V' \subseteq V$ such that $|V'| \geqslant J$ and every two vertices in V' are joined by an edge in E?

HAMILTONIAN CIRCUIT (HC)
INSTANCE: A graph $G = (V,E)$.
QUESTION: Does G contain a Hamiltonian circuit, that is, an ordering $\langle v_1, v_2, \ldots, v_n \rangle$ of the vertices of G, where $n = |V|$, such that $\{v_n, v_1\} \in E$ and $\{v_i, v_{i+1}\} \in E$ for all i, $1 \leqslant i < n$?

PARTITION
INSTANCE: A finite set A and a "size" $s(a) \in Z^+$ for each $a \in A$.
QUESTION: Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a) \; ?$$

One reason for the popularity of these six problems is that they all appeared in the original list of 21 NP-complete problems presented in [Karp, 1972]. We shall begin our illustration of the techniques for proving NP-completeness by proving that each of these six problems is NP-complete, noting, whenever appropriate, variants of these problems whose NP-completeness follows more or less directly from that of the basic problems.

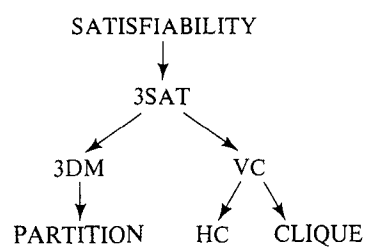

Figure 3.1 Diagram of the sequence of transformations used to prove that the six basic problems are NP-complete.

Our initial transformation will be from SATISFIABILITY, since it is the only "known" NP-complete problem we have so far. However, as we proceed through these six proofs, we will be enlarging our collection of known NP-complete problems, and all problems proved NP-complete before a problem Π will be available for use in proving that Π is NP-complete. The diagram of Figure 3.1 shows which problems we will be transforming to each of our six basic problems, where an arrow is drawn from one problem to another if the first is transformed to the second. This sequence of

is clear that $s(a_i)$ can be constructed from the given 3DM instance in polynomial time.

The important thing to observe about this part of the construction is that, if we sum up all the entries in any zone, over all elements of $\{a_i: 1 \leqslant i \leqslant k\}$, the total can never exceed $k = 2^p - 1$. Hence, in adding up $\sum_{a \in A'} s(a)$ for any subset $A' \subseteq \{a_i: 1 \leqslant i \leqslant k\}$, there will never be any "carries" from one zone to the next. It follows that if we let

$$B = \sum_{j=0}^{3q-1} 2^{pj}$$

(which is the number whose binary representation has a 1 in the rightmost position of every zone), then any subset $A' \subseteq \{a_i: 1 \leqslant i \leqslant k\}$ will satisfy

$$\sum_{a \in A'} s(a) = B$$

if and only if $M' = \{m_i: a_i \in A'\}$ is a matching for M.

The final step of the construction specifies the last two elements of A. These are denoted by b_1 and b_2 and have sizes defined by

$$s(b_1) = 2\left(\sum_{i=1}^{k} s(a_i)\right) - B$$

and

$$s(b_2) = \left(\sum_{i=1}^{k} s(a_i)\right) + B$$

Both of these can be specified in binary with no more than $(3pq+1)$ bits and thus can be constructed in time polynomial in the size of the given 3DM instance.

Now suppose we have a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$$

Then both of these sums must be equal to $2\sum_{i=1}^{k} s(a_i)$, and one of the two sets, A' or $A - A'$, contains b_1 but not b_2. It follows that the remaining elements of that set form a subset of $\{a_i: 1 \leqslant i \leqslant k\}$ whose sizes sum to B, and hence, by our previous comments, that subset corresponds to a matching M' in M. Conversely, if $M' \subseteq M$ is a matching, then the set $\{b_1\} \cup \{a_i: m_i \in M'\}$ forms the desired set A' for the PARTITION instance. Therefore, 3DM $\propto$ PARTITION, and the theorem is proved. ■

3.2 Some Techniques for Proving NP-Completeness

The techniques used for proving NP-completeness results vary almost as widely as the NP-complete problems themselves, and we cannot hope to illustrate them all here. However, there are several general types of proofs that occur frequently and that can provide a suggestive framework for deciding how to go about proving a new problem NP-complete. We call these (a) restriction, (b) local replacement, and (c) component design.

In this section we shall indicate what we mean by each of these proof types, primarily by giving examples. It would be sheer folly to attempt to define them explicitly. Many proofs can be interpreted in ways that would place them arbitrarily in any one of the three categories. Other proofs depend on decidedly problem-specific methods, so that no such limited set of categories could possibly include them in a natural way. Thus, we caution the reader *not* to interpret this as a way to classify all NP-completeness proofs. Rather, our sole intent is to illustrate several ways of thinking about NP-completeness proofs that the authors (and others) have found to be both intuitively appealing and constructive.

For brevity in what follows, we shall be omitting from all our proofs the verification that the given problem is in NP. Each of the problems we consider is easily seen to be solvable in polynomial time by a nondeterministic algorithm, and the reader should have no difficulty supplying such an algorithm whenever required.

3.2.1 Restriction

Proof by restriction is the simplest, and perhaps the most frequently applicable, of our three proof types. An NP-completeness proof by restriction for a given problem $\Pi \in$ NP consists simply of showing that Π contains a known NP-complete problem Π' as a special case. The heart of such a proof lies in the specification of the additional restrictions to be placed on the instances of Π so that the resulting restricted problem will be identical to Π'. We do not require that the restricted problem and the known NP-complete problem be *exact* duplicates of one another, but rather that there be an "obvious" one-to-one correspondence between their instances that preserves "yes" and "no" answers. This one-to-one correspondence, which provides the required transformation from Π' to Π, is usually so apparent that it need not even be given explicitly.

