Models of computation (MOD) 2015/16
Exam – Feb. 10, 2017

[Ex. 1] Let the transition relation $\to^x$ with $x \in \mathbf{Loc}$ be defined by the following inference rules

$$\frac{\sigma(x) = 0}{\langle c, \sigma\rangle \to^x \sigma} \qquad \frac{\sigma(x) \neq 0 \quad \langle c, \sigma\rangle \to \sigma'' \quad \langle c, \sigma''\rangle \to^x \sigma'}{\langle c, \sigma\rangle \to^x \sigma'}$$

1. Prove the determinacy of $\to^x$ (for any $x \in \mathbf{Loc}$).
2. Prove that for all c, σ, σ', x:

$$\langle \mathbf{while}\ (x \neq 0)\ \mathbf{do}\ c, \sigma\rangle \to \sigma' \;\Rightarrow\; \langle c, \sigma\rangle \to^x \sigma'.$$

[Ex. 2] Let S be a non-empty, finite set. A *neighbourhood* over S is a function $\eta : S \to \wp(S)$ that assigns to each element in S the set of its neighbours, with the constraint that $\forall x, y \in S.\ x \in \eta(y) \Leftrightarrow y \in \eta(x)$. Let $\mathcal{N}(S)$ denote the set of all neighbourhoods over S, ordered by the relation

$$\eta \sqsubseteq \eta' \quad \stackrel{\text{def}}{=} \quad \forall s \in S.\ \eta(s) \subseteq \eta'(s)$$

1. Prove that $(\mathcal{N}(S), \sqsubseteq)$ is a complete partial order with bottom.
2. Let $\rho : (\mathcal{N}(S), \sqsubseteq) \to (\mathbb{N}, \leq)$ be defined by $\rho(\eta) \stackrel{\text{def}}{=} |\{s \in S \mid s \in \eta(s)\}|$. Prove that ρ is monotone.

[Ex. 3] Consider the HOFL term

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ f.\ \lambda x.\ \lambda y.\ \mathbf{if}\ (x + y)\ \mathbf{then}\ 0\ \mathbf{else}\ (\ (f\ (x+1))\ (y-1)\ )$$

1. Under which hypothesis is t typable?
2. Compute the (lazy) denotational semantics of t.

[Ex. 4] Let K be a positive natural number and consider the Markov chain with $K + 1$ states, numbered from 0 to K, and transition probabilities

$$p_{i,j} = \begin{cases} p & \text{if } i = j = 0 \\ q & \text{if } j = i + 1 \\ p & \text{if } j = i - 1 \\ q & \text{if } i = j = K \\ 0 & \text{otherwise} \end{cases}$$

for some fixed value $p \in (0, 1)$ and $q = 1 - p$.

1. Draw the transition system for $K = 2$.
2. Define the corresponding DTMC (for $K = 2$). Is it ergodic?
3. What is the probability to be in state 0 on the long run (for $K = 2$)? Does it depend on K?