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Models of Computation

– Monograph –

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Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity.

Alan Turing[1]

[1] The purpose of ordinal logics (from Systems of Logic Based on Ordinals), Proceedings of the London Mathematical Society, series 2, vol. 45, 1939.

Contents

Acronyms

$\sim$	operational equivalence in IMP (see Definition 3.3)
$\equiv_{den}$	denotational equivalence in HOFL (see Definition 10.4)
$\equiv_{op}$	operational equivalence in HOFL (see Definition 10.3)
$\simeq$	CCS strong bisimilarity (see Definition 11.5)
$\approx$	CCS weak bisimilarity (see Definition 11.16)
$\cong$	CCS weak observational congruence (see Section 11.7.2)
$\approx_d$	CCS dynamic bisimilarity (see Definition 11.17)
$\overset{\circ}{\sim}_E$	π-calculus early bisimilarity (see Definition 13.3)
$\overset{\circ}{\sim}_L$	π-calculus late bisimilarity (see Definition 13.4)
$\sim_E$	π-calculus strong early full bisimilarity (see Section 13.5.3)
$\sim_L$	π-calculus strong late full bisimilarity (see Section 13.5.3)
$\overset{\bullet}{\approx}_E$	π-calculus weak early bisimilarity (see Section 13.5.4)
$\overset{\bullet}{\approx}_L$	π-calculus weak late bisimilarity (see Section 13.5.4)
$\mathscr{A}$	interpretation function for the denotational semantics of IMP arithmetic expressions (see Section 6.2.1)
ack	Ackermann function (see Example 4.18)
Aexp	set of IMP arithmetic expressions (see Chapter 3)
$\mathscr{B}$	interpretation function for the denotational semantics of IMP boolean expressions (see Section 6.2.2)
Bexp	set of IMP boolean expressions (see Chapter 3)
$\mathbb{B}$	set of booleans
$\mathscr{C}$	interpretation function for the denotational semantics of IMP commands (see Section 6.2.3)
CCS	Calculus of Communicating Systems (see Chapter 11)
Com	set of IMP commands (see Chapter 3)
CPO	Complete Partial Order (see Definition 5.11)
$\mathrm{CPO}_\bot$	Complete Partial Order with bottom (see Definition 5.12)
CSP	Communicating Sequential Processes (see Section 16.2)
CTL	Computation Tree Logic (see Section 12.1.2)
CTMC	Continuous Time Markov Chain (see Definition 14.15)

DTMC	Discrete Time Markov Chain (see Definition 14.14)
Env	set of HOFL environments (see Chapter 9)
fix	(least) fixpoint (see Definition 5.2.2)
FIX	(greatest) fixpoint
gcd	greatest common divisor
HML	Hennessy-Milner modal Logic (see Section 11.5)
HM-Logic	Hennessy-Milner modal Logic (see Section 11.5)
HOFL	A Higher-Order Functional Language (see Chapter 7)
IMP	A simple IMPerative language (see Chapter 3)
int	integer type in HOFL (see Definition 7.2)
Loc	set of locations (see Chapter 3)
LTL	Linear Temporal Logic (see Section 12.1.1)
LTS	Labelled Transition System (see Definition 11.2)
lub	least upper bound (see Definition 5.7)
$\mathbb{N}$	set of natural numbers
$\mathscr{P}$	set of closed CCS processes (see Definition 11.1)
PEPA	Performance Evaluation Process Algebra (see Chapter 16)
Pf	set of partial functions on natural numbers (see Example 5.13)
PI	set of partial injective functions on natural numbers (see Problem 5.12)
PO	Partial Order (see Definition 5.1)
PTS	Probabilistic Transition System (see Section 14.3.2)
$\mathbb{R}$	set of real numbers
$\mathscr{T}$	set of HOFL types (see Definition 7.2)
Tf	set of total functions from $\mathbb{N}$ to $\mathbb{N}_\perp$ (see Example 5.14)
Var	set of HOFL variables (see Chapter 7)
$\mathbb{Z}$	set of integers

Part III
HOFL: a higher-order functional language

This part focuses on models for sequential computations that are associated to HOFL, a higher-order declarative language that follows the functional style. Chapter 7 presents the syntax, typing and operational semantics of HOFL, while Chapter 9 defines its denotational semantics. The two are related in Chapter 10. Chapter 8 extends the theory presented in Chapter 5 to allow the construction of more complex domains, as needed by the type-constructors available in HOFL.

Chapter 8
Domain Theory

Order, unity and continuity are human inventions just as truly as catalogues and encyclopedias. (Bertrand Russell)

Abstract As done for IMP we would like to introduce the denotational semantics of HOFL, for which we need to develop a proper domain theory that is more sophisticated than the one presented in Chapter 5. In order to define the denotational semantics of IMP we have shown that the semantic domain of commands, for which we need to apply fixpoint theorem, has the required properties. The situation is more complicated for HOFL, because HOFL provides constructors for infinitely many term types, so there are infinitely many domains to be considered. We will handle this problem by showing, using structural induction, that the type constructors of HOFL correspond to domains which are equipped with adequate $CPO_\perp$ structures and that we can define useful continuous functions between them.

8.1 The Flat Domain of Integer Numbers $\mathbb{Z}_\perp$

The first domain we introduce is very simple: it consists of all the integers numbers together with a distinguished bottom element. It relies on a flat order in the sense of Example 5.5.

Definition 8.1 ($\mathbb{Z}_\perp$). We define the CPO with bottom $\mathbb{Z}_\perp = (\mathbb{Z} \cup \{\perp\}, \sqsubseteq)$ as follows:

- $\mathbb{Z}$ is the set of integer numbers;
- $\perp$ is a distinguished bottom element that we add to the purpose;
- $\forall x \in \mathbb{Z} \cup \{\perp\}.\ \perp \sqsubseteq x$ and $x \sqsubseteq x$

It is immediate to check that $\mathbb{Z}_\perp$ is a CPO with bottom, where $\perp$ is the bottom element and each chain has a lub because chains are all finite: they either contain 1 or 2 different elements.

