



PSC 2021/22 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni

<http://www.di.unipi.it/~bruni/>

<http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/start>

22a - Temporal logic

Testing

how do you guarantee that your code is correct?

testing can show the presence of bugs

not their absence

coverage of all cases: difficult to achieve

especially in concurrent systems!
(because of nondeterminism)

Formal logics

what does it mean to be correct? to satisfy some properties

how are these properties expressed? in some syntax

formal logics serve to express properties about programs

safety: something bad will not happen

liveness: something good will happen

model checking are certain properties satisfied
(by a model of the program)?

Temporal logics

notion of time (discrete, infinite)

properties of states (atomic proposition)

linear operators at the next instant

always

never

eventually

path quantifiers (nondeterministic systems)

for all possible futures

in a possible future

Modal logics

notion of time (discrete, infinite)

properties of states (atomic proposition)

modal operators at the next step
at any next step
(like HM logic)

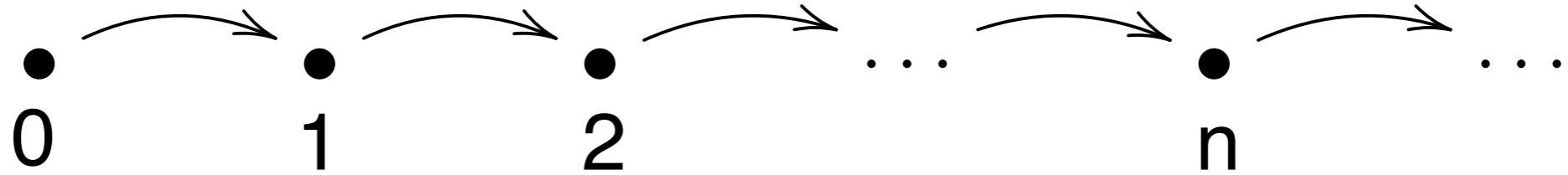
fix point operators recursively defined formulas
minimal / maximal fixpoint
(meaning of a formula:
the set of states where it holds)

LTL

Linear temporal logic

Linear Temporal Logic

models



syntax

ψ	$::=$	tt ff $\neg\psi$ $\psi_0 \wedge \psi_1$ $\psi_0 \vee \psi_1$
		p atomic proposition $p \in P$
		$O\psi$ NEXT: ψ holds at the next instant of time
		$F\psi$ FINALLY: ψ holds sometimes in the future
		$G\psi$ GLOBALLY: ψ holds always in the future
		$\psi_0 U \psi_1$ UNTIL: ψ_0 holds until ψ_1 is true

$O\psi$ sometimes written $X\psi$ or $N\psi$

Linear Structure

$$S : P \rightarrow \wp(\mathbb{N})$$

set of atomic propositions

$S(p)$ is the set of time instants
in which p holds

$$S(p) = \{n \mid p \text{ holds at } n\}$$

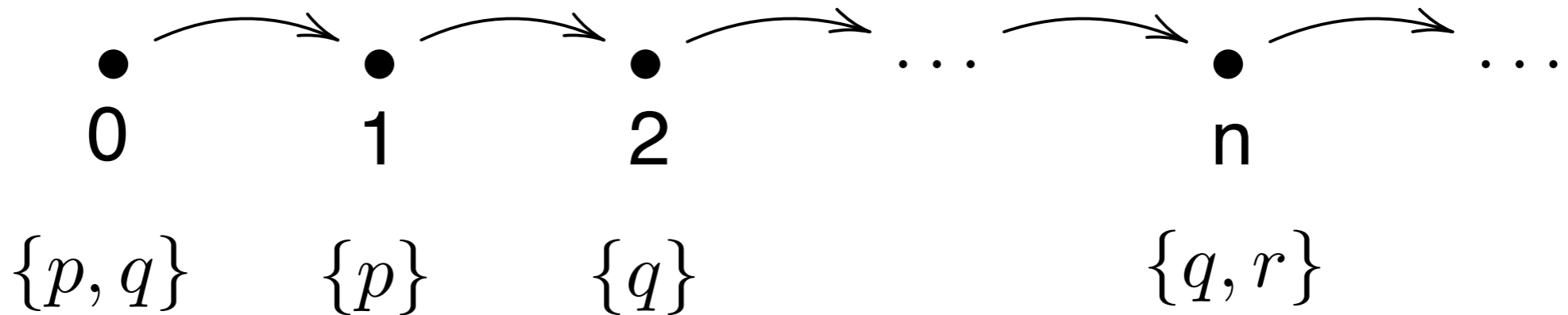
Shift $S^k : P \rightarrow \wp(\mathbb{N})$

