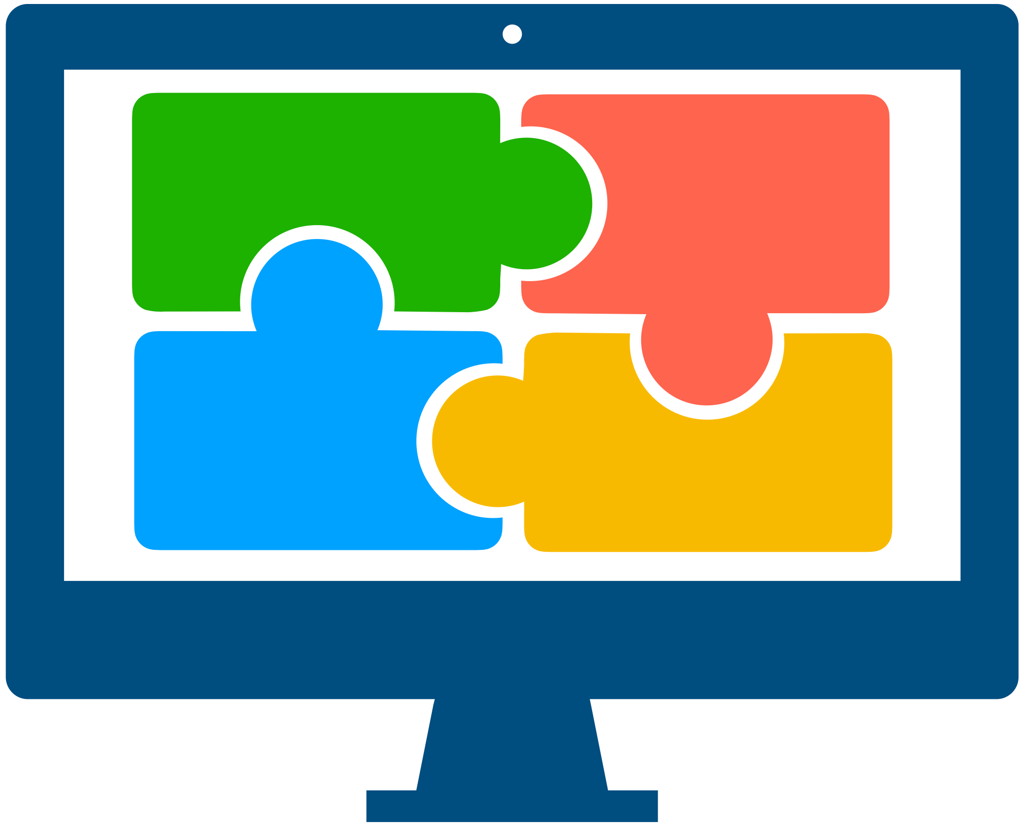

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**PSC 2020/21** (375AA, 9CFU)

Principles for Software Composition

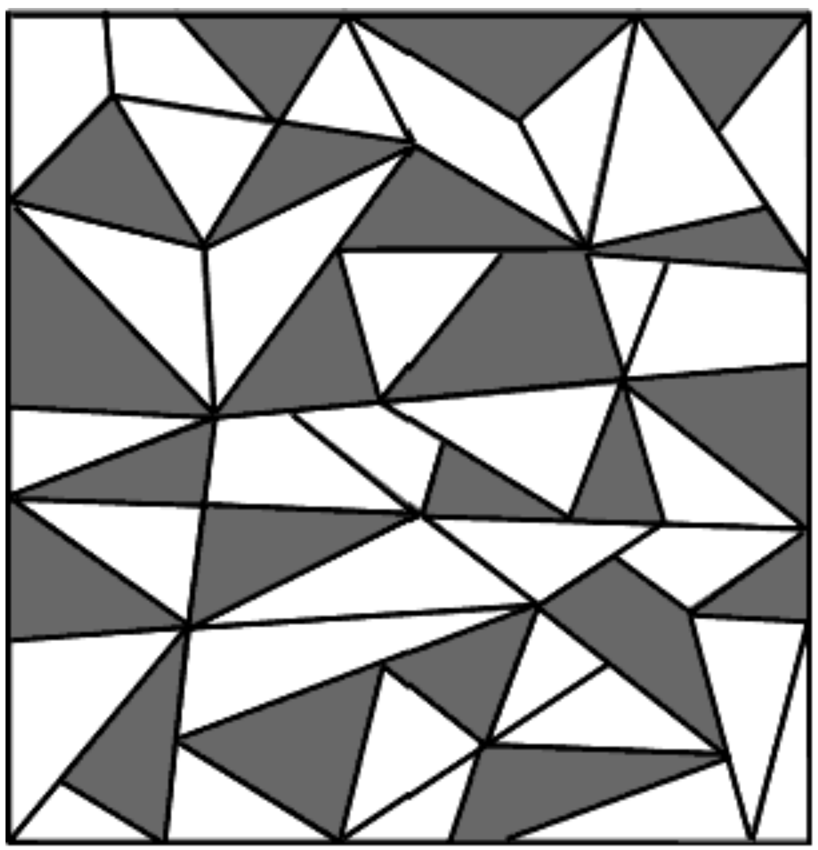
Roberto Bruni
http://www.di.unipi.it/~bruni/