Remark 8.1. Since in this chapter we present several different domains, each coming with its proper order relation and bottom element, we find it useful to annotate them with the name of the domain as a subscript to avoid ambiguities. For example, we can write $\perp_{\mathbb{Z}_\perp}$ to make explicit that we are referring to the bottom element of the domain $\mathbb{Z}_\perp$. Also note that the subscript $\perp$ we attach to the name of the domain $\mathbb{Z}$ is just a tag and it should not be confused with the name of the bottom element itself: it is the standard way to indicate that the domain $\mathbb{Z}$ is enriched with a bottom element (e.g., we could have used a different notation like $\underline{\mathbb{Z}}$ to the same purpose).

8.2 Cartesian Product of Two Domains

Given two $\text{CPO}_\perp$s we can combine them to obtain another $\text{CPO}_\perp$ whose elements are pairs formed with one element from each $\text{CPO}_\perp$.

Definition 8.2. Let:

$$\mathscr{D} = (D, \sqsubseteq_D) \qquad \mathscr{E} = (E, \sqsubseteq_E)$$

be two $\text{CPO}_\perp$s. Now we define their Cartesian product domain

$$\mathscr{D} \times \mathscr{E} = (D \times E, \sqsubseteq_{D\times E})$$

1. whose elements are the pairs of elements from D and E; and
2. whose order $\sqsubseteq_{D\times E}$ is defined as follows:[1]

$$\forall d_0, d_1 \in D,\ \forall e_0, e_1 \in E.\ (d_0, e_0) \sqsubseteq_{D\times E} (d_1, e_1) \ \Leftrightarrow\ d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

Proposition 8.1. $(D \times E, \sqsubseteq_{D\times E})$ *is a partial order with bottom.*

Proof. We need to show that the relation $\sqsubseteq_{D\times E}$ is reflexive, antisymmetric and transitive:

reflexivity: since $\sqsubseteq_D$ and $\sqsubseteq_E$ are reflexive we have $\forall e \in E.\ e \sqsubseteq_E e$ and $\forall d \in D.\ d \sqsubseteq_D d$ so by definition of $\sqsubseteq_{D\times E}$ we have

$$\forall d \in D\ \forall e \in E.\ (d, e) \sqsubseteq_{D\times E} (d, e).$$

antisymmetry: let us assume $(d_0, e_0) \sqsubseteq_{D\times E} (d_1, e_1)$ and $(d_1, e_1) \sqsubseteq_{D\times E} (d_0, e_0)$ so by definition of $\sqsubseteq_{D\times E}$ we have $d_0 \sqsubseteq_D d_1$ (using the first relation) and $d_1 \sqsubseteq_D d_0$ (by using the second relation) so it must be $d_0 = d_1$ and similarly $e_0 = e_1$, hence $(d_0, e_0) = (d_1, e_1)$.

transitivity: let us assume $(d_0, e_0) \sqsubseteq_{D\times E} (d_1, e_1)$ and $(d_1, e_1) \sqsubseteq_{D\times E} (d_2, e_2)$. By definition of $\sqsubseteq_{D\times E}$ we have $d_0 \sqsubseteq_D d_1$, $d_1 \sqsubseteq_D d_2$, $e_0 \sqsubseteq_E e_1$ and $e_1 \sqsubseteq_E e_2$. By transitivity of $\sqsubseteq_D$ and $\sqsubseteq_E$ we have $d_0 \sqsubseteq_D d_2$ and $e_0 \sqsubseteq_E e_2$. By definition of $\sqsubseteq_{D\times E}$ we get $(d_0, e_0) \sqsubseteq_{D\times E} (d_2, e_2)$.

Finally, we show that the there is a bottom element. Let $\perp_{D\times E} = (\perp_D, \perp_E)$. In fact $\forall d \in D, e \in E.\ \perp_D \sqsubseteq d \wedge \perp_E \sqsubseteq e$, thus $(\perp_D, \perp_E) \sqsubseteq_{D\times E} (d, e)$. $\square$

It remains to show the completeness of $\mathscr{D} \times \mathscr{E}$.

[1] Note that the order is different from the lexicographic one considered in Example 4.9.

Theorem 8.1 (Completeness of $\mathscr{D} \times \mathscr{E}$). *The PO $\mathscr{D} \times \mathscr{E}$ defined above is complete.*

Proof. We prove that for each chain $(d_i, e_i)_{i\in\mathbb{N}}$ it holds:

$$\bigsqcup_{i\in\mathbb{N}} (d_i, e_i) = \left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i \right)$$

Obviously $(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i)$ is an upper bound, indeed for each $j \in \mathbb{N}$ we have $d_j \sqsubseteq_D \bigsqcup_{i\in\mathbb{N}} d_i$ and $e_j \sqsubseteq_E \bigsqcup_{i\in\mathbb{N}} e_i$ so by definition of $\sqsubseteq_{D\times E}$ it holds $(d_j, e_j) \sqsubseteq_{D\times E} (\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i)$.
Moreover $(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i)$ is also the least upper bound. Indeed, let (d, e) be an upper bound of $\{(d_i, e_i)\}_{i\in\mathbb{N}}$, since $\bigsqcup_{i\in\mathbb{N}} d_i$ is the lub of $\{d_i\}_{i\in\mathbb{N}}$ we have $\bigsqcup_{i\in\mathbb{N}} d_i \sqsubseteq_D d$, furthermore we have that $\bigsqcup_{i\in\mathbb{N}} e_i$ is the lub of $\{e_i\}_{i\in\mathbb{N}}$ then $\bigsqcup_{i\in\mathbb{N}} e_i \sqsubseteq_E e$. So by definition of $\sqsubseteq_{D\times E}$ we have $(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i) \sqsubseteq_{D\times E} (d, e)$. Thus $(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i)$ is the least upper bound. □

We can now define suitable projection operators over $\mathscr{D} \times \mathscr{E}$.

