



PSC 2021/22 (375AA, 9CFU)

Principles for Software Composition

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08a - Complete Partial Orders

Partial orders

Partially ordered set

(Poset or just PO)

a set

(P, \sqsubseteq) a binary relation $\sqsubseteq \subseteq P \times P$

reflexive

$$\forall p \in P. \quad p \sqsubseteq p$$

antisymmetric

$$\forall p, q \in P. \quad p \sqsubseteq q \wedge q \sqsubseteq p \Rightarrow p = q$$

transitive

$$\forall p, q, r \in P. \quad p \sqsubseteq q \wedge q \sqsubseteq r \Rightarrow p \sqsubseteq r$$

q



p

$$p \sqsubseteq q$$

means that p and q are **comparable** and that p is less than (or equal to) q

$$p \sqsubset q$$

means $p \sqsubseteq q \wedge p \neq q$

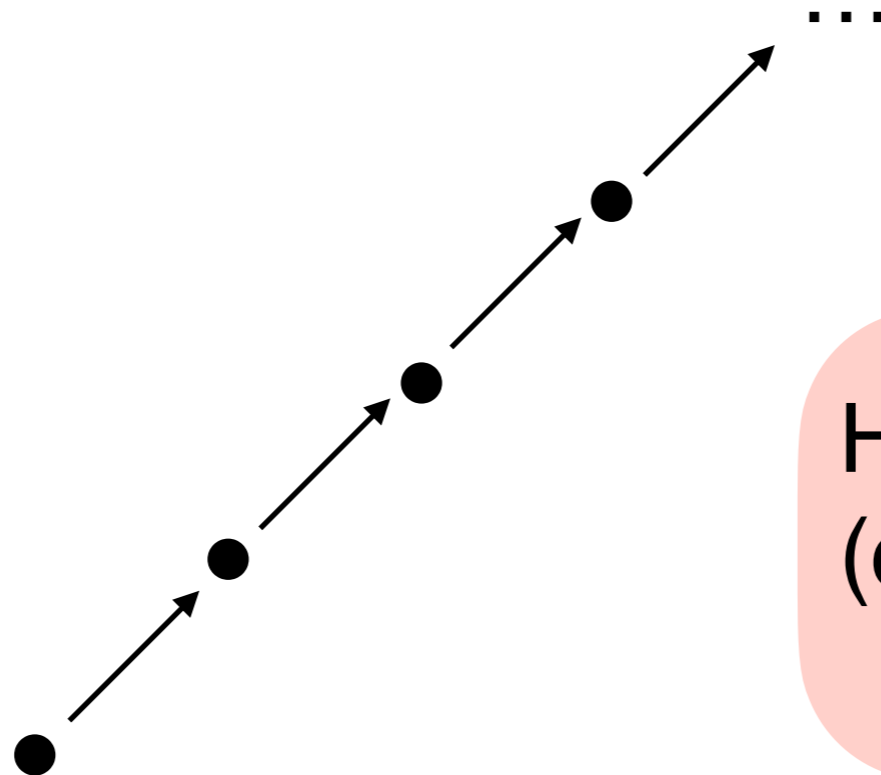
Total orders

(P, \sqsubseteq) PO

total

$$\forall p, q \in P. \quad p \sqsubseteq q \vee q \sqsubseteq p$$

a PO where every two elements are **comparable**



Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Discrete orders

(P, \sqsubseteq) PO

discrete

$$\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$$

any element is **comparable** only to itself

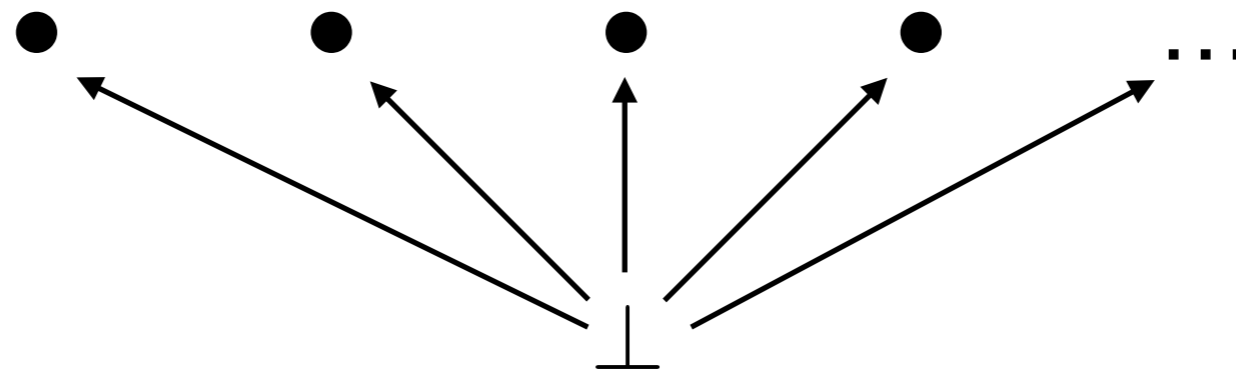


Flat orders

(P, \sqsubseteq) PO

flat $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q \vee p = \perp$

any element is **comparable** only to itself
and with a distinguished (smaller) element \perp





Exercise

(\mathbb{N}, \leq)

PO?

Total?

Discrete?

Flat?



...



3



2



1



0



Exercise

$(\wp(S), \subseteq)$

PO?



Total?