# 13a - Cartesian domains

# A metaphor

# Hidden star

Can you spot a regular 5-point star shape inside the picture?



some find it immediately: for them it is impossible that others cannot see it

some spend many efforts in finding it: for them the success is rewarding

some just wait for someone to help

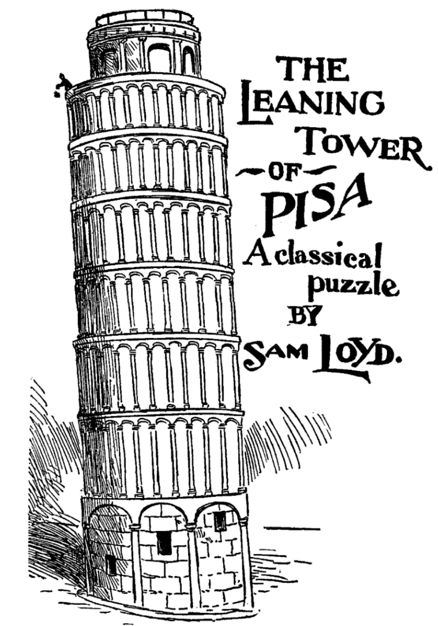
in all cases: once shown, the star will never be hidden again

the metaphor: star = mathematical way to problem solving

# The creator

Samuel Loyd (1841-1911)

chess player, puzzle maker, and recreational mathematician


***How far does the ball travel?***

IF AN elastic ball is dropped from the Leaning Tower of Pisa at a height of 179 feet from the ground, and on each rebound the ball rises exactly one tenth of its previous height, what distance will it travel before it comes to rest?

maybe (?) the inventor of the famous 15 puzzle

inventor of "*the leaning tower of Pisa*" puzzle

# HOFL
# Towards a denotational semantics

# Imp

three syntactic categories (types) $Aexp, Bexp, Com$

one interpretation function each

$$\mathcal{A}[\![\cdot]\!] : Aexp \to \Sigma \to \mathbb{Z}$$

$$\mathcal{B}[\![\cdot]\!] : Bexp \to \Sigma \to \mathbb{B}$$

$$\mathcal{C}[\![\cdot]\!] : Com \to \Sigma \to \Sigma_\perp$$

$$\Sigma \triangleq Ide \to \mathbb{Z}$$

environment

semantic values

meaning of identifiers

syntax

one semantic domain each

# HOFL

one syntactic category for pre-terms $T$

infinitely many types $\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$

infinitely many categories for typeable terms $T_\tau$

one semantic domain each $D_\tau$

one parametric interpretation function $[\![\cdot]\!]$

variables also have different types $x : \tau$

the environment must be type-sensitive $\rho$

# Requirements

a domain for each type!

$t : \tau$ $\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$t$ may diverge (e.g. $\mathbf{rec}\ x.\ x$ ) $\Rightarrow$ $D_\tau$ must include a bottom element $\bot_{D_\tau}$

# Requirements

a domain for each type!