Definition 8.3 (Projection operators π_1 and π_2). Let $(d, e) \in D \times E$ be a pair, we define the left and right projection functions $\pi_1 : D \times E \to D$ and $\pi_2 : D \times E \to E$ as follows.

$$\pi_1((d, e)) \stackrel{\text{def}}{=} d \qquad \text{and} \qquad \pi_2((d, e)) \stackrel{\text{def}}{=} e.$$

Recall that in order to use a function in domain theory we have to show that it is continuous; this ensures that the function respects the domain structure (i.e., the function preserves the order and limits) and so we can calculate its fixpoints to solve recursive equations. So we have to prove that each function which we use on $\mathscr{D} \times \mathscr{E}$ is continuous. The proof that projections are monotone is immediate and left as an exercise (see Problem 8.1).

Theorem 8.2 (Continuity of π_1 and π_2). *Let π_1 and π_2 be the projection functions in Definition 8.3 and let $\{(d_i, e_i)\}_{i\in\mathbb{N}}$ be a chain of elements in $\mathscr{D} \times \mathscr{E}$, then:*

$$\pi_1\left(\bigsqcup_{i\in\mathbb{N}} (d_i, e_i)\right) = \bigsqcup_{i\in\mathbb{N}} \pi_1((d_i, e_i)) \qquad \pi_2\left(\bigsqcup_{i\in\mathbb{N}} (d_i, e_i)\right) = \bigsqcup_{i\in\mathbb{N}} \pi_2((d_i, e_i))$$

Proof. Let us prove the first statement:

$$\begin{aligned}
\pi_1\left(\bigsqcup_{i\in\mathbb{N}} (d_i, e_i)\right) &= \pi_1\left(\left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i\right)\right) && \text{(by definition of limit in } D \times E) \\
&= \bigsqcup_{i\in\mathbb{N}} d_i && \text{(by definition of projection)} \\
&= \bigsqcup_{i\in\mathbb{N}} \pi_1((d_i, e_i)) && \text{(by definition of projection).}
\end{aligned}$$

For the second statement the proof is completely analogous. □

8.3 Functional Domains

Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be two CPOs. In the following we denote by $D \to E \stackrel{\text{def}}{=} \{f \mid f : D \to E\}$ the set of all functions from D to E (where the order relations are not important), while we denote by $[D \to E] \subseteq D \to E$ the set of all continuous functions from D to E (i.e., $[D \to E]$ contains just the functions that preserve order and limits). As for Cartesian product, we can define a suitable order on the set $[D \to E]$ to get a $\text{CPO}_\perp$. Note that as usual we require the continuity of the functions to preserve the applicability of fixpoint theory.

Definition 8.4. Let us consider the $\text{CPO}_\perp$s:

$$\mathscr{D} = (D, \sqsubseteq_D) \qquad \mathscr{E} = (E, \sqsubseteq_E)$$

We define an order on the set of continuous functions from D to E as follows:

$$[\mathscr{D} \to \mathscr{E}] = \big([D \to E], \sqsubseteq_{[D \to E]}\big)$$

where:

1. $[D \to E] = \{f \mid f : D \to E, f \text{ is continuous}\}$
2. $f \sqsubseteq_{[D \to E]} g \Leftrightarrow \forall d \in D. f(d) \sqsubseteq_E g(d)$

We leave as an exercise the proof that $[\mathscr{D} \to \mathscr{E}]$ is a PO with bottom, namely that the relation $\sqsubseteq_{[D \to E]}$ is reflexive, antisymmetric, transitive and that the function $\perp_{[D \to E]} : D \to E$ defined by letting, for any $d \in D$:

$$\perp_{[D \to E]}(d) \stackrel{\text{def}}{=} \perp_E$$

is continuous and that it is also the bottom element of $[\mathscr{D} \to \mathscr{E}]$ (see Problem 8.2).

We show that the PO $[\mathscr{D} \to \mathscr{E}]$ is complete. In order to simplify the proof we introduce first the following lemmas.

Lemma 8.1 (Switch Lemma). *Let $(E, \sqsubseteq_E)$ be a CPO whose elements are of the form $e_{n,m}$ with $n, m \in \mathbb{N}$. If $\sqsubseteq_E$ is such that:*

$$e_{n,m} \sqsubseteq_E e_{n',m'} \text{ if } n \le n' \text{ and } m \le m'$$

then, it holds:

$$\bigsqcup_{n,m \in \mathbb{N}} e_{n,m} = \bigsqcup_{n \in \mathbb{N}} \left(\bigsqcup_{m \in \mathbb{N}} e_{n,m} \right) = \bigsqcup_{m \in \mathbb{N}} \left(\bigsqcup_{n \in \mathbb{N}} e_{n,m} \right) = \bigsqcup_{k \in \mathbb{N}} e_{k,k}$$

Proof. The relation between the elements of E can be summarized as follows:

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}$$

We show that all the following sets have the same upper bounds:

$$\{e_{n,m}\}_{n,m\in\mathbb{N}} \qquad \left\{\bigsqcup_{m\in\mathbb{N}} e_{n,m}\right\}_{n\in\mathbb{N}} \qquad \left\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\right\}_{m\in\mathbb{N}} \qquad \{e_{k,k}\}_{k\in\mathbb{N}}$$

- Let us consider the first two sets. For any $n \in \mathbb{N}$, let $e_n = \bigsqcup_{j\in\mathbb{N}} e_{n,j}$. This amounts to consider each row of the above diagram and compute the least upper bound for the elements in the same row. Clearly, $e_{n_1} \sqsubseteq e_{n_2}$ when $n_1 \leq n_2$ because for any $j \in \mathbb{N}$ an upper bound of $e_{n_2,j}$ is also an upper bound of $e_{n_1,j}$.

$$\begin{array}{ccccccccccccl}
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\sqcup\!\! \mid & & \sqcup\!\! \mid & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid & & & & \sqcup\!\! \mid \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}$$

Let e be an upper bound of $\{e_i\}_{i\in\mathbb{N}}$, we want to show that e is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$. Take any $n, m \in \mathbb{N}$. Then

$$e_{n,m} \sqsubseteq \bigsqcup_{j\in\mathbb{N}} e_{n,j} = e_n \sqsubseteq e$$

since $e_{n,m}$ is an element of the chain $\{e_{n,j}\}_{j\in\mathbb{N}}$ whose limit is $e_n = \bigsqcup_{j\in\mathbb{N}} e_{n,j}$. Thus e is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$.