$$S^k(p) = \{n - k \mid n \geq k \wedge n \in S(p)\}$$

$$S^k(p) = \{m \mid m + k \in S(p)\}$$

Example

$$S : P \rightarrow \wp(\mathbb{N})$$

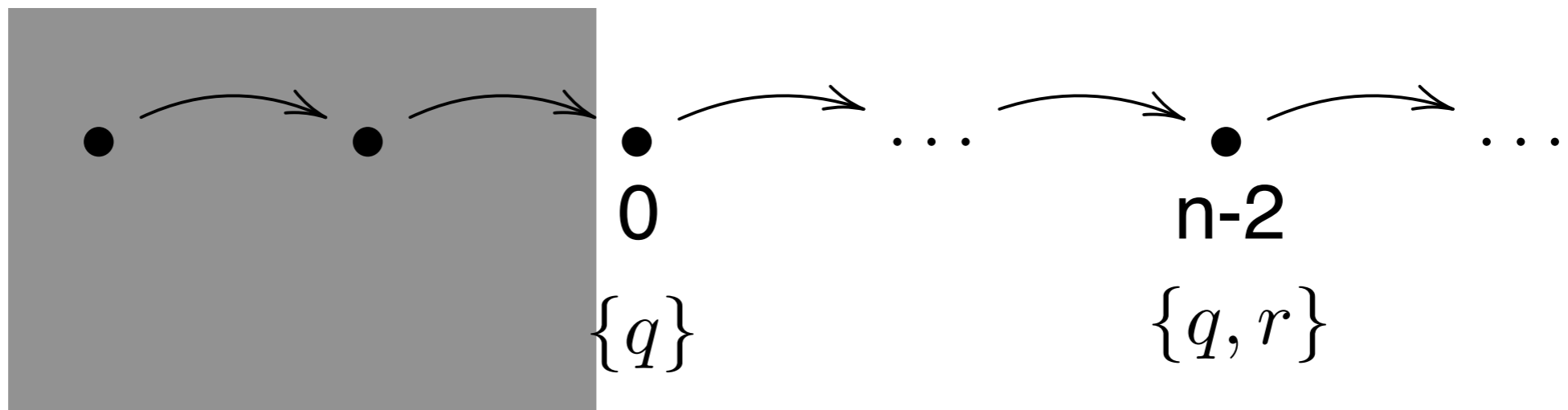


$$S(p) = \{0, 1, \dots\}$$

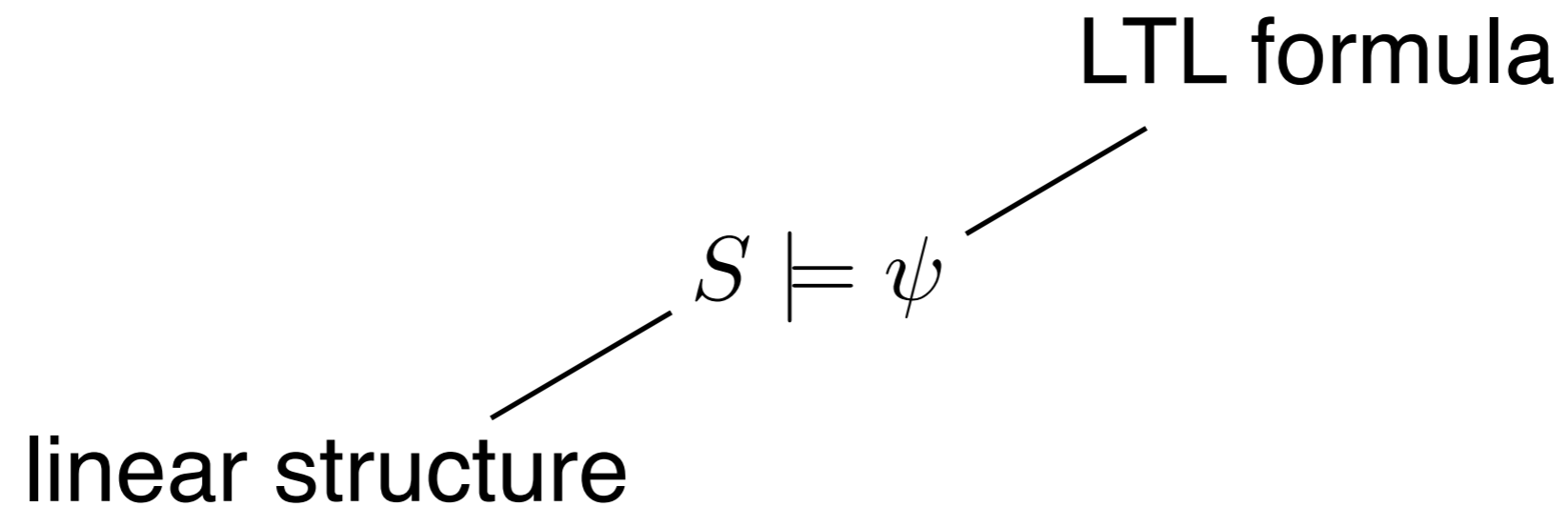
$$S(q) = \{0, 2, n, \dots\}$$

$$S(r) = \{n, \dots\}$$

$$S^2 : P \rightarrow \wp(\mathbb{N})$$



LTL: satisfaction



LTL: satisfaction

$$S \models \mathbf{tt}$$

current time: 0

$$S \models \neg\psi \quad \text{iff } S \not\models \psi$$

$$S \models \psi_0 \wedge \psi_1 \quad \text{iff } S \models \psi_0 \text{ and } S \models \psi_1$$

$$S \models \psi_0 \vee \psi_1 \quad \text{iff } S \models \psi_0 \text{ or } S \models \psi_1$$

$$S \models p \quad \text{iff } 0 \in S(p)$$

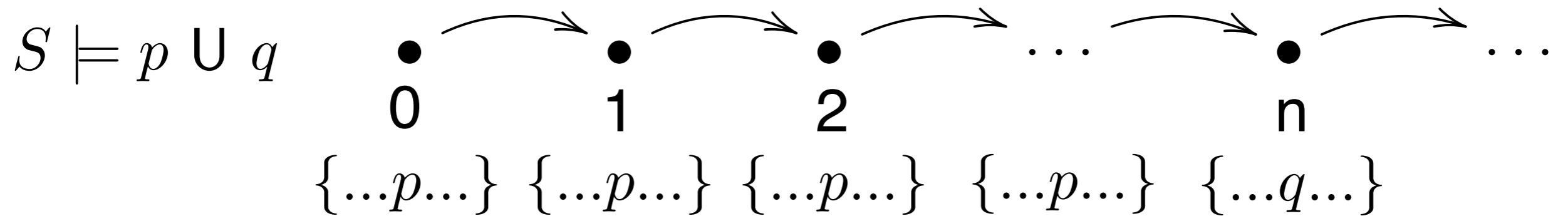
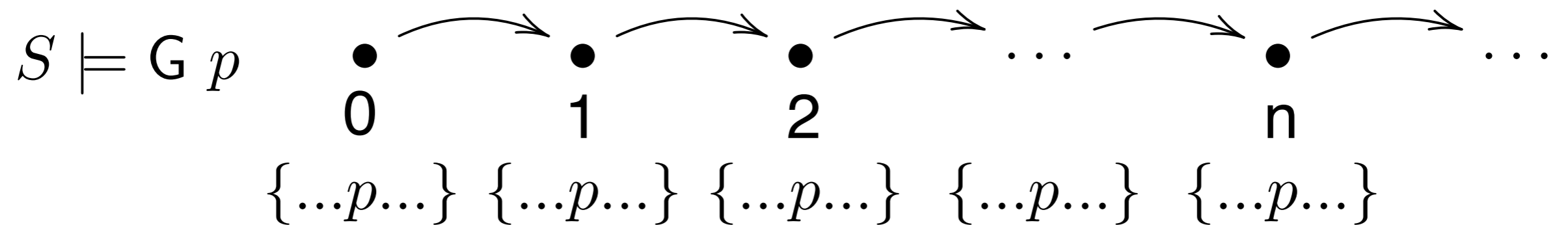
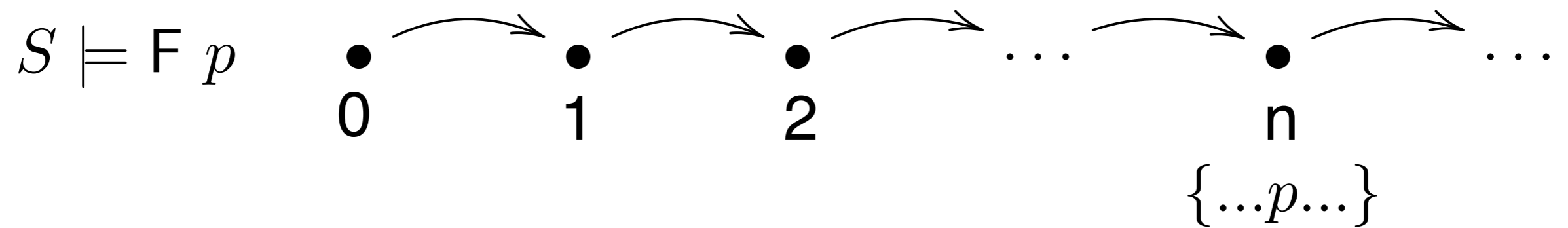
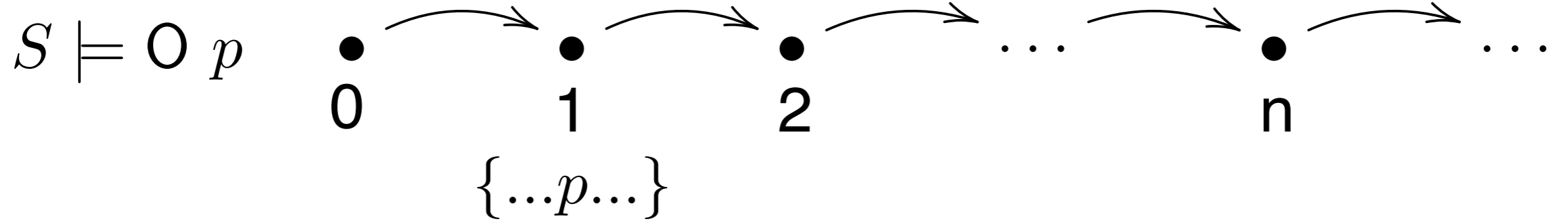
$$S \models \mathbf{O}\psi \quad \text{iff } S^1 \models \psi$$

$$S \models \mathbf{F}\psi \quad \text{iff } \exists k \in \mathbb{N}. S^k \models \psi$$

$$S \models \mathbf{G}\psi \quad \text{iff } \forall k \in \mathbb{N}. S^k \models \psi$$

$$S \models \psi_0 \mathbf{U}\psi_1 \quad \text{iff } \exists k \in \mathbb{N}. S^k \models \psi_1 \text{ and } \forall i < k. S^i \models \psi_0$$