$$|S| < 2$$

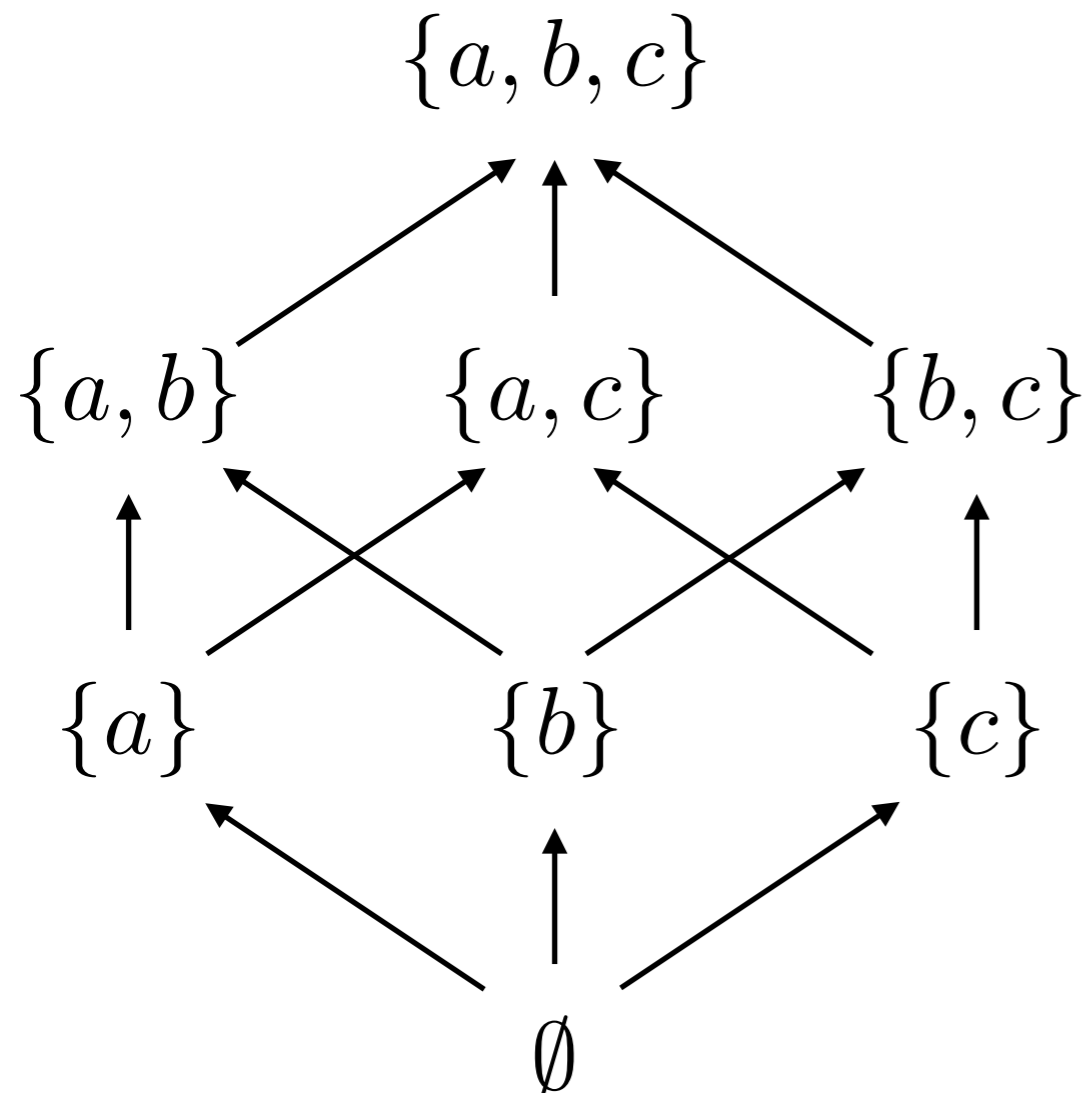
Discrete?

$$S = \emptyset$$

Flat?

$$|S| < 2$$

example: $S = \{a, b, c\}$



$$\{a, b\} \not\subseteq \{b, c\}$$

$$\{a\} \not\subseteq \{b\}$$



Exercise

$(\mathbb{N}, =)$

PO?

Total?

Discrete?

Flat?



0

1

2

3

...



Exercise

$(\mathbb{N} \cup \{\perp\}, \{(\perp, n) \mid n \in \mathbb{N}\}^*)$

PO?



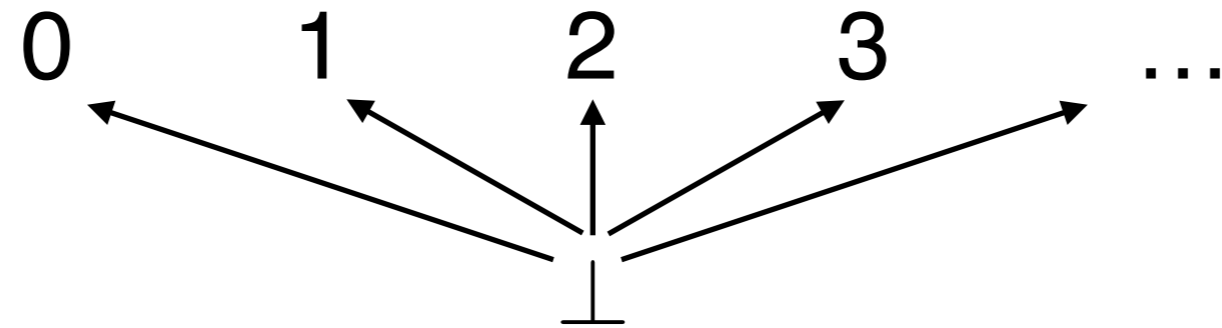
Total?



Discrete?



Flat?





Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO?



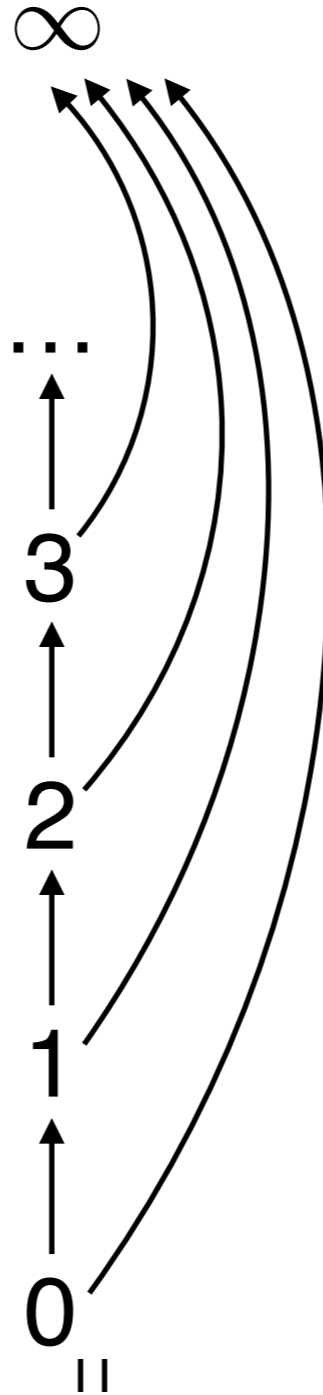
Total?



Discrete?



Flat?