Vice versa, let e be an upper bound of $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ and consider $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$ for some n. Since $\{e_{n,m}\}_{m\in\mathbb{N}} \subseteq \{e_{i,j}\}_{i,j\in\mathbb{N}}$, obviously e is an upper bound for $\{e_{n,m}\}_{m\in\mathbb{N}}$ and therefore $e_n \sqsubseteq e$, because e_n is the lub of $\{e_{n,m}\}_{m\in\mathbb{N}}$.

- The correspondence between the sets of upper bounds of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ and $\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\}_{m\in\mathbb{N}}$ can be proved analogously.
- Finally, let us consider the sets $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ and $\{e_{k,k}\}_{k\in\mathbb{N}}$ and show that they have the same set of upper bounds.
  Taken any $n,m \in \mathbb{N}$ the element, let $k = \max\{n,m\}$. We have

$$e_{n,m} \sqsubseteq e_{n,k} \sqsubseteq e_{k,k}$$

  thus any upper bound of $\{e_{k,k}\}_{k\in\mathbb{N}}$ is also an upper bound of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$.
  Vice versa, it is immediate to check that $\{e_{k,k}\}_{k\in\mathbb{N}}$ is a subset of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ so any upper bound of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ is also an upper bound of $\{e_{k,k}\}_{k\in\mathbb{N}}$.

We conclude by noting that the set of upper bounds $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ has a least element. In fact, $\{\bigsqcup_{m\in\mathbb{N}} e_{n,m}\}_{n\in\mathbb{N}}$ is a chain, and it has a lub because E is a CPO. □

Lemma 8.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a chain of functions*[2] *in $\mathscr{D} \to \mathscr{E}$. Then the lub $\bigsqcup_{n\in\mathbb{N}} f_n$ exists and it is defined as:*

$$\left(\bigsqcup_{n\in\mathbb{N}} f_n\right)(d) = \bigsqcup_{n\in\mathbb{N}} (f_n(d))$$

Proof. The function

$$h \stackrel{\text{def}}{=} \lambda d. \bigsqcup_{n\in\mathbb{N}} (f_n(d))$$

is clearly an upper bound for $\{f_n\}_{n\in\mathbb{N}}$ since for every $k \in \mathbb{N}$ and $d \in D$ we have $f_k(d) \sqsubseteq_E \bigsqcup_{n\in\mathbb{N}} f_n(d)$.

The function h is also the lub of $\{f_n\}_{n\in\mathbb{N}}$. In fact, given any other upper bound g, i.e., such that $f_n \sqsubseteq_{D\to E} g$ for any $n \in \mathbb{N}$, we have that for any $d \in D$ the element $g(d)$ is an upper bound of the chain $\{f_n(d)\}_{n\in\mathbb{N}}$ and therefore $\bigsqcup_{n\in\mathbb{N}}(f_n(d)) \sqsubseteq_E g(d)$. □

Lemma 8.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a chain of continuous functions in $[\mathscr{D} \to \mathscr{E}]$ and let $\{d_n\}_{n\in\mathbb{N}}$ be a chain of $\mathscr{D}$. Then, the function*

$$h \stackrel{\text{def}}{=} \lambda d. \bigsqcup_{n\in\mathbb{N}} (f_n(d))$$

is continuous, namely

$$h\left(\bigsqcup_{m\in\mathbb{N}} d_m\right) = \bigsqcup_{m\in\mathbb{N}} h(d_m)$$

Furthermore, h is the lub of $\{f_n\}_{n\in\mathbb{N}}$ not only in $\mathscr{D} \to \mathscr{E}$ as stated by Lemma 8.2, but also in $[\mathscr{D} \to \mathscr{E}]$.

[2] Note that the f_n are not necessarily continuous, because we select $\mathscr{D} \to \mathscr{E}$ and not $[\mathscr{D} \to \mathscr{E}]$.

Proof.

$$
\begin{aligned}
h\left(\bigsqcup_{m\in\mathbb{N}} d_m\right) &= \bigsqcup_{n\in\mathbb{N}}\left(f_n\left(\bigsqcup_{m\in\mathbb{N}} d_m\right)\right) && \text{(by definition of } h\text{)}\\
&= \bigsqcup_{n\in\mathbb{N}}\left(\bigsqcup_{m\in\mathbb{N}} (f_n(d_m))\right) && \text{(by continuity of } f_n\text{)}\\
&= \bigsqcup_{m\in\mathbb{N}}\left(\bigsqcup_{n\in\mathbb{N}} (f_n(d_m))\right) && \text{by Lemma 8.1 (switch lemma)}\\
&= \bigsqcup_{m\in\mathbb{N}} h(d_m) && \text{(by definition of } h\text{)}
\end{aligned}
$$

The upper bounds of $\{f_n\}_{n\in\mathbb{N}}$ in the PO $\mathscr{D}\to\mathscr{E}$ are a larger set then those in $[\mathscr{D}\to\mathscr{E}]$, thus if h is the lub in $\mathscr{D}\to\mathscr{E}$, it is also the lub in $[\mathscr{D}\to\mathscr{E}]$. □

Theorem 8.3 ($[\mathscr{D}\to\mathscr{E}]$ is a CPO$_\perp$). *The PO $[\mathscr{D}\to\mathscr{E}]$ is a CPO$_\perp$*

Proof. The statement follows immediately from the previous lemmas. □

8.4 Lifting

In IMP we introduced a lifting operator (see Definition 6.9) on functions $f:\Sigma\to\Sigma_\perp$ to derive a function $f^*:\Sigma_\perp\to\Sigma_\perp$ defined over the lifted domain $\Sigma_\perp$, and thus able to handle the argument $\perp_{\Sigma_\perp}$. In the semantics of HOFL we need the same operator in a more general fashion: we need to apply the lifting operator to any domain, not just Σ.