Examples



LTL: equivalent formulas

$$\psi_0 \equiv \psi_1 \quad \text{iff} \quad \forall S. S \models \psi_0 \Leftrightarrow S \models \psi_1$$

$$F \psi \equiv \mathbf{tt} \mathbf{U} \psi$$

$$\begin{aligned} G \psi &\equiv \neg(F \neg\psi) \\ &\equiv \neg(\mathbf{tt} \mathbf{U} \neg\psi) \end{aligned}$$

$$\psi_0 \Rightarrow \psi_1 \triangleq \psi_1 \vee \neg\psi_0$$

Examples

$G \neg error$

error will never arise

$press \Rightarrow F error$

if you press now, an error will arise in the future

$G F enter$

enter will happen infinitely often (fairness)

$F G idle$

the system will stay idle from some time in the future onward

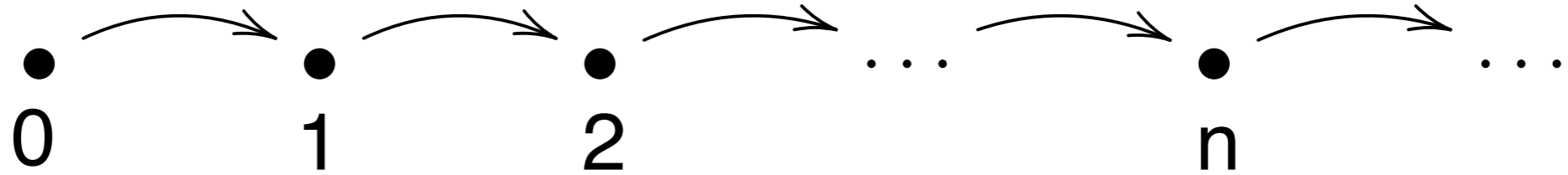
$G (req \Rightarrow (req U eval))$

whenever a request is made, it holds until evaluated

LTL automata-like models

LTL, again

models

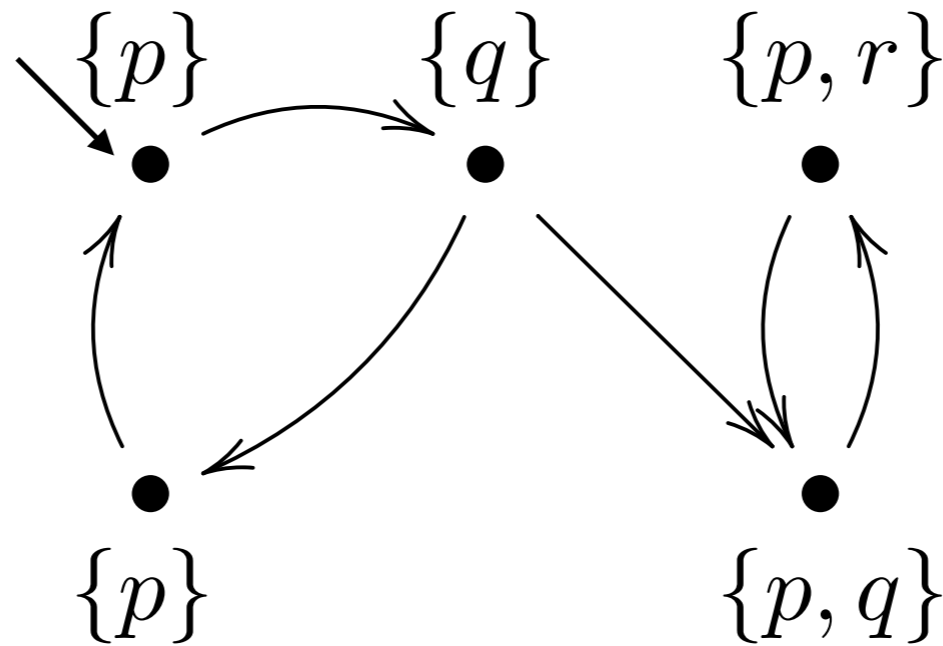


syntax

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		p atomic proposition $p \in P$
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$O\psi$ sometimes written $X\psi$ or $N\psi$

Automata-like models

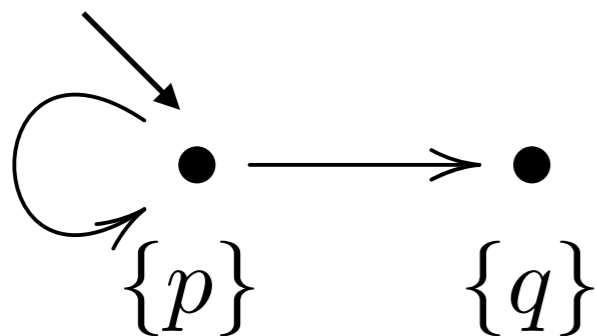


the formula must be satisfied along all (infinite) traces

(if we enter a deadlock state, the last state is repeated forever)



Exercise



$\not\models F q$ ⊗ $\{p\} \{p\} \{p\} \dots$

$\not\models G p$ ⊗ $\{p\} \{q\} \{q\} \dots$

$\not\models p U q$ ⊗ $\{p\} \{p\} \{p\} \dots$

$\models q U p$ ✔

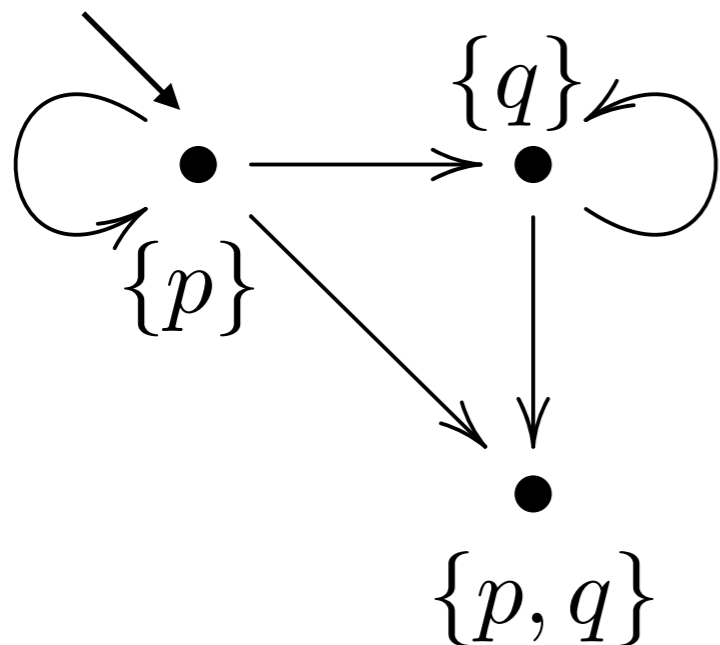
$\models G(q \Rightarrow G q)$ ✔

the formula must be satisfied along all (infinite) traces

(if we enter a deadlock state, the last state is repeated forever)