Exercise

| | PO? | Total? | Discrete? | Flat? |
|---|-----|--------|-----------|-------|
| $(\mathbb{N}, <)$ | ✗ | ✗ | ✗ | ✗ |
| (\mathbb{Z}, \leq) | ✓ | ✓ | ✗ | ✗ |
| $(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$ | ✓ | ✓ | ✗ | ✗ |
| (T_Σ, \prec) | ✗ | ✗ | ✗ | ✗ |
| (\mathbb{N}, \neq) | ✗ | ✗ | ✗ | ✗ |

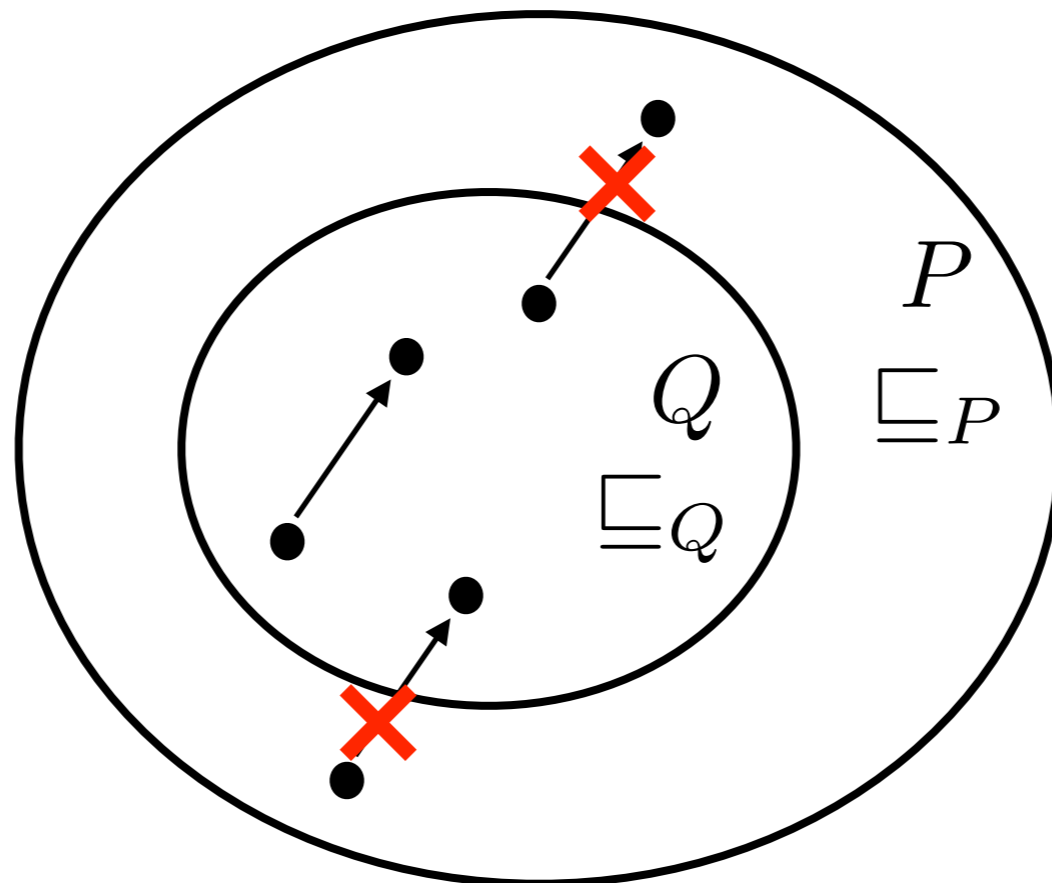
Subset of a PO

(P, \sqsubseteq_P) PO $Q \subseteq P$

let $\sqsubseteq_Q \triangleq \sqsubseteq_P \cap (Q \times Q)$

TH. (Q, \sqsubseteq_Q) is a PO

TH. if (P, \sqsubseteq_P) is total, then (Q, \sqsubseteq_Q) is total



PO \sqsubseteq

w.f. \prec

reflexive

not reflexive (otherwise cycle!)

antisymmetric

antisymmetric (otherwise cycle!)
 $p \prec q \wedge q \prec p$ is always false

transitive

can be transitive (\prec^+ w.f.)

has infinite descending chains
(if nonempty)

no infinite descending chain

\prec^* is always a PO

\sqsubseteq can be w.f.

Element properties (least, minimal, ...)

Least element

(P, \sqsubseteq) PO $Q \subseteq P$ $l \in Q$

l is a **least** element of Q if $\forall q \in Q. l \sqsubseteq q$

TH. (uniqueness of least element)

(P, \sqsubseteq) PO $Q \subseteq P$ l_1, l_2 least elements of Q implies $l_1 = l_2$

$$\left. \begin{array}{l} l_1 \text{ least element of } Q \Rightarrow l_1 \sqsubseteq l_2 \\ l_2 \text{ least element of } Q \Rightarrow l_2 \sqsubseteq l_1 \end{array} \right\} \Rightarrow l_1 = l_2$$


by antisymmetry

Bottom

(P, \sqsubseteq) PO the least element of P
(if it exists) is called **bottom** and denoted \perp

sometimes written \perp_P

Examples

| PO | bottom? |
|--------------------------------------|---|
| $(\mathbb{N} \cup \{\infty\}, \leq)$ | 0 |
| $(\wp(S), \subseteq)$ | \emptyset |
| (\mathbb{Z}, \leq) |  |

Minimal element

(P, \sqsubseteq) PO $Q \subseteq P$ $m \in Q$

m is a **minimal** element of Q if $\forall q \in Q. q \sqsubseteq m \Rightarrow q = m$

(no smaller element can be found in Q)

least $\forall q \in Q. l \sqsubseteq q$

minimal $\forall q \in Q. q \sqsubseteq m \Rightarrow q = m$

unique

not necessarily unique

minimal

not necessarily least
can be least

Reverse order

TH. (P, \sqsubseteq) PO implies (P, \sqsupseteq) PO

note:
 $\sqsupseteq = \sqsubseteq^{-1}$

proof. it is immediate to check that \sqsupseteq

- is reflexive
- is antisymmetric
- is transitive

(P, \sqsubseteq) PO $Q \subseteq P$

greatest element: least element of Q w.r.t. (P, \sqsupseteq)

top element: \top greatest element of P (if it exists)