$t : \tau$ $\qquad [\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = [\![t]\!]\rho[^{[\![\mathbf{rec}\ x.\ t]\!]\rho}/_x]$$

$$\Gamma_{x,t} \triangleq \lambda d.\ [\![t]\!]\rho[^d/_x]$$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = \Gamma_{x,t}\ ([\![\mathbf{rec}\ x.\ t]\!]\rho)$$

$$[\![\mathbf{rec}\ x.\ t]\!] = \mathit{fix}\ \Gamma_{x,t}$$

to solve recursive equations:

$D_\tau$ must be a $\mathrm{CPO}_\perp$

$\Gamma_{x,t}$ must be continuous

# Requirements

$t : \tau$ $\qquad$ $[\![t]\!]\rho \in D_\tau$ — a domain for each type!

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$$

we must be able to combine $\mathrm{CPO}_\perp$

using cartesian product and function spaces

# Requirements

a domain for each type!

$t : \tau$ $\quad\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \; ::= \; int \;\mid\; \tau_0 * \tau_1 \;\mid\; \tau_0 \to \tau_1$$

choose $D_{int}$

given $D_{\tau_0}, D_{\tau_1}$ build $D_{\tau_0 * \tau_1}$ $\quad$ $D_{\tau_0 \to \tau_1}$

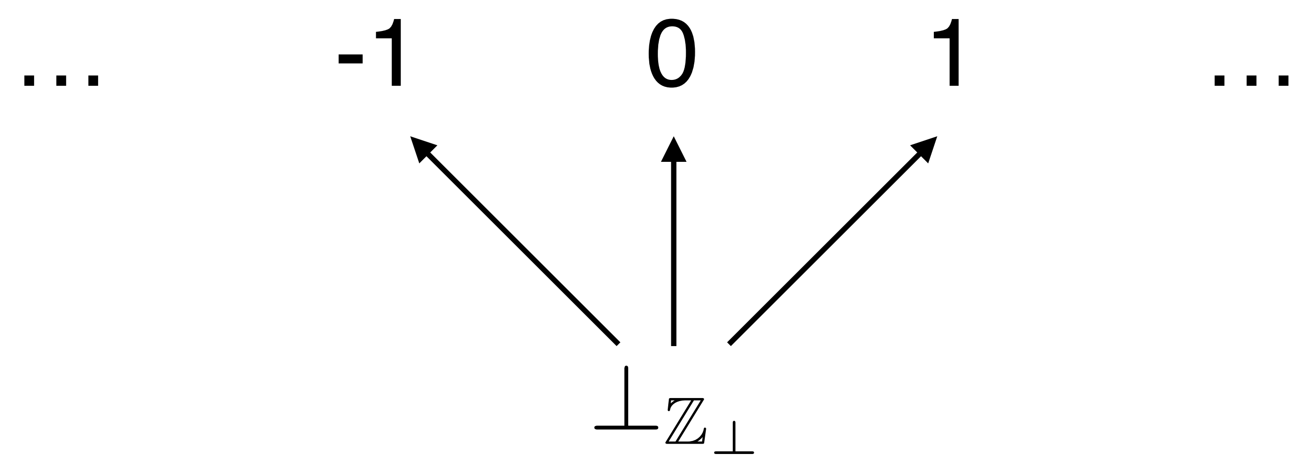
# Flat domain of Integers

# Flat domain of Integers

$\mathbb{Z}$ … -1 0 1 …

# Flat domain of Integers

$\mathbb{Z}_\perp$




PO: flat order

bottom: any flat order has bottom

completeness: any flat order is complete
(only finite chains are possible)

# Strict extensions

$\mathsf{op} \in \{+, - \times\}$

$\mathsf{op} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$\underline{\mathsf{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$

$1 \underline{+}_\perp \perp = \perp$

$\perp \underline{\times}_\perp 5 = \perp$

$\perp \underline{-}_\perp \perp = \perp$

$$v_1 \ \underline{\mathsf{op}}_\perp \ v_2 \triangleq \begin{cases} v_1 \ \underline{\mathsf{op}} \ v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

called *strict* extension

to prove: $\underline{\mathsf{op}}_\perp$ is monotone and continuous

is $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$ a $\mathrm{CPO}_\perp$ ?

# Cartesian product of domains

# Cartesian product

$\mathcal{D} = (D, \sqsubseteq_D)$

$\mathcal{E} = (E, \sqsubseteq_E)$

$\text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$

how to order pairs?