Exercise



$$\not\models G(q \cup p) \quad \times \{p\} \cdots \{p\} \{q\} \{q\} \cdots$$

$$\models G p \vee F q \quad \checkmark$$

$$\not\models F q \Rightarrow \neg G p \quad \times \{p\} \{p, q\} \{p, q\} \cdots$$

$$\models G(q \Rightarrow O q) \quad \checkmark$$

the formula must be satisfied along all (infinite) traces

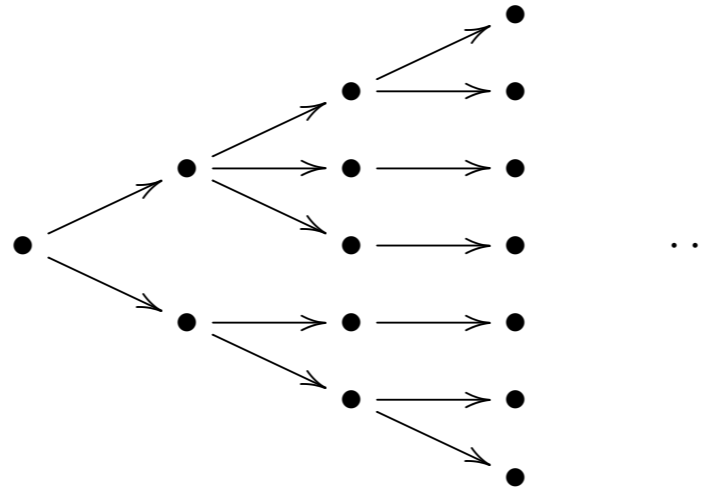
(if we enter a deadlock state, the last state is repeated forever)

CTL*, CTL

Computational tree logic

Computational Tree Logic

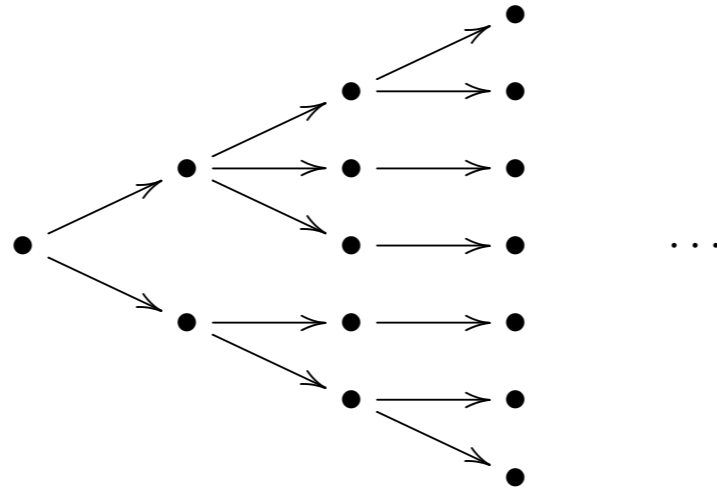
models



syntax (CTL*)

ψ	$::=$	tt ff $\neg\psi$ $\psi_0 \wedge \psi_1$ $\psi_0 \vee \psi_1$	classical ops
		p O ψ F ψ G ψ $\psi_0 \text{U} \psi_1$	linear ops
		E ψ	POSSIBLY : there is a path that satisfies
		A ψ	ALWAYS : every path satisfies ψ

Infinite Tree



$T = (V, \rightarrow)$ directed graph

tree

$v_0 \in V$ root: a distinguished vertex (no incoming arc)