We have already seen several examples of this type of proof. In Section 3.1.2, the problem EXACT COVER BY 3-SETS was shown to be NP-complete by restricting its instances to 3-sets that contain one element from a set W, one from a set X, and one from a set Y, where W, X, and Y are disjoint sets having the same cardinality, thereby obtaining a problem identical to the 3DM problem. In Section 3.1.4, DIRECTED HAMILTONIAN

CIRCUIT was shown to be NP-complete by restricting its instances to directed graphs in which each arc (u,v) occurs only in conjunction with the oppositely directed arc (v,u), thereby obtaining a problem identical to the undirected HAMILTONIAN CIRCUIT problem.

Thus proofs by restriction can be seen to embody a different way of looking at things than the standard NP-completeness proofs. Instead of trying to discover a way of transforming a known NP-complete problem to our target problem, we focus on the target problem itself and attempt to restrict away its "inessential" aspects until a known NP-complete problem appears.

We now give a number of additional examples of problems proved NP-complete by restriction, stating each proof with the brevity it deserves.

(1) **MINIMUM COVER**
INSTANCE: Collection C of subsets of a set S, positive integer K.
QUESTION: Does C contain a *cover* for S of size K or less, that is, a subset $C' \subseteq C$ with $|C'| \leq K$ and such that $\bigcup_{c \in C'} c = S$?
Proof: Restrict to X3C by allowing only instances having $|c|=3$ for all $c \in C$ and having $K = |S|/3$.

(2) **HITTING SET**
INSTANCE: Collection C of subsets of a set S, positive integer K.
QUESTION: Does S contain a *hitting set* for C of size K or less, that is, a subset $S' \subseteq S$ with $|S'| \leq K$ and such that S' contains at least one element from each subset in C?
Proof: Restrict to VC by allowing only instances having $|c|=2$ for all $c \in C$.

(3) **SUBGRAPH ISOMORPHISM**
INSTANCE: Two graphs, $G=(V_1,E_1)$ and $H=(V_2,E_2)$.
QUESTION: Does G contain a subgraph *isomorphic* to H, that is, a subset $V \subseteq V_1$ and a subset $E \subseteq E_1$ such that $|V|=|V_2|, |E|=|E_2|$, and there exists a one-to-one function $f: V_2 \rightarrow V$ satisfying $\{u,v\} \in E_2$ if and only if $\{f(u),f(v)\} \in E$?
Proof: Restrict to CLIQUE by allowing only instances for which H is a complete graph, that is, E_2 contains all possible edges joining two members of V_2.

(4) **BOUNDED DEGREE SPANNING TREE**
INSTANCE: A graph $G=(V,E)$ and a positive integer $K \leq |V|-1$.
QUESTION: Is there a *spanning tree* for G in which no vertex has degree exceeding K, that is, a subset $E' \subseteq E$ such that $|E'|=|V|-1$, the graph $G'=(V,E')$ is connected, and no vertex in V is included in more than K edges from E'?
Proof: Restrict to HAMILTONIAN PATH by allowing only instances in which $K=2$.

(5) **MINIMUM EQUIVALENT DIGRAPH**
INSTANCE: A directed graph $G=(V,A)$ and a positive integer $K \leq |A|$.
QUESTION: Is there a directed graph $G'=(V,A')$ such that $A' \subseteq A$, $|A'| \leq K$, and such that, for every pair of vertices u and v in V, G' contains a directed path from u to v if and only if G contains a directed path from u to v.
Proof: Restrict to DIRECTED HAMILTONIAN CIRCUIT by allowing only instances in which G is strongly connected, that is, contains a path from every vertex u to every vertex v, and $K=|V|$. Note that this is actually a restriction to DIRECTED HAMILTONIAN CIRCUIT FOR STRONGLY CONNECTED DIGRAPHS, but the NP-completeness of that problem follows immediately from the constructions we gave for HC and DIRECTED HC.

(6) **KNAPSACK**
INSTANCE: A finite set U, a "size" $s(u) \in Z^+$ and a "value" $v(u) \in Z^+$ for each $u \in U$, a size constraint $B \in Z^+$, and a value goal $K \in Z^+$.
QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s(u) \leq B \quad \text{and} \quad \sum_{u \in U'} v(u) \geq K$$

Proof: Restrict to PARTITION by allowing only instances in which $s(u)=v(u)$ for all $u \in U$ and $B=K=\frac{1}{2}\sum_{u \in U} s(u)$.

(7) **MULTIPROCESSOR SCHEDULING**
INSTANCE: A finite set A of "tasks," a "length" $l(a) \in Z^+$ for each $a \in A$, a number $m \in Z^+$ of "processors," and a "deadline" $D \in Z^+$.
QUESTION: Is there a partition $A = A_1 \cup A_2 \cup \cdots \cup A_m$ of A into m disjoint sets such that

$$\max \left\{ \sum_{a \in A_i} l(a) : 1 \leq i \leq m \right\} \leq D \quad ?$$

Proof: Restrict to PARTITION by allowing only instances in which $m=2$ and $D=\frac{1}{2}\sum_{a \in A} l(a)$.

As a final comment, we observe that, of all the approaches to proving NP-completeness we shall discuss, proof by restriction is the one that would profit most from an extensive knowledge of the class of known NP-complete problems — beyond the basic six and their variants. Many problems that arise in practice are simply more complicated versions of problems