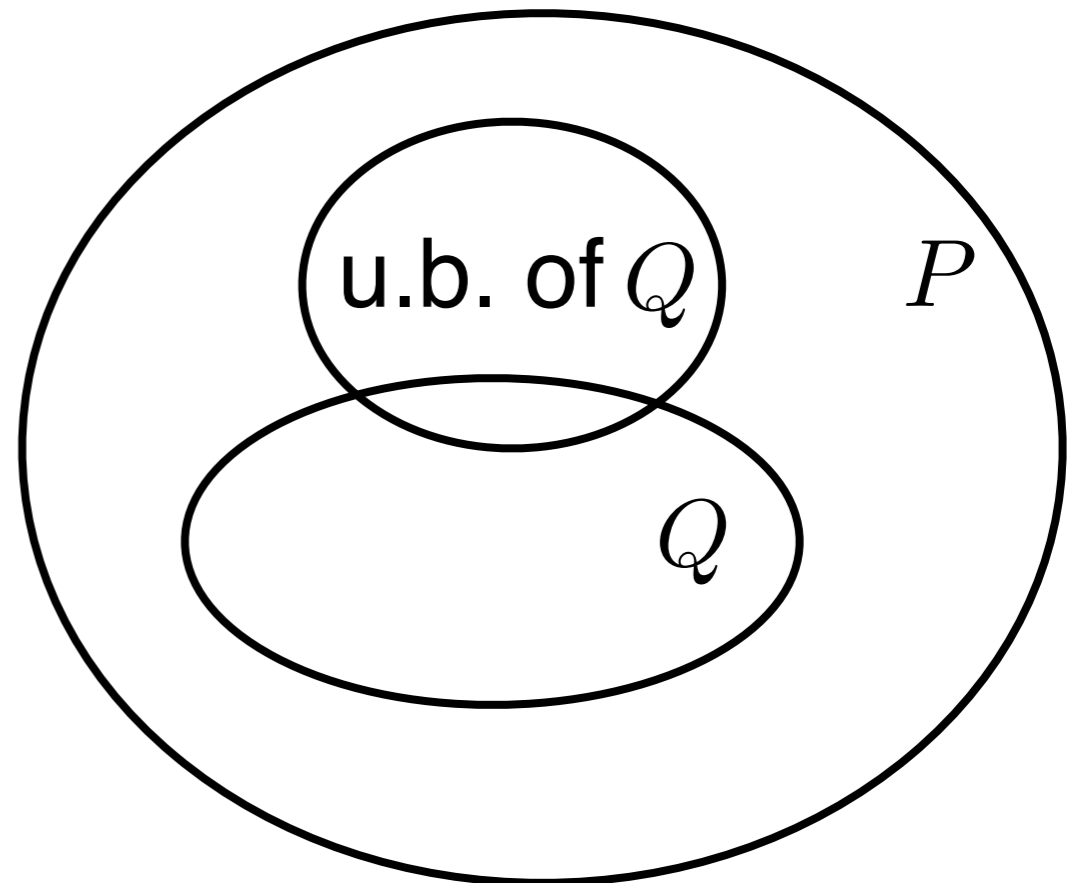
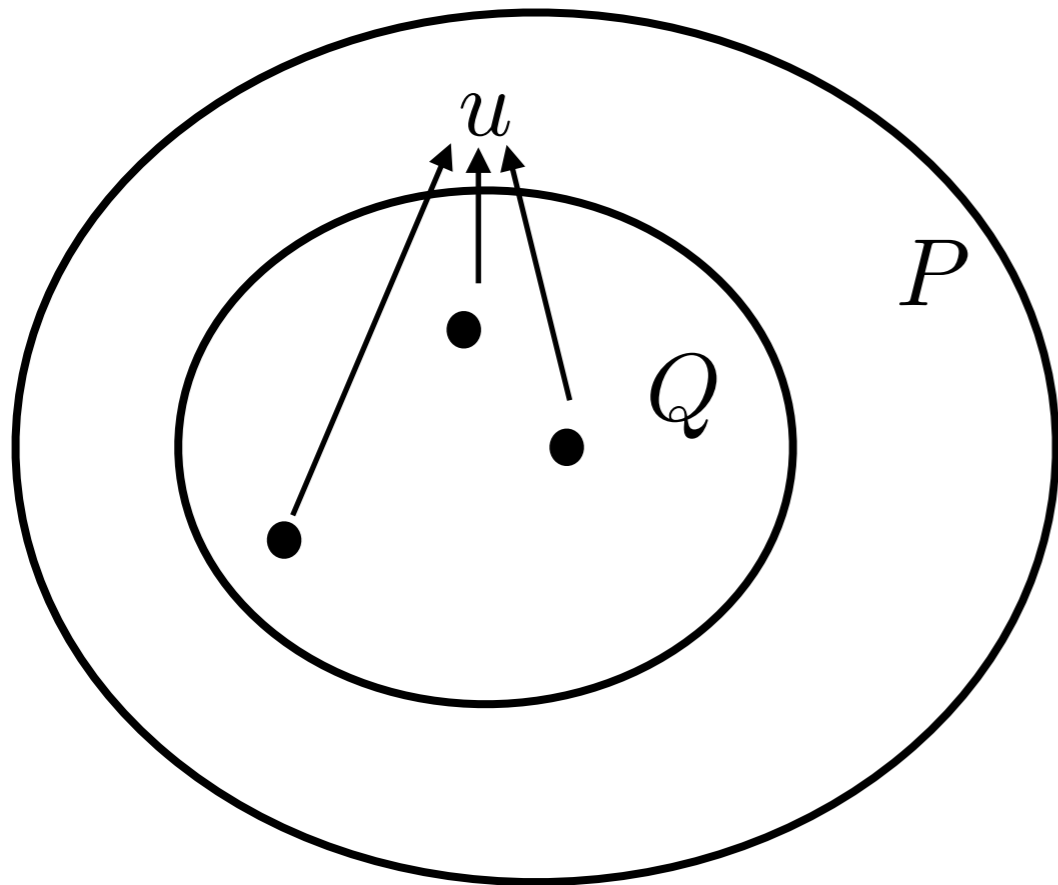
maximal element: minimal element of Q w.r.t. (P, \sqsupseteq)

Upper bound

(P, \sqsubseteq) PO $Q \subseteq P$ $u \in P$

u is an **upper bound** of Q if $\forall q \in Q. q \sqsubseteq u$
(all the elements of Q are smaller than u)

Q may have many upper bounds



Least upper bound

(P, \sqsubseteq) PO $Q \subseteq P$ $p \in P$

p is the **least upper bound (lub)** of Q if

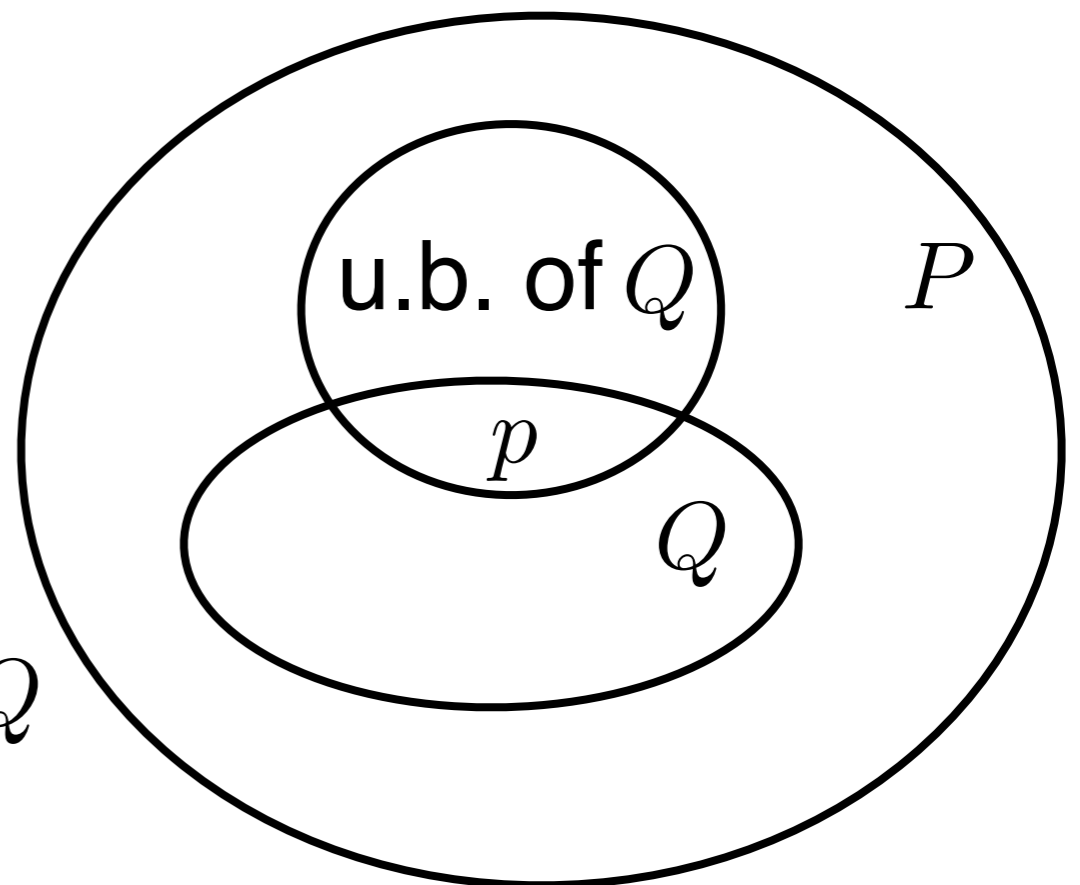
1. it is an upper bound of Q $\forall q \in Q. q \sqsubseteq p$
2. it is smaller than any other upper bound of Q