exactly one directed path from v_0 to any other vertex $v \in V$

infinite

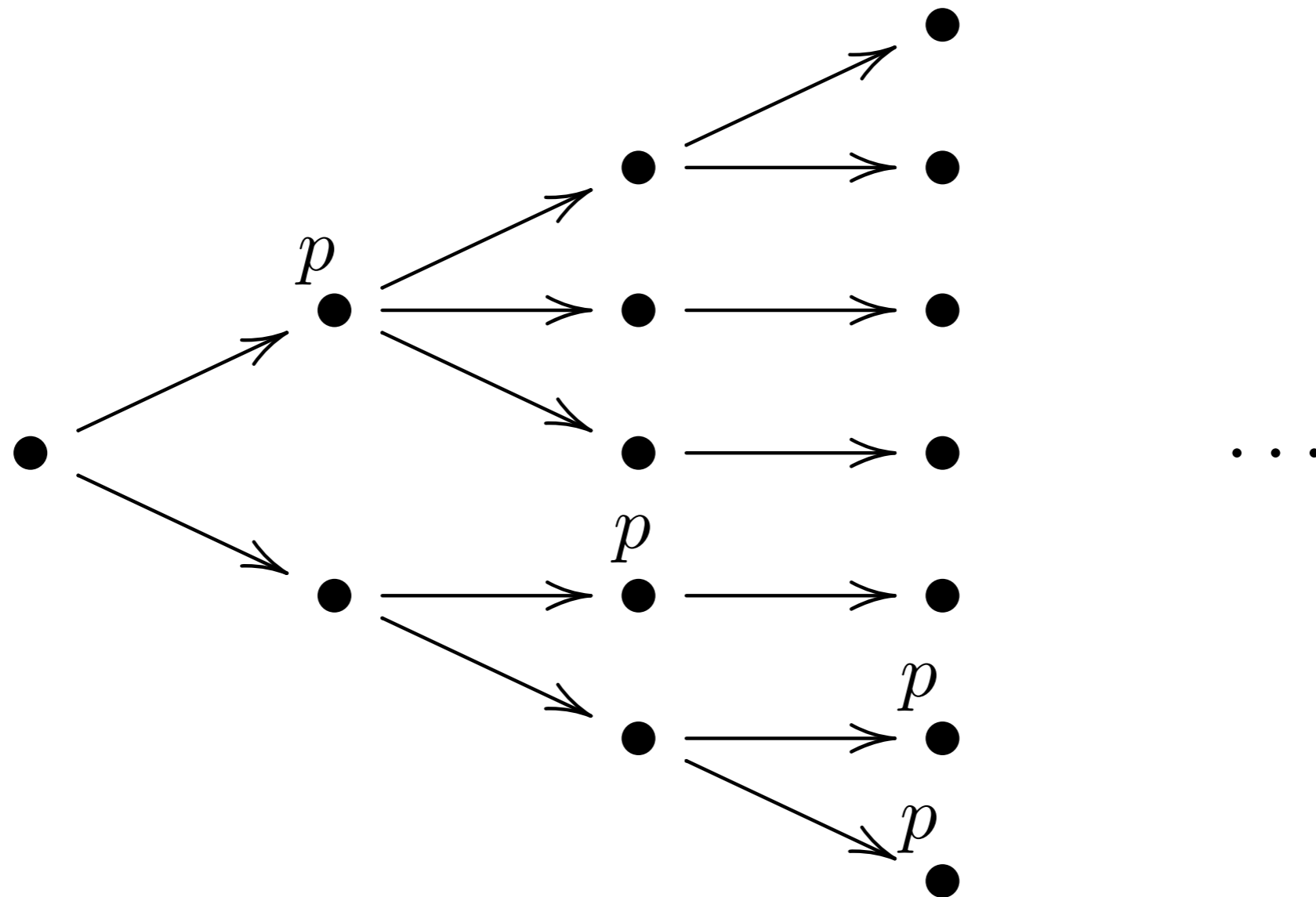
every node has a child

Branching Structure

$T = (V, \rightarrow)$ infinite tree

$S : P \rightarrow \wp(V)$

$$S(p) = \{x \in V \mid x \text{ satisfies } p\}$$



Infinite Path

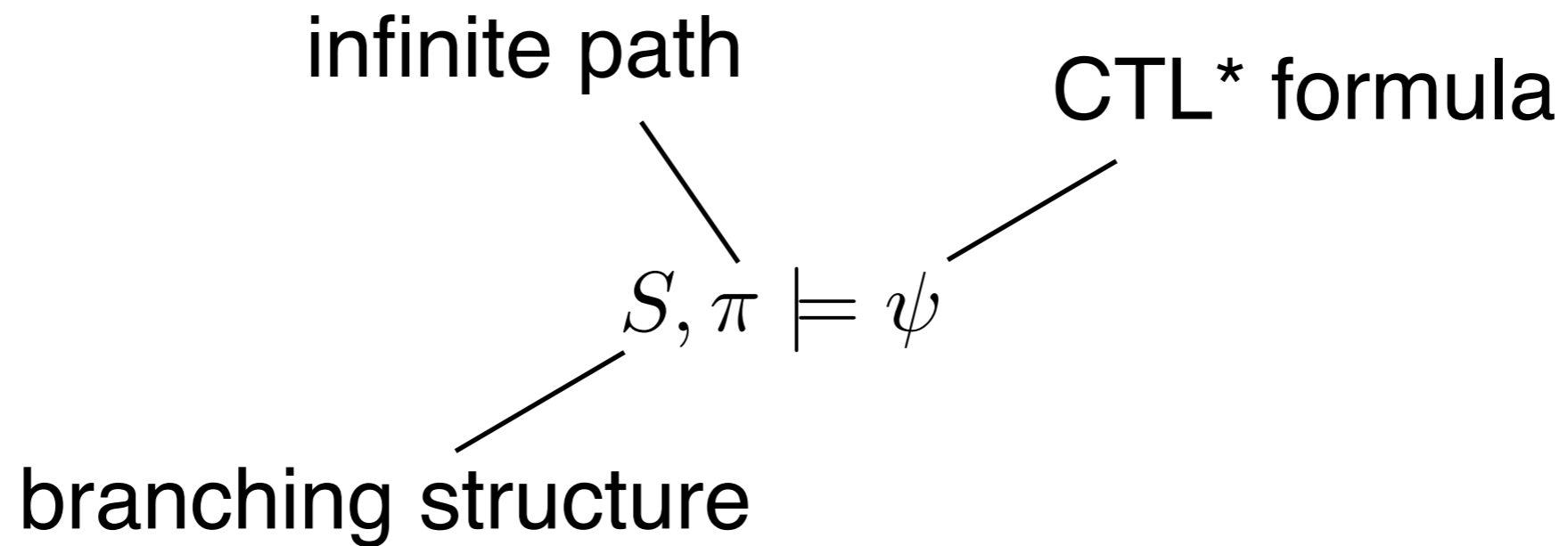
$T = (V, \rightarrow)$ $S : P \rightarrow \wp(V)$ branching structure

infinite path $T = (V, \rightarrow)$ $\pi : \mathbb{N} \rightarrow V$ ($\pi = v_0 v_1 \dots$)

such that $\forall k \in \mathbb{N}. v_k \rightarrow v_{k+1}$

path shifting $\pi = v_0 v_1 \dots$ $\pi^k = v_k v_{k+1} \dots$
 $\pi : \mathbb{N} \rightarrow V$ $\pi^k : \mathbb{N} \rightarrow V$
 $\pi^k(i) = \pi(k + i)$

CTL*: satisfaction



CTL*: satisfaction

$$S, \pi \models \mathbf{tt}$$

$$S, \pi \models \neg\psi \quad \text{iff } S, \pi \not\models \psi$$

$$S, \pi \models \psi_0 \wedge \psi_1 \quad \text{iff } S, \pi \models \psi_0 \text{ and } S, \pi \models \psi_1$$

$$S, \pi \models \psi_0 \vee \psi_1 \quad \text{iff } S, \pi \models \psi_0 \text{ or } S, \pi \models \psi_1$$

$$S, \pi \models p \quad \text{iff } \pi(0) \in S(p)$$

$$S, \pi \models \mathbf{O}\psi \quad \text{iff } S, \pi^1 \models \psi$$

state ops

$$S, \pi \models \mathbf{F}\psi \quad \text{iff } \exists k \in \mathbb{N}. S, \pi^k \models \psi$$

$$S, \pi \models \mathbf{G}\psi \quad \text{iff } \forall k \in \mathbb{N}. S, \pi^k \models \psi$$

$$S, \pi \models \psi_0 \mathbf{U} \psi_1 \quad \text{iff } \exists k \in \mathbb{N}. S, \pi^k \models \psi_1 \text{ and } \forall i < k. S, \pi^i \models \psi_0$$

$$S, \pi \models \mathbf{E}\psi \quad \text{iff } \exists \pi'. \pi'(0) = \pi(0) \text{ and } S, \pi' \models \psi$$

path ops

$$S, \pi \models \mathbf{A}\psi \quad \text{iff } \forall \pi'. \pi'(0) = \pi(0) \text{ implies } S, \pi' \models \psi$$

CTL*: equivalent formulas

$$\psi_0 \equiv \psi_1 \quad \text{iff} \quad \forall S. \forall \pi. S, \pi \models \psi_0 \Leftrightarrow S, \pi \models \psi_1$$