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

example $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$

$$(0, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2)$$

$$(\perp_{\mathbb{Z}_\perp}, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1)$$

$$(2, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0)$$

$$(0, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\perp_{\mathbb{Z}_\perp}, 0)$$

# Cartesian CPO

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{D \times E}$
follow immediately from those of $\sqsubseteq_D$ $\sqsubseteq_E$

is there a bottom element?

let $\bot_{D \times E} = (\bot_D, \bot_E)$

take any pair $(d, e) \in D \times E$

since $\bot_D \sqsubseteq_D d$ and $\bot_E \sqsubseteq_E e$ then $\bot_{D \times E} = (\bot_D, \bot_E) \sqsubseteq_{D \times E} (d, e)$

# Cartesian CPO (ctd)

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it complete?

take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$ we need to find its lub

we prove its lub is $\left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

# Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

1. $\left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is an upper bound of the chain

take a generic element of the chain $(d_j, e_j)$

we have $d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i$ and $e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i$ thus $(d_j, e_j) \sqsubseteq_{D \times E} \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

# Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

2. $\left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is the least among upper bounds

take a generic upper bound $(d, e)$: $\forall i \in \mathbb{N}.\ (d_i, e_i) \sqsubseteq_{D \times E} (d, e)$

by def $\forall i \in \mathbb{N}.\ d_i \sqsubseteq_D d\ \wedge\ \forall i \in \mathbb{N}.\ e_i \sqsubseteq_E e$

i.e., $d$ is an upper bound of $\{d_i\}_{i \in \mathbb{N}}$

$e$ is an upper bound of $\{e_i\}_{i \in \mathbb{N}}$

$\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i \sqsubseteq_D d$ and $\bigsqcup_{i \in \mathbb{N}} e_i \sqsubseteq_E e$

hence $\left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \sqsubseteq_{D \times E} (d, e)$

# Cartesian CPO: recap

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

$$\bot_{D \times E} \triangleq (\bot_D, \bot_E)$$

$$\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$$

is $\mathbb{Z}_\bot \times \mathbb{Z}_\bot$ a $\mathrm{CPO}_\bot$ ? ✓

# Projections

$\pi_1 : D \times E \to D$ $\pi_2 : D \times E \to E$

$\pi_1(d, e) = d$ $\pi_2(d, e) = e$

**TH.** projections are monotone

*proof.* take $(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$

we want to prove $\pi_1(d_0, e_0) \sqsubseteq_D \pi_1(d_1, e_1)$

$\pi_2(d_0, e_0) \sqsubseteq_E \pi_2(d_1, e_1)$

$$\pi_1(d_0, e_0) = d_0 \sqsubseteq_D d_1 = \pi_1(d_1, e_1)$$

$$\Uparrow$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$$

the case of $\pi_2$ is analogous

# Projections (ctd)

$$\pi_1 : D \times E \to D \qquad \pi_2 : D \times E \to E$$

$$\pi_1(d,e) = d \qquad \pi_2(d,e) = e$$

**TH.** projections are continuous

*proof.* take $\{(d_i, e_i)\}_{i\in\mathbb{N}}$

we want to prove $\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i,e_i)\right) = \bigsqcup_{i\in\mathbb{N}}\pi_1(d_i,e_i)$

$$\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i,e_i)\right) = \pi_1\left(\bigsqcup_{i\in\mathbb{N}}d_i, \bigsqcup_{i\in\mathbb{N}}e_i\right) = \bigsqcup_{i\in\mathbb{N}}d_i = \bigsqcup_{i\in\mathbb{N}}\pi_1(d_i,e_i)$$

by def of lub; by def of $\pi_1$; by def of $\pi_1$

the case of $\pi_2$ is analogous

# Exercise: smashed prod

$$\mathcal{D} = (D, \sqsubseteq_D) \qquad \mathcal{E} = (E, \sqsubseteq_E) \qquad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \otimes \mathcal{E} = (\ D \otimes E\ ,\ \sqsubseteq_{D \otimes E}\ )$$

$$D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \Leftrightarrow e = \perp_E\}$$

how to order pairs?

bottom element?

complete order?