$$\forall u \in P. (\forall q \in Q. q \sqsubseteq u) \Rightarrow p \sqsubseteq u$$

we write $p = \text{lub } Q$

intuitively, it is the least element that represents all of Q

p not necessarily an element of Q

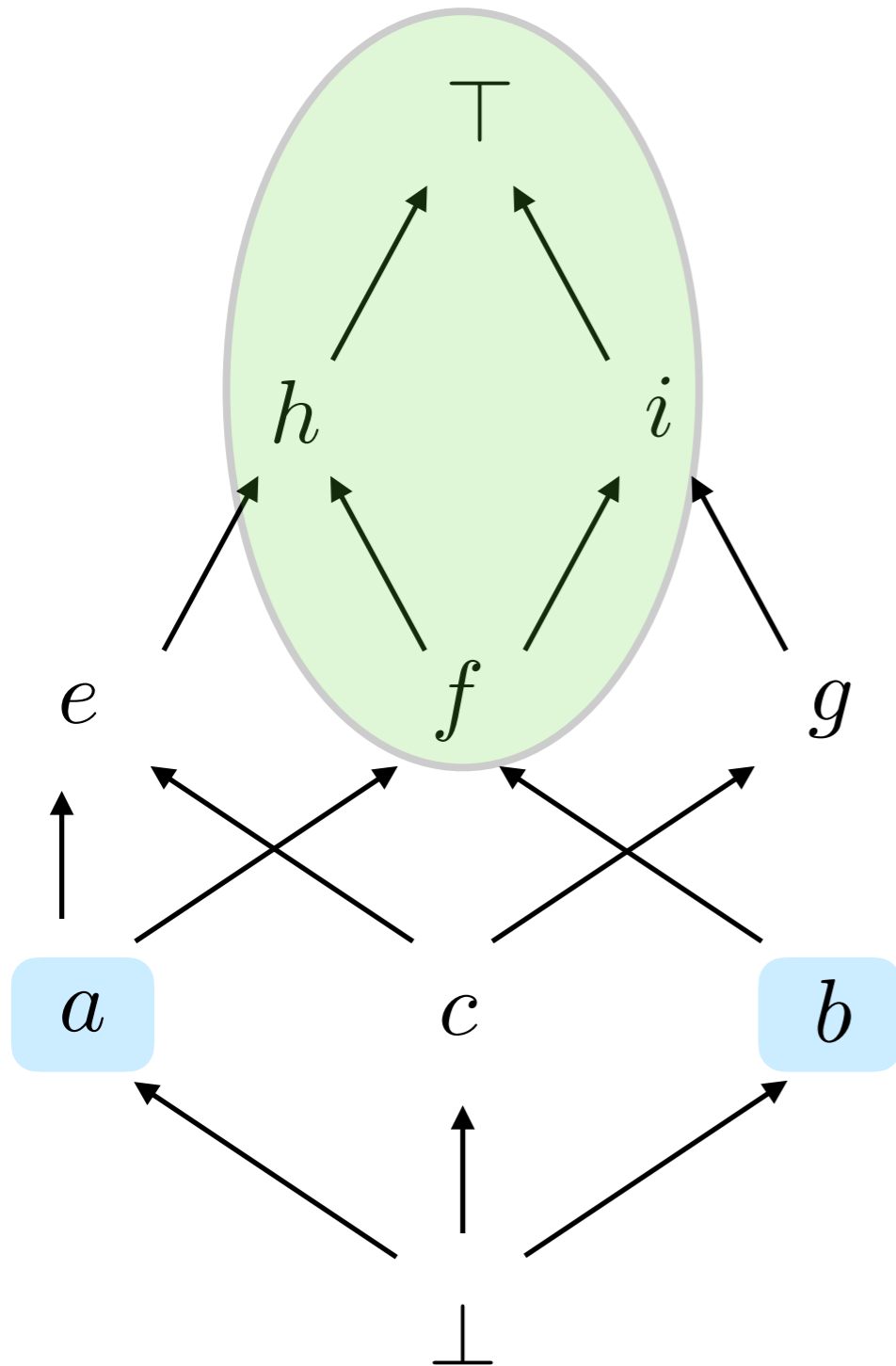




Exercise

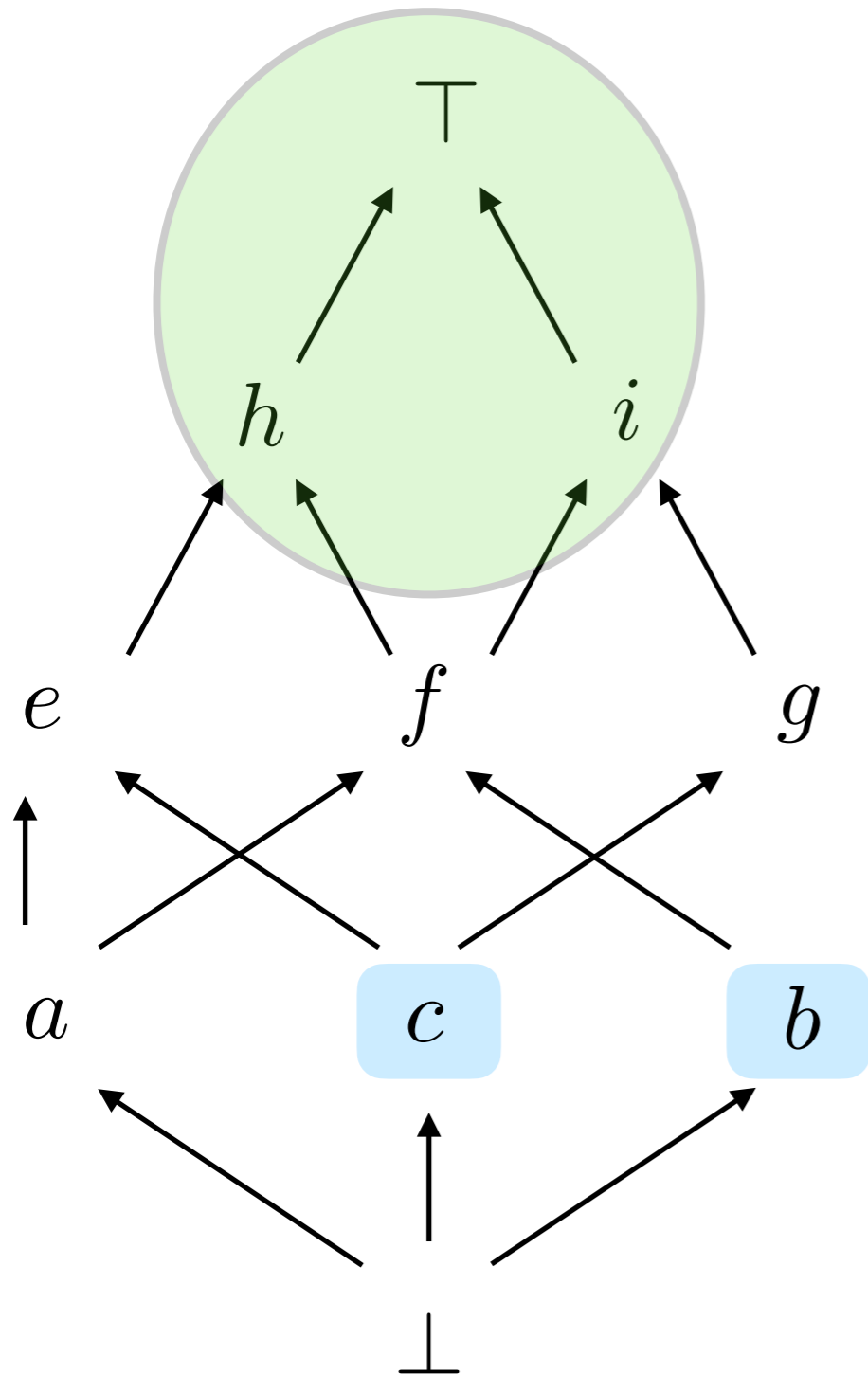
Upper bounds of $\{a, b\}$? $\{f, h, i, \top\}$