$$A \psi \equiv \neg(E \neg \psi)$$

$$A A \psi \equiv A \psi$$

$$A E \psi \equiv E \psi$$

LTL formulas as CTL* ones

ψ

$A \psi$

Examples

$$E O \psi$$

analogous to HML formula $\diamond\psi$

$$A G p$$

p holds at any reachable state

$$E F p$$

p holds at some reachable state

$$A F p$$

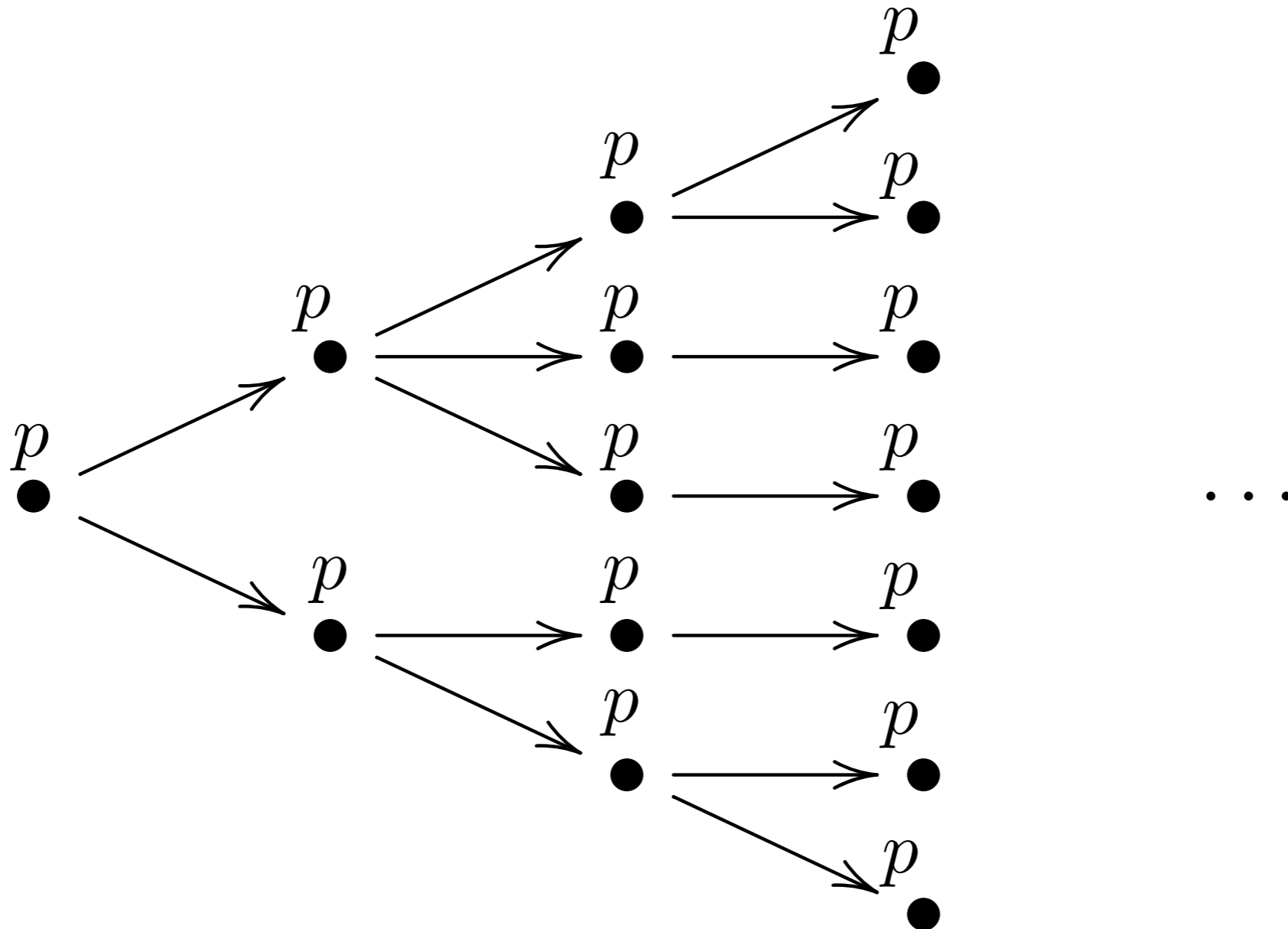
on every path there is a state where p holds