Definition 8.5 (Lifted domain). Let $\mathscr{D}=(D,\sqsubseteq_D)$ be a CPO and let $\perp$ be an element not in D. We define the lifted domain $\mathscr{D}_\perp=(D_\perp,\sqsubseteq_{D_\perp})$ as follows:

- $D_\perp \stackrel{\text{def}}{=} \{\perp\}\uplus D = \{(0,\perp)\}\cup\{1\}\times D$
- $\perp_{D_\perp} \stackrel{\text{def}}{=} (0,\perp)$
- $\forall x\in D_\perp.\ \perp_{D_\perp}\sqsubseteq_{D_\perp} x$
- $\forall d_1,d_2\in D.\ d_1\sqsubseteq_D d_2 \Rightarrow (1,d_1)\sqsubseteq_{D_\perp}(1,d_2)$

We leave it as an exercise to show that $\mathscr{D}_\perp$ is a CPO$_\perp$ (see Problem 8.3).

We define a lifting function $\lfloor\cdot\rfloor: D\to D_\perp$ by letting, for any $d\in D$:

$$\lfloor d\rfloor \stackrel{\text{def}}{=} (1,d)$$

As it was the case for Σ in the IMP semantics, when we add a bottom element to a domain $\mathscr{D}$ we would like to extend the continuous functions in $[D\to E]$ to continuous functions in $[D_\perp\to E]$. The function defining the extension should itself be continuous.

Definition 8.6 (Lifting). Let $\mathscr{D}$ be a CPO and let $\mathscr{E}$ be a $CPO_\perp$. We define the lifting operator $(\cdot)^* : [D \to E] \to [D_\perp \to E]$ as follows:

$$\forall f \in [D \to E]\,.\; f^*(x) \stackrel{\text{def}}{=} \begin{cases} \perp_E & \text{if } x = \perp_{D_\perp} \\ f(d) & \text{if } x = \lfloor d \rfloor \end{cases}$$

We need to prove that the definition is well-given and that the lifting operator is continuous.

Theorem 8.4. *Let $\mathscr{D}, \mathscr{E}$ be two CPOs.*

1. *If $f : D \to E$ is continuous, then f^* is continuous.*
2. *The operator $(\cdot)^*$ is continuous.*

Proof. We prove the two statements separately.

1. We need to prove that if $f \in [D \to E]$, then $f^* \in [D_\perp \to E]$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a chain in $\mathscr{D}_\perp$. We have to prove $f^*(\bigsqcup_{n\in\mathbb{N}} x_n) = \bigsqcup_{n\in\mathbb{N}} f^*(x_n)$.
   If $\forall n \in \mathbb{N}.\ x_n = \perp_{D_\perp}$, then this is obvious.
   Otherwise, for some $k \in \mathbb{N}$ there must exist a set of elements $\{d_{n+k}\}_{n\in\mathbb{N}}$ in D such that for all $m \geq k$ we have $x_m = \lfloor d_m \rfloor$ and also $\bigsqcup_{n\in\mathbb{N}} x_n = \bigsqcup_{n\in\mathbb{N}} x_{n+k} = \lfloor \bigsqcup_{n\in\mathbb{N}} d_{n+k} \rfloor$ (by prefix independence of the limit, Lemma 5.1). Then:

$$\begin{aligned}
f^*\left(\bigsqcup_{n\in\mathbb{N}} x_n\right) &= f^*\left(\left\lfloor \bigsqcup_{n\in\mathbb{N}} d_{n+k} \right\rfloor\right) && \text{by the above argument} \\
&= f\left(\bigsqcup_{n\in\mathbb{N}} d_{n+k}\right) && \text{by definition of lifting} \\
&= \bigsqcup_{n\in\mathbb{N}} f(d_{n+k}) && \text{by continuity of } f \\
&= \bigsqcup_{n\in\mathbb{N}} f^*(\lfloor d_{n+k} \rfloor) && \text{by definition of lifting} \\
&= \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k}) && \text{by definition of } x_{n+k} \\
&= \bigsqcup_{n\in\mathbb{N}} f^*(x_n) && \text{by Lemma 5.1}
\end{aligned}$$

2. We leave the proof that $(\cdot)^*$ is monotone as an exercise (see Problem 8.4).
   Let $\{f_i\}_{i\in\mathbb{N}}$ be a chain of functions in $[\mathscr{D} \to \mathscr{E}]$. We will prove that for all $x \in D_\perp$:

$$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^*(x) = \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right)(x)$$

   if $x = \perp_{D_\perp}$ both sides of the equation simplify to $\perp_E$. So let us assume $x = \lfloor d \rfloor$ for some $d \in D$ we have:

$$\begin{aligned}
\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^*(\lfloor d\rfloor) &= \left(\bigsqcup_{i\in\mathbb{N}} f_i\right)(d) && \text{by definition of lifting}\\
&= \bigsqcup_{i\in\mathbb{N}} (f_i(d)) && \text{by def. of lub in a functional domain}\\
&= \bigsqcup_{i\in\mathbb{N}} (f_i^*(\lfloor d\rfloor)) && \text{by definition of lifting}\\
&= \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right)(\lfloor d\rfloor) && \text{by def. of lub in a functional domain}
\end{aligned}$$

□

8.5 Function's Continuity Theorems

In this section we show some theorems which allow to prove the continuity of some functions. We start proving that the composition of two continuous functions is continuous.

Theorem 8.5 (Continuity of composition). *Let $f \in [D \to E]$ and $g \in [E \to F]$. Their composition*

$$f;g = g \circ f \stackrel{\text{def}}{=} \lambda d.\ g(f(d)) : D \to F$$

is a continuous function, i.e., $g \circ f \in [D \to F]$.

Proof. The statement is just a rephrasing of Theorem 5.5. □

Now we consider a function whose outcome is a pair of values. So the function has a single CPO as domain but it returns a result over a product of CPOs.

$$f : D \to E_1 \times E_2$$

For this type of functions we introduce a theorem which allows to prove the continuity of f in a convenient way. We will consider f as the pairing of two simpler functions $g_1 : D \to E_1$ and $g_2 : D \to E_2$, such that $f(d) = (g_1(d), g_2(d))$ for any $d \in D$. Then we can prove the continuity of f from the continuity of g_1 and g_2 (and vice versa).