lub? f





Exercise



Upper bounds of $\{b, c\}$? $\{h, i, \top\}$

lub? no lub!



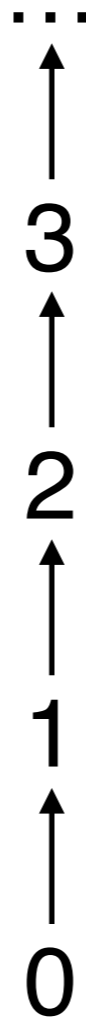
Exercise

(\mathbb{N}, \leq)

$Q \subseteq \mathbb{N}$

lub?

if Q finite $\text{lub } Q = \max Q$
otherwise no lub

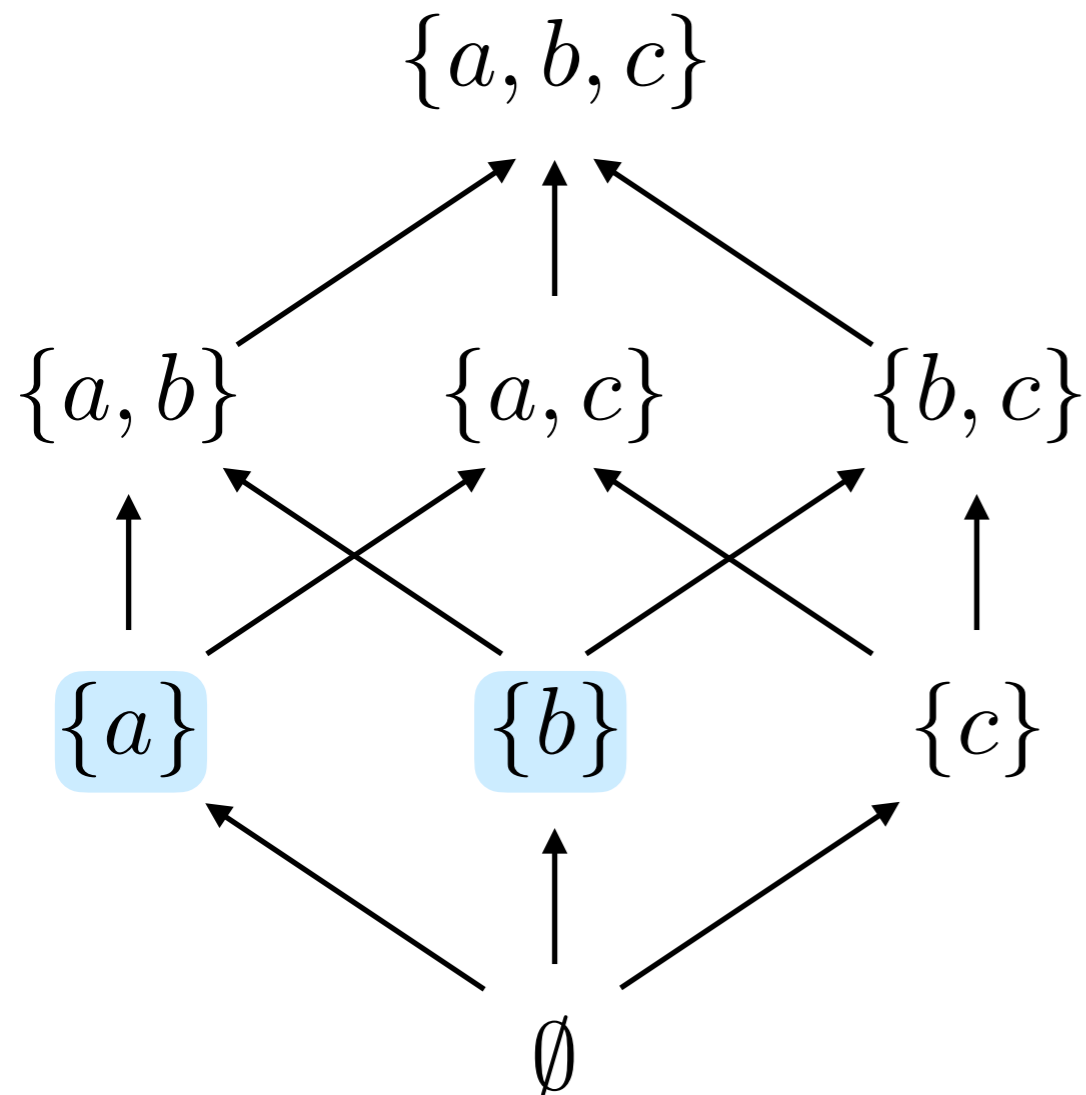




Exercise

$(\wp(S), \subseteq)$ $Q \subseteq \wp(S)$ lub?