$$E (p U q)$$

there is a path where p holds until q

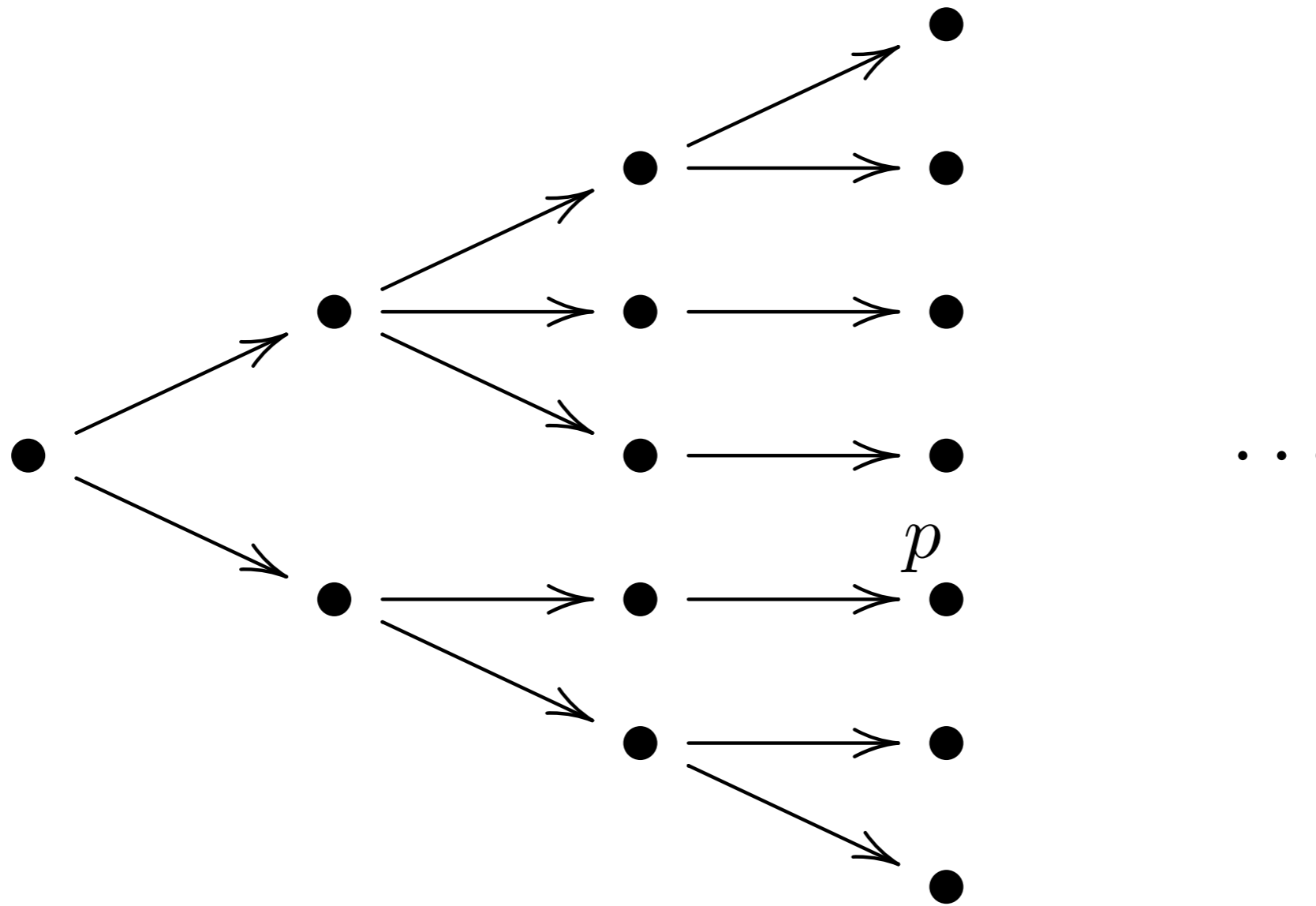
Example

A G p



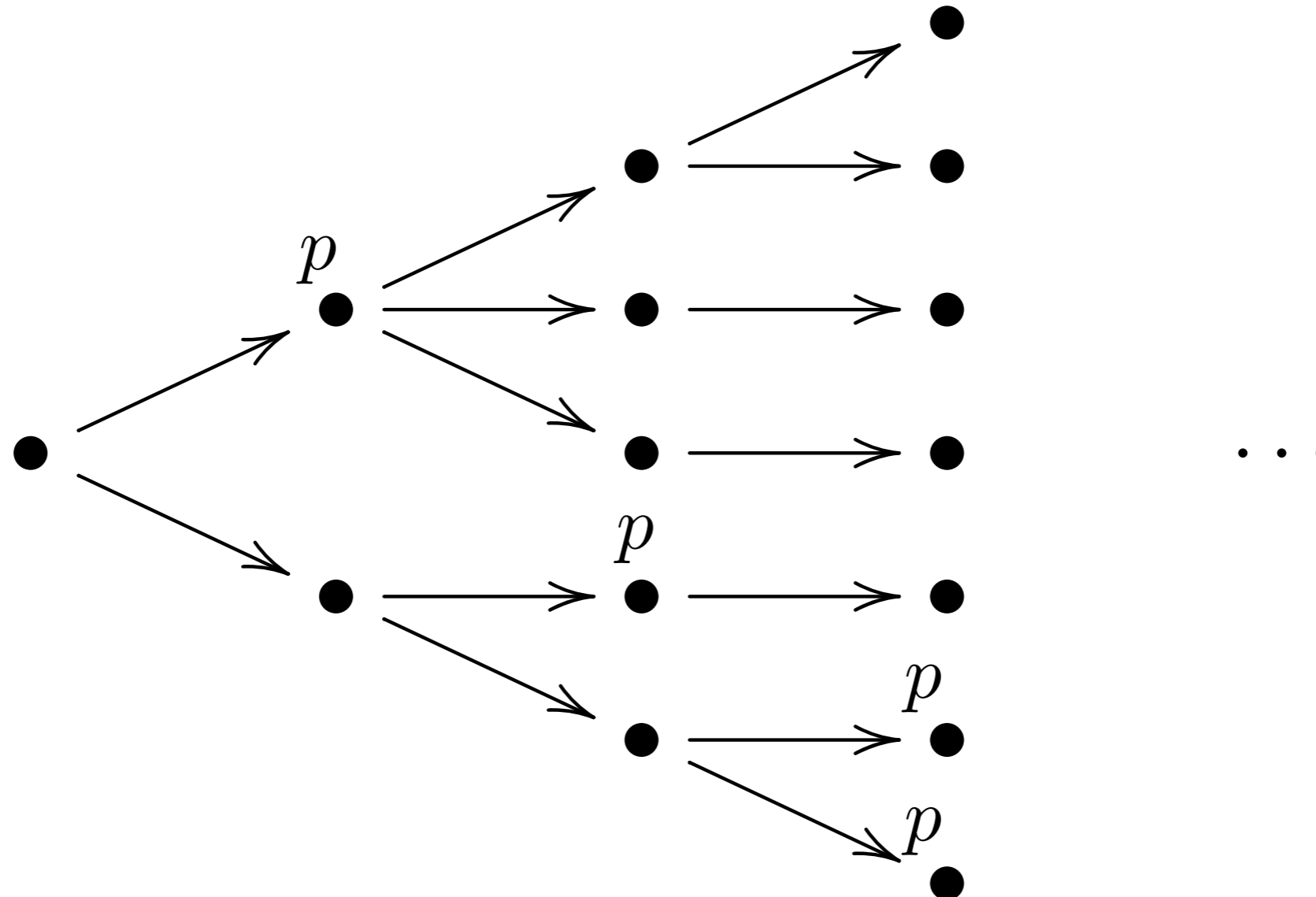
Example

$E F p$



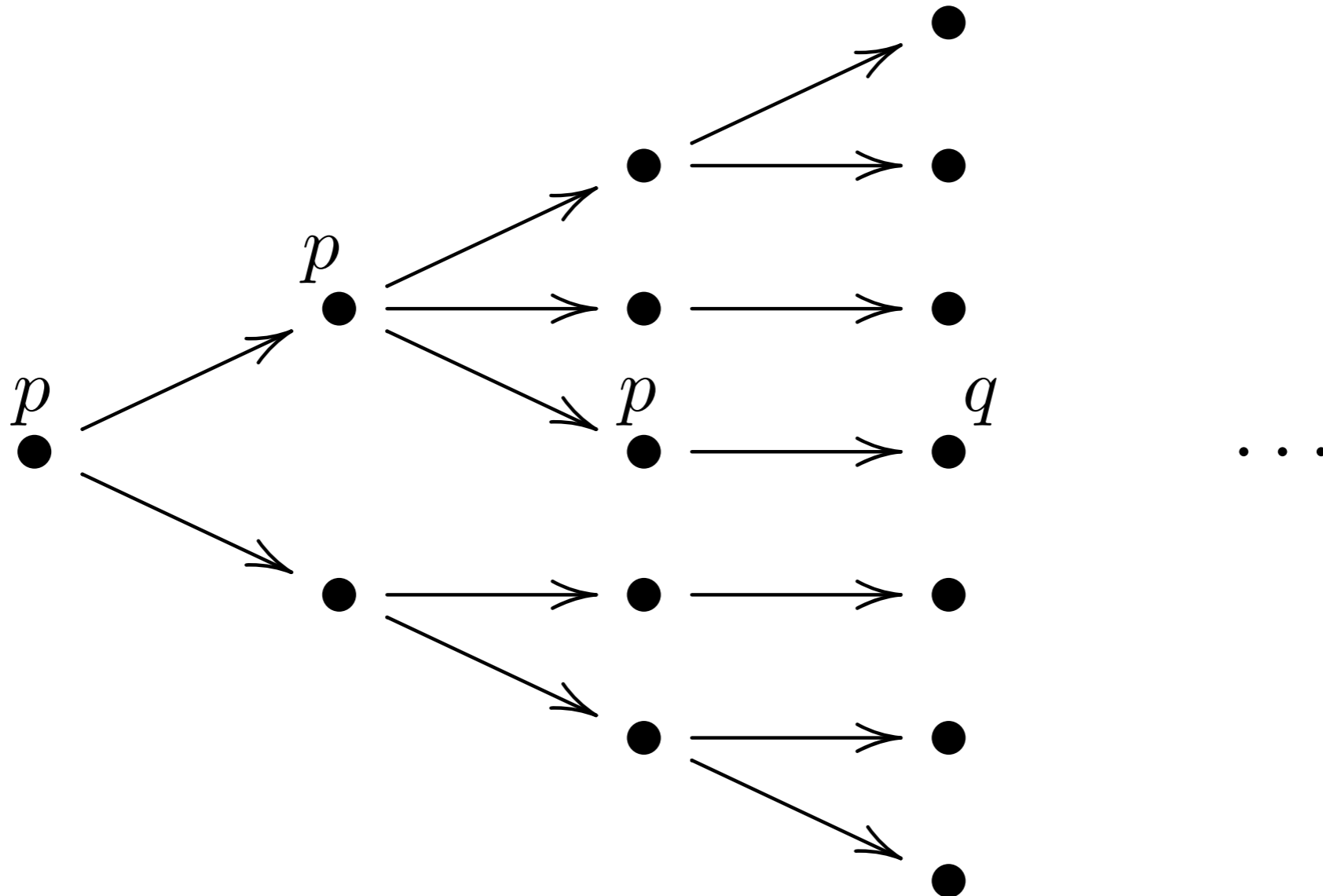
Example

A F p



Example

$$E (p \cup q)$$



CTL

CTL formulas

each path op (A/E) appears immediately before a linear op

each linear op (O/F/G/U) appears immediately after a path op

$E O \psi$

$E F \psi$

$E G \psi$

$E (\psi_0 U \psi_1)$

$A O \psi$

$A F \psi$

$A G \psi$

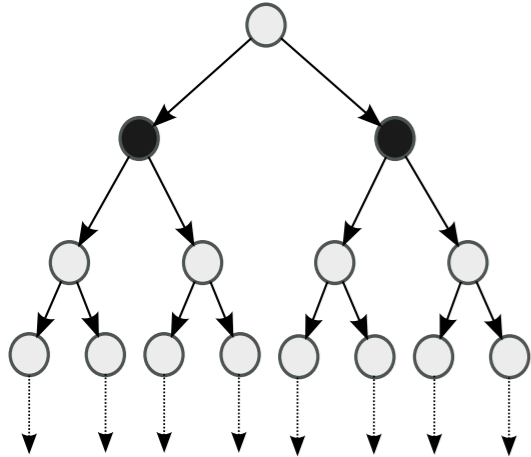
$A (\psi_0 U \psi_1)$

$A G F \psi$

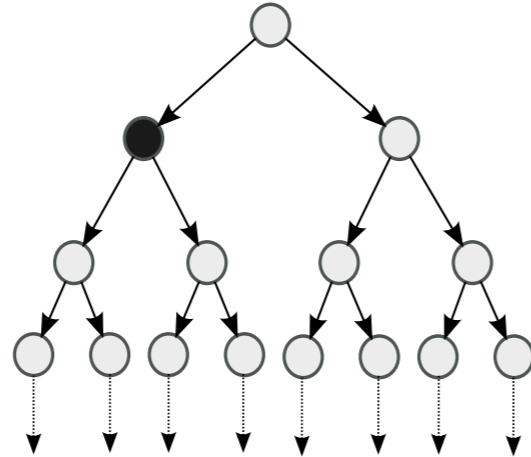
CTL*, not CTL

CTL formulas

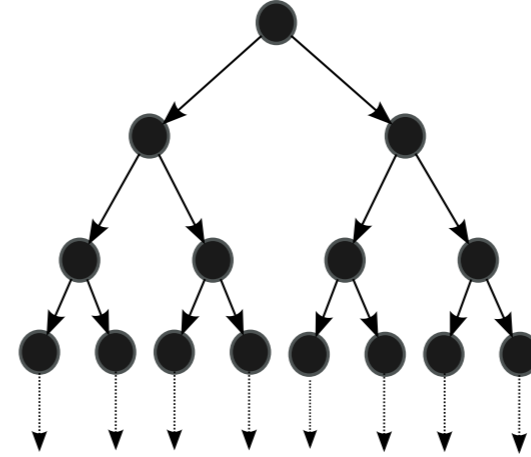
$AO \emptyset$



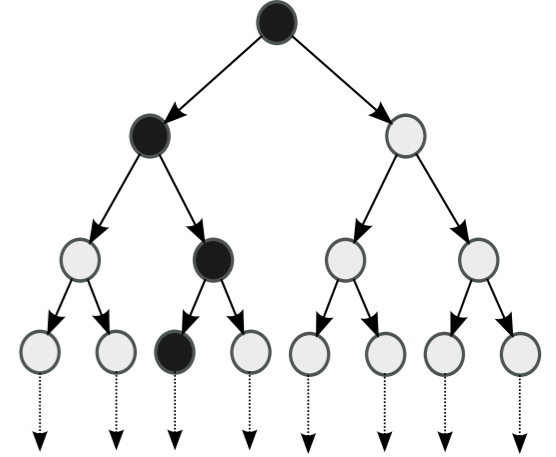
$EO \emptyset$



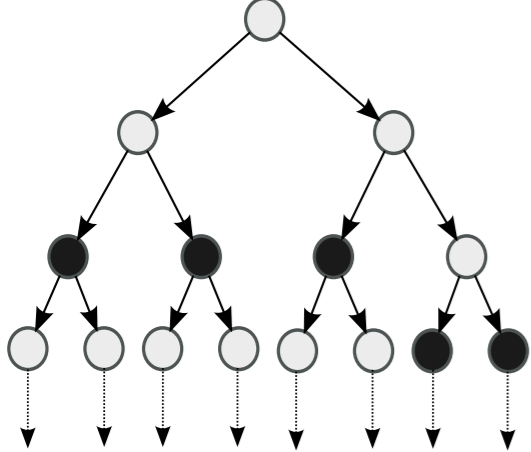
$AG \emptyset$