Theorem 8.6. *Let $f : D \to E_1 \times E_2$ be a function over CPOs and let*

$$g_1 \stackrel{\text{def}}{=} f;\pi_1 : D \to E_1 \qquad g_2 \stackrel{\text{def}}{=} f;\pi_2 : D \to E_2$$

where $f;\pi_i = \lambda x.\ \pi_i(f(x))$ is the composition of f and π_i for $i = 1,2$. Then: f is continuous if and only if g_1 and g_2 are continuous.

Proof. Notice that we have

$$\forall d \in D.\ f(d) = (g_1(d), g_2(d))$$

We prove the two implications separately.

⇒) Since f is continuous, by Theorem 8.5 (continuity of composition) and Theorem 8.2 (continuity of projections), also g_1 and g_2 are continuous, because they are obtained as the composition of continuous functions.

⇐) We assume the continuity of g_1 and g_2 and we want to prove that f is continuous. Let $\{d_i\}_{i\in\mathbb{N}}$ be a chain in D. We want to prove:

$$f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} f(d_i)$$

So we have:

$$\begin{aligned}
f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) &= \left(g_1\left(\bigsqcup_{i\in\mathbb{N}} d_i\right), g_2\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)\right) && \text{(by definition of } g_1, g_2\text{)}\\
&= \left(\bigsqcup_{i\in\mathbb{N}} g_1(d_i), \bigsqcup_{i\in\mathbb{N}} g_2(d_i)\right) && \text{(by continuity of } g_1 \text{ and } g_2\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}} (g_1(d_i), g_2(d_i)) && \text{(by definition of lub of pairs)}\\
&= \bigsqcup_{i\in\mathbb{N}} f(d_i) && \text{(by definition of } g_1, g_2\text{)}
\end{aligned}$$

□

Now let us consider the case of a function $f : D_1 \times D_2 \to E$ over CPOs which takes a pair of arguments in D_1 and D_2 and then returns an element of E. The following theorem allows us to study the continuity of f by analysing each parameter separately.

Theorem 8.7. *Let $f : D_1 \times D_2 \to E$ be a function over CPOs. Then f is continuous if and only if all the functions in the following two classes are continuous:*

1. $\forall d' \in D_1.\ f_{d'} : D_2 \to E$ *is defined as* $f_{d'} \stackrel{\text{def}}{=} \lambda y. f(d', y)$;
2. $\forall d'' \in D_2.\ f_{d''} : D_1 \to E$ *is defined as* $f_{d''} \stackrel{\text{def}}{=} \lambda x. f(x, d'')$.

Proof. We prove the two implications separately:

⇒) If f is continuous then for all $d' \in D_1, d'' \in D_2$ the functions $f_{d'}$ and $f_{d''}$ are continuous, since we are considering only certain chains (where one element of the pair is fixed). For example, let us fix $d' \in D_1$ and consider a chain $\{d''_i\}_{i\in\mathbb{N}}$ in D_2. Then we prove that $f_{d'}$ is continuous as follows:

$$
\begin{aligned}
f_{d'}\left(\bigsqcup_{i\in\mathbb{N}} d''_i\right) &= f\left(d', \bigsqcup_{i\in\mathbb{N}} d''_i\right) && \text{(by definition of } f_{d'}\text{)}\\
&= f\left(\bigsqcup_{i\in\mathbb{N}} (d', d''_i)\right) && \text{(by definition of lub)}\\
&= \bigsqcup_{i\in\mathbb{N}} f(d', d''_i) && \text{(by continuity of } f\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}} f_{d'}(d''_i) && \text{(by definition of } f_{d'}\text{)}
\end{aligned}
$$

Similarly, if we fix $d'' \in D_2$ and take a chain $\{d'_i\}_{i\in\mathbb{N}}$ in D_1 we have $f_{d''}(\bigsqcup_{i\in\mathbb{N}} d'_i) = \bigsqcup_{i\in\mathbb{N}} f_{d''}(d'_i)$.

$\Leftarrow$) In the opposite direction, assume that $f_{d'}$ and $f_{d''}$ are continuous for all elements $d' \in D_1$ and $d'' \in D_2$. We want to prove that f is continuous. Take a chain $\{(d'_k, d''_k)\}_{k\in\mathbb{N}}$. By definition of lub on pairs, we have

$$\bigsqcup_{k\in\mathbb{N}} (d'_k, d''_k) = \left(\bigsqcup_{i\in\mathbb{N}} d'_i, \bigsqcup_{j\in\mathbb{N}} d''_j\right)$$

Let $d'' \stackrel{\text{def}}{=} \bigsqcup_{j\in\mathbb{N}} d''_j$. It follows:

$$\begin{aligned}
f\left(\bigsqcup_{k\in\mathbb{N}}(d'_k,d''_k)\right) &= f\left(\bigsqcup_{i\in\mathbb{N}}d'_i,\bigsqcup_{j\in\mathbb{N}}d''_j\right) && \text{(by definition of lub on pairs)}\\
&= f\left(\bigsqcup_{i\in\mathbb{N}}d'_i,d''\right) && \text{(by definition of } d''\text{)}\\
&= f_{d''}\left(\bigsqcup_{i\in\mathbb{N}}d'_i\right) && \text{(by definition of } f_{d''}\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}f_{d''}(d'_i) && \text{(by continuity of } f_{d''}\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}f(d'_i,d'') && \text{(by definition of } f_{d''}\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}f_{d'_i}(d'') && \text{(by definition of } f_{d'_i}\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}f_{d'_i}\left(\bigsqcup_{j\in\mathbb{N}}d''_j\right) && \text{(by definition of } d''\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}\bigsqcup_{j\in\mathbb{N}}f_{d'_i}(d''_j) && \text{(by continuity of } f_{d'_i}\text{)}\\
&= \bigsqcup_{i\in\mathbb{N}}\bigsqcup_{j\in\mathbb{N}}f(d'_i,d''_j) && \text{(by definition of } f_{d'_i}\text{)}\\
&= \bigsqcup_{k\in\mathbb{N}}f(d'_k,d''_k) && \text{(by Lemma 8.1 (switch lemma))}
\end{aligned}$$

□

8.6 Apply, Curry and Fix

As done for IMP we will use the λ-notation as meta-language for the denotational semantics of HOFL. In Section 8.2 we have already defined two new continuous functions for our meta-language (π_1 and π_2). In this section we introduce some additional functions that will form the kernel of our meta-language.