$$\text{lub } Q = \bigcup_{T \in Q} T$$



$$\text{lub } \{\{a\}, \{b\}\} = \{a, b\}$$

Complete partial orders (CPO)

Completeness: the idea

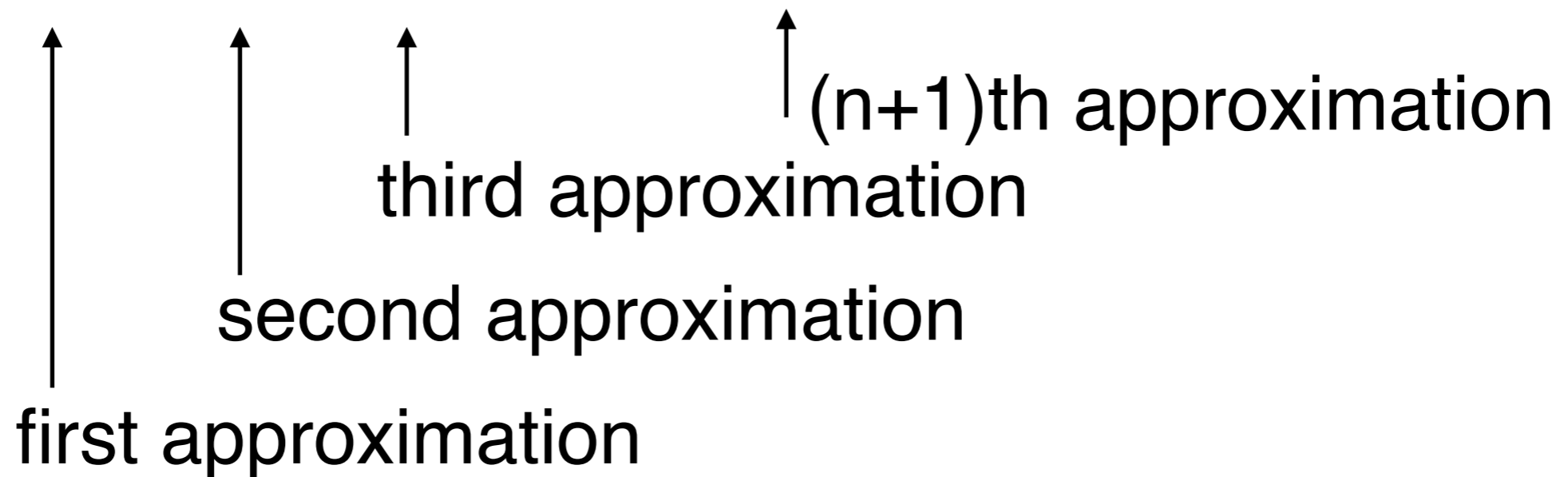
D a domain

\sqsubseteq a way to compare element

$x \sqsubseteq y$ x is a (less precise) approximation of y
 x and y are consistent,
but y is more accurate than x

} PO

$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \dots \sqsubseteq x_n \sqsubseteq \dots$



does any sequence of approximations tend to some limit?

Chain

(P, \sqsubseteq) PO $\{d_i\}_{i \in \mathbb{N}}$ is a **chain** if $\forall i \in \mathbb{N}. d_i \sqsubseteq d_{i+1}$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

any chain is an infinite list

finite chain: there are only finitely many distinct elements

$$\exists k \in \mathbb{N}. \forall i \geq k. d_i = d_{i+1}$$

or equivalently

$$\exists k \in \mathbb{N}. \forall i \geq k. d_i = d_k$$

Example

(\mathbb{N}, \leq)

$0 \leq 2 \leq 4 \leq \dots \leq 2n \leq \dots$ is an infinite chain

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \dots \leq 5 \leq \dots$ is a finite chain