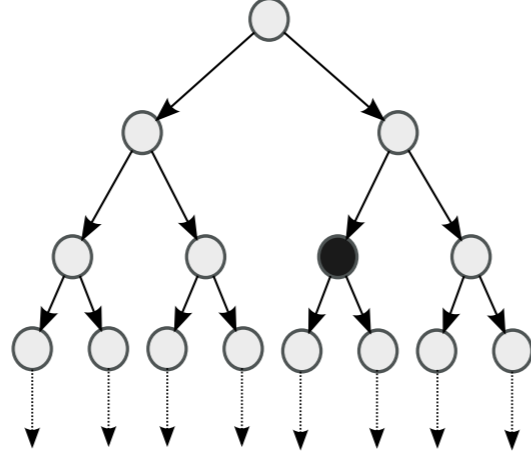
$EG \emptyset$



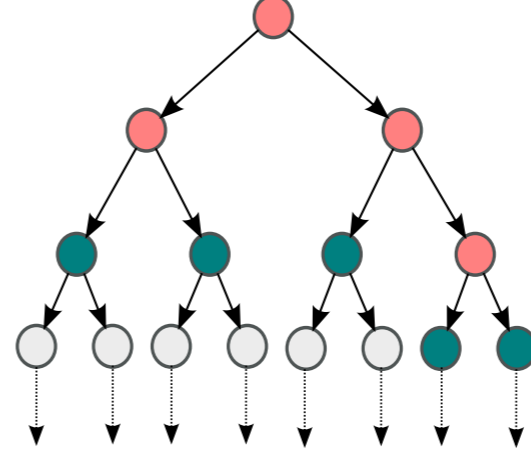
$AF \emptyset$



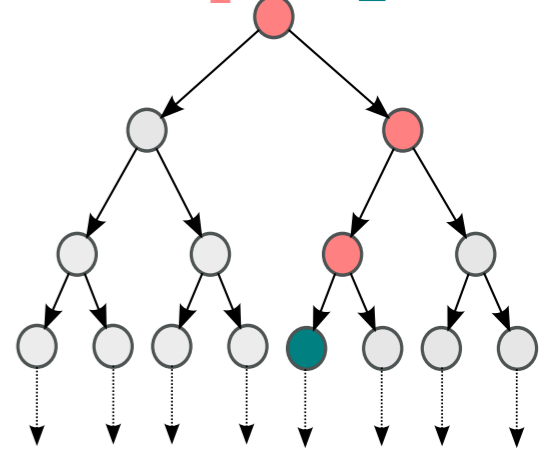
$EF \emptyset$



$A[\phi_1 U \phi_2]$



$E[\phi_1 U \phi_2]$



CTL: minimal set of ops

$\neg \cdot \quad \cdot \vee \cdot \quad \text{EO} \cdot \quad \text{EG} \cdot \quad \text{E}(\cdot \text{U} \cdot)$

$$\text{AO } \psi \equiv \neg(\text{EO } \neg\psi)$$

$$\text{AF } \psi \equiv \neg(\text{EG } \neg\psi)$$

$$\text{EF } \psi \equiv \text{E}(\text{tt U } \psi)$$

$$\begin{aligned} \text{AG } \psi &\equiv \neg(\text{EF } \neg\psi) \\ &\equiv \neg\text{E}(\text{tt U } \neg\psi) \end{aligned}$$

$$\text{A } (\psi_0 \text{ U } \psi_1) \equiv \neg(\text{EG } \neg\psi_1 \vee \text{E}(\neg\psi_1 \text{ U } \neg(\psi_0 \vee \psi_1)))$$

Expressiveness