Definition 8.7 (Apply). Let D and E be two CPOs. We define a function apply : $[D \to E] \times D \to E$ as follows:

$$\text{apply}(f,d) \stackrel{\text{def}}{=} f(d)$$

The function apply represents the application of a function in our meta-language: it takes a continuous function $f : D \to E$ and an element $d \in D$ and then returns $f(d)$ as a result. We leave it as an exercise to prove that apply is monotone (see Problem 8.5). We prove that it is also continuous.

Theorem 8.8 (Continuity of apply). *Let* apply : $[D \to E] \times D \to E$ *be the function defined above and let* $\{(f_n, d_n)\}_{n\in\mathbb{N}}$ *be a chain in the CPO*$_\perp$ $[\mathscr{D} \to \mathscr{E}] \times \mathscr{D}$ *then:*

$$\text{apply}\left(\bigsqcup_{n\in\mathbb{N}} (f_n, d_n)\right) = \bigsqcup_{n\in\mathbb{N}} \text{apply}(f_n, d_n)$$

Proof. By Theorem 8.7 we can prove the continuity on each parameter separately.

- Let us fix $d \in D$ and take a chain $\{f_n\}_{n\in\mathbb{N}}$ in $[D \to E]$. We have:

$$\begin{aligned}
\text{apply}\left(\left(\bigsqcup_{n\in\mathbb{N}} f_n\right), d\right) &= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)(d) && \text{(by definition)}\\
&= \bigsqcup_{n\in\mathbb{N}} (f_n(d)) && \text{(by definition of lub of functions)}\\
&= \bigsqcup_{n\in\mathbb{N}} \text{apply}(f_n, d) && \text{(by definition)}
\end{aligned}$$

- Now we fix $f \in [D \to E]$ and take a chain $\{d_n\}_{n\in\mathbb{N}}$ in D. We have:

$$\begin{aligned}
\text{apply}\left(f, \bigsqcup_{n\in\mathbb{N}} d_n\right) &= f\left(\bigsqcup_{n\in\mathbb{N}} d_n\right) && \text{(by definition)}\\
&= \bigsqcup_{n\in\mathbb{N}} f(d_n) && \text{(by continuity of } f\text{)}\\
&= \bigsqcup_{n\in\mathbb{N}} \text{apply}(f, d_n) && \text{(by definition)}
\end{aligned}$$

So apply is a continuous function. □

Currying is the name of a technique for transforming a function that takes a pair (or, more generally, a tuple) of arguments into a function that takes each argument separately but computes the same result.

Definition 8.8 (Curry and un-curry). We define the function

$$\text{curry} : (D \times E \to F) \to (D \to E \to F)$$

by letting, for any $g : D \times E \to F$, $d \in D$ and $e \in E$:

$$\text{curry } g\ d\ e \stackrel{\text{def}}{=} g(d, e)$$

And we define the function

$$\text{un-curry} : (D \to E \to F) \to (D \times E \to F)$$

by letting, for any $h : D \to E \to F$, $d \in D$ and $e \in E$:

$$\text{un-curry } h\ (d,e) \stackrel{\text{def}}{=} h\ d\ e$$

Theorem 8.9 (Continuity of curry). *Let D,E,F be CPOs and $g : D \times E \to F$ be a continuous function. Then $(\text{curry}\, g) : D \to (E \to F)$ is a continuous function, namely given any chain $\{d_i\}_{i\in\mathbb{N}}$ in D:*

$$(\text{curry } g)\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} (\text{curry } g)(d_i).$$

Proof. Let us note that since g is continuous, by Theorem 8.7, g is continuous separately on each argument. Then let us take $e \in E$ we have:

$$\begin{aligned}
(\text{curry } g)\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)(e) &= g\left(\left(\bigsqcup_{i\in\mathbb{N}} d_i\right), e\right) && \text{(by definition of curry } g) \\
&= \bigsqcup_{i\in\mathbb{N}} g(d_i, e) && \text{(by continuity of } g) \\
&= \bigsqcup_{i\in\mathbb{N}} ((\text{curry } g)(d_i)(e)) && \text{(by definition of curry } g)
\end{aligned}$$

□

To define the denotational semantics of recursive definitions we need to provide a fixpoint operator. So it seems useful to introduce fix in our meta-language.

Definition 8.9 (Fix). Let D be a $CPO_\perp$. We define $\text{fix} : [D \to D] \to D$ as:

$$\text{fix} \stackrel{\text{def}}{=} \bigsqcup_{i\in\mathbb{N}} \lambda f. f^i(\perp_D)$$

Note that, since $\{\lambda f.\ f^i(\perp_D)\}_{i\in\mathbb{N}}$ is a chain of functions and $[D \to D] \to D$ is complete, we are guaranteed that the lub $\bigsqcup_{i\in\mathbb{N}} \lambda f. f^i(\perp_D)$ exists.

Theorem 8.10 (Continuity of fix). *The function* $\text{fix} : [D \to D] \to D$ *is continuous, namely* $\text{fix} \in [[D \to D] \to D]$.

Proof. We know that $[[D \to D] \to D]$ is complete, thus if for all $i \in \mathbb{N}$ the function $\lambda f.\ f^i(\perp_D)$ is continuous, then $\text{fix} = \bigsqcup_{i\in\mathbb{N}} \lambda f.\ f^i(\perp_D)$ is also continuous. We prove that $\forall i \in \mathbb{N}.\ \lambda f.\ f^i(\perp_D)$ is continuous by mathematical induction on i.

Base case: $\lambda f.\ f^0(\perp_D) = \lambda f.\ \perp_D$ is a constant, and thus continuous, function.