any chain has infinite length

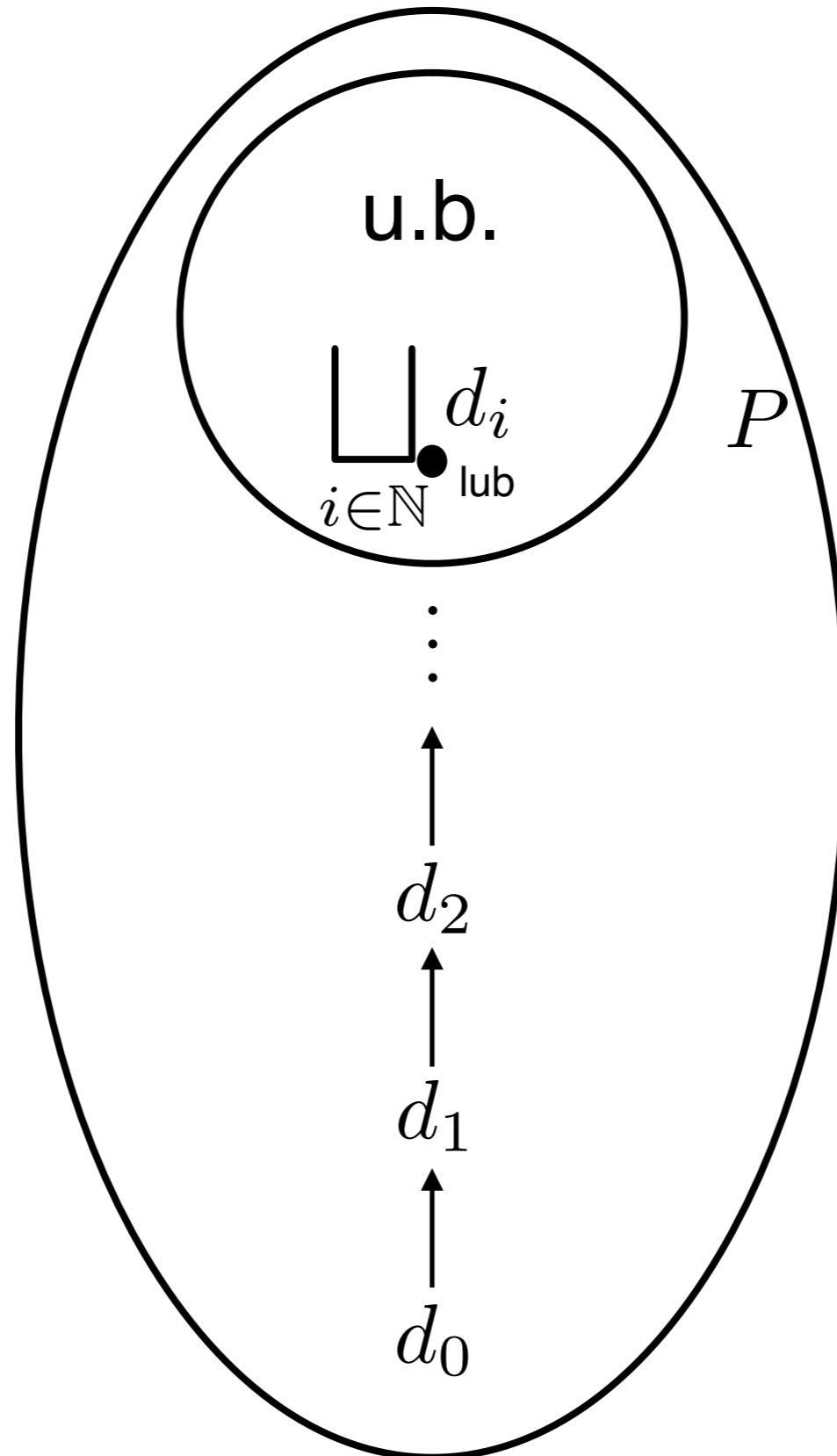
Limit of a chain

(P, \sqsubseteq) PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

we denote by $\bigsqcup_{i \in \mathbb{N}} d_i$ the lub of $\{d_i\}_{i \in \mathbb{N}}$ if it exists

and call it the **limit** of the chain

Limit illustrated



Example

(\mathbb{N}, \leq)

$0 \leq 2 \leq 4 \leq \dots \leq 2n \leq \dots$ has no lub
(empty set of upper bounds)

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \dots \leq 5 \leq \dots$ has lub 5
(which upper bounds?)

Lemma on finite chains

Lemma (any finite chain has a limit)

(P, \sqsubseteq) PO $\{d_i\}_{i \in \mathbb{N}}$ a finite chain $\Rightarrow \bigsqcup_{i \in \mathbb{N}} d_i$ exists

proof.

$\{d_i\}_{i \in \mathbb{N}}$ finite $\Rightarrow \exists k. \forall i. d_{i+k} = d_k$

the elements of the chain are totally ordered

d_k is the greatest element of the chain

d_k is an upper bound $\forall i. d_i \sqsubseteq d_k$

d_k is the least upper bound

take u such that $\forall i. d_i \sqsubseteq u$ then $d_k \sqsubseteq u$

Prefix independence

Lemma (prefix independence) (P, \sqsubseteq) PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

$$\text{if } \bigsqcup_{i \in \mathbb{N}} d_i \text{ exists } \Rightarrow \forall k. \bigsqcup_{i \in \mathbb{N}} d_{i+k} = \bigsqcup_{i \in \mathbb{N}} d_i$$

$$\begin{array}{ccc} d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots & & \bigsqcup_{i \in \mathbb{N}} d_i \\ & & = \\ & & \bigsqcup_{i \in \mathbb{N}} d_{i+k} \end{array}$$

Prefix independence

Lemma (prefix independence) (P, \sqsubseteq) PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

$$\text{if } \bigsqcup_{i \in \mathbb{N}} d_i \text{ exists } \Rightarrow \forall k. \bigsqcup_{i \in \mathbb{N}} d_{i+k} = \bigsqcup_{i \in \mathbb{N}} d_i$$

proof.

take a generic k

we prove that $\{d_i\}_{i \in \mathbb{N}}$ and $\{d_{i+k}\}_{i \in \mathbb{N}}$ have the same u.b.
(and thus the same lub)

1. if u is an u.b. of $\{d_i\}_{i \in \mathbb{N}}$ then is an u.b. of $\{d_{i+k}\}_{i \in \mathbb{N}}$

because $\{d_{i+k}\}_{i \in \mathbb{N}} \subseteq \{d_i\}_{i \in \mathbb{N}}$

2. if u is an u.b. of $\{d_{i+k}\}_{i \in \mathbb{N}}$ we need to show $\forall j. d_j \sqsubseteq u$

for $j \geq k$ it is obvious

if $j < k$ then $d_j \sqsubseteq d_k \sqsubseteq u$ because $d_k \in \{d_{i+k}\}_{i \in \mathbb{N}}$

Complete partial order

(P, \sqsubseteq) PO P is **complete** if each chain has a limit (lub)

TH. Any finite chain has a limit
(the last element in the sequence)

If P has only finite chains it is complete

If P is finite it is complete

Any discrete order is complete

Any flat order is complete

Example

(\mathbb{N}, \leq) is not complete
(it is enough to exhibit a chain with no limit)

$0 \leq 2 \leq 4 \leq \dots \leq 2n \leq \dots$ has no lub
(empty set of u.b.)

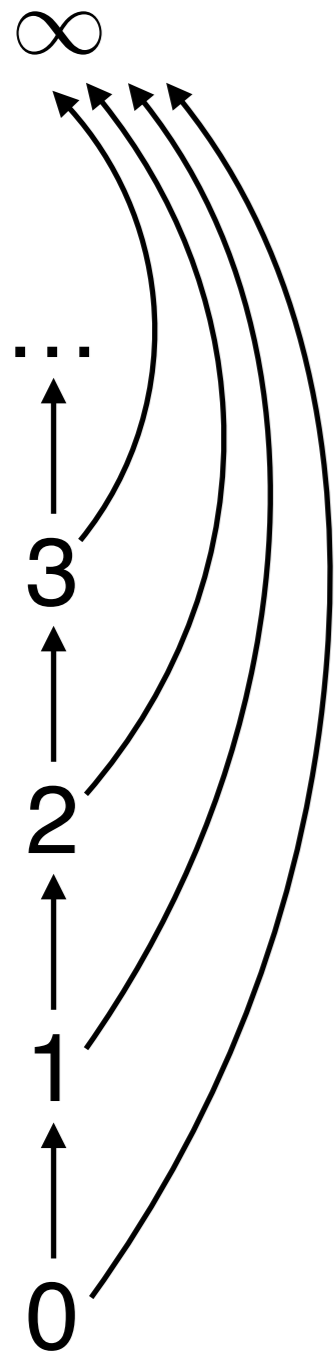


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

complete? 

any infinite chain has limit ∞
(set of u.b. $\{\infty\}$)





Exercise

$(\wp(S), \subseteq)$

complete? 

$\{S_i\}_{i \in \mathbb{N}}$

$$\bigsqcup_{i \in \mathbb{N}} S_i = \bigcup_{i \in \mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$$



Exercise

$(\mathbb{N} \cup \{\infty_1, \infty_2\}, \leq)$ complete? ✘

any infinite chain has no limit
(set of u.b. $\{\infty_1, \infty_2\}$)