Inductive case: Let us assume that $g \stackrel{\text{def}}{=} \lambda f.\ f^i(\perp_D)$ is continuous, i.e., that given a chain $\{f_n\}_{n\in\mathbb{N}}$ in $[D \to D]$ we have $g\left(\bigsqcup_{n\in\mathbb{N}} f_n\right) = \bigsqcup_{n\in\mathbb{N}} g(f_n)$, and let us prove that $h \stackrel{\text{def}}{=} \lambda f.\ f^{i+1}(\perp_D)$ is continuous, namely that $h\left(\bigsqcup_{n\in\mathbb{N}} f_n\right) = \bigsqcup_{n\in\mathbb{N}} h(f_n)$. In fact we have:

$$
\begin{aligned}
h\left(\bigsqcup_{n\in\mathbb{N}} f_n\right) &= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)^{i+1}(\bot_D) && \text{(by def. of } h\text{)}\\
&= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)\left(\left(\bigsqcup_{n\in\mathbb{N}} f_n\right)^{i}(\bot_D)\right) && \text{(by def. of } (\cdot)^{i+1}\text{)}\\
&= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)\left(g\left(\bigsqcup_{n\in\mathbb{N}} f_n\right)\right) && \text{(by def. of } g\text{)}\\
&= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)\left(\bigsqcup_{n\in\mathbb{N}} g(f_n)\right) && \text{(by ind. hyp.)}\\
&= \left(\bigsqcup_{n\in\mathbb{N}} f_n\right)\left(\bigsqcup_{n\in\mathbb{N}} f_n^i(\bot_D)\right) && \text{(by def of } g\text{)}\\
&= \bigsqcup_{n\in\mathbb{N}}\left(f_n\left(\bigsqcup_{m\in\mathbb{N}} f_m^i(\bot_D)\right)\right) && \text{(by def. of lub)}\\
&= \bigsqcup_{n\in\mathbb{N}}\bigsqcup_{m\in\mathbb{N}} f_n\left(f_m^i(\bot_D)\right) && \text{(by cont. of } f_n\text{)}\\
&= \bigsqcup_{k\in\mathbb{N}} f_k\left(f_k^i(\bot_D)\right) && \text{(by Lemma 8.1)}\\
&= \bigsqcup_{k\in\mathbb{N}} f_k^{i+1}(\bot_D) && \text{(by def. of } (\cdot)^{i+1}\text{)}\\
&= \bigsqcup_{n\in\mathbb{N}} h(f_n) && \text{(by def. of } h\text{)}
\end{aligned}
$$

□

Finally we introduce the **let** operator, whose role is that of binding a name x to a de-lifted expression. Note that the continuity of the **let** operator directly follows from the continuity of the lifting operator.

Definition 8.10 (Let operator). Let $\mathscr{E}$ be a $CPO_\bot$ and $\lambda x.\ e$ a function in $[D \to E]$. We define the *let* operator as follows, where $d' \in D_\bot$:

$$
\mathbf{let}\ x \Leftarrow d'.\ e \stackrel{\text{def}}{=} \underbrace{\underbrace{\underbrace{(\lambda x.\ e)}_{D\to E}{}^*}_{D_\bot\to E}\ (\underbrace{d'}_{D_\bot})}_{E} = \begin{cases} \bot_E & \text{if } d' = \bot_{D_\bot} \\ e\left[{}^{d}/_{x}\right] & \text{if } d' = \lfloor d \rfloor \text{ for some } d \in D \end{cases}
$$

Intuitively, taken $d' \in D_\bot$, if $d' = \bot$ then **let** $x \Leftarrow d'.\ e$ returns $\bot_E$, otherwise it means that $d' = \lfloor d \rfloor$ for some $d \in D$ and thus it returns $e\left[{}^{d}/_{x}\right]$, as if $\lambda x.\ e$ was applied to d, i.e., $d' = \lfloor d \rfloor$ is de-lifted so that $\lambda x.\ e$ can be used.

Problems

8.1. Prove that the projection functions in Definition 8.3 are monotone.

8.2. Prove that the domain $[\mathscr{D} \to \mathscr{E}]$ from Definition 8.4 is a $\text{CPO}_\perp$.

8.3. Prove that the lifted domain $\mathscr{D}_\perp$ from Definition 8.5 is a $\text{CPO}_\perp$.

8.4. Complete the proof of Theorem 8.4 for what is concerned with the monotonicity of the lifting function $(\cdot)^*$.

8.5. Prove that the function apply : $[D \to E] \times D \to E$ introduced in Definition 8.7 is monotone.

8.6. Let D be a CPO and $f : D \to D$ be a continuous function. Prove that the set of fixpoints of f is itself a CPO (under the order inherited from D).

8.7. Let D and E be two $\text{CPO}_\perp$s. A function $f : D \to E$ is called *strict* if $f(\perp_D) = \perp_E$. Prove that the set of strict functions from D to E is a $\text{CPO}_\perp$ under the usual order.

8.8. Let D and E be two CPOs. Prove that the following two definitions of the order between continuous functions $f, g : D \to E$ are equivalent.

1. $f \sqsubseteq g \ \Leftrightarrow\ \forall d \in D.\ f(d) \sqsubseteq_E g(d)$.
2. $f \preceq g \ \Leftrightarrow\ \forall d_1, d_2 \in D.\ (d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E g(d_2))$

8.9. Let $\mathscr{D} = (D, \sqsubseteq_D)$ and $\mathscr{E} = (E, \sqsubseteq_E)$ be two CPOs. Their sum $\mathscr{D} + \mathscr{E}$ has:

1. The set of elements

$$\{\perp_{D+E}\} \cup D \uplus E = \{\perp_{D+E}\} \cup (\{0\} \times D) \cup (\{1\} \times E)$$

2. The order relation $\sqsubseteq_{D+E}$ defined by letting:

- $(0, d_1) \sqsubseteq_{D+E} (0, d_2)$ if $d_1 \sqsubseteq_D d_2$;
- $(1, e_1) \sqsubseteq_{D+E} (1, e_2)$ if $e_1 \sqsubseteq_E e_2$;
- $\forall x \in \{\perp_{D+E}\} \cup D \uplus E.\ \perp_{D+E} \sqsubseteq_{D+E} x$.

Prove that $\mathscr{D} + \mathscr{E}$ is a $\text{CPO}_\perp$.

8.10. Prove that un-curry is continuous and inverse to curry (see Definition 8.8).