PSC 2023/24 (375AA, 9CFU)

Principles for Software Composition

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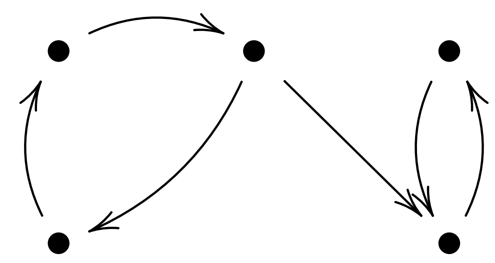
22b - Mu-calculus

μ-calculus

mu-calculus

models



syntax

<table>
<tr><td>ψ</td><td>$::=$</td><td>$\mathbf{tt} \mid \mathbf{ff} \mid \psi_0 \wedge \psi_1 \mid \psi_0 \vee \psi_1$</td><td>classical ops</td></tr>
<tr><td></td><td>$\mid$</td><td>$p \mid \neg p$</td><td>atomic propositions</td></tr>
<tr><td></td><td>$\mid$</td><td>$\Diamond\psi$</td><td>there is a next state where ψ holds</td></tr>
<tr><td></td><td>$\mid$</td><td>$\Box\psi$</td><td>ψ holds at every next state</td></tr>
<tr><td></td><td>$\mid$</td><td>x</td><td>predicate variable, for recursive def</td></tr>
<tr><td></td><td>$\mid$</td><td>$\mu x.\ \psi$</td><td>LEAST FIXPOINT of $x =_{\text{min}} \psi$</td></tr>
<tr><td></td><td>$\mid$</td><td>$\nu x.\ \psi$</td><td>GREATEST FIXPOINT of $x =_{\text{max}} \psi$</td></tr>
</table>

mu-calculus: semantics

$$G = (V, \to)$$

$[\![\psi]\!]\rho$ set of nodes where ψ holds

$\rho : P \cup X \to \wp(V)$

assignment

$(\wp(V), \subseteq)$ is a complete lattice

for monotone functions: least / greatest fixpoint exist

for this reason negation is not present in the syntax

the formulas we consider: *positive normal form*

otherwise:

even number of negations before each variable occurrence

Positive normal form

$$\neg\Diamond\psi \equiv \Box\neg\psi$$

$$\neg\Box\psi \equiv \Diamond\neg\psi$$

$$\neg\mu x.\ \psi \equiv \nu x.\ \neg\psi[{}^{\neg x}/{}_{x}]$$

$$\neg\nu x.\ \psi \equiv \mu x.\ \neg\psi[{}^{\neg x}/{}_{x}]$$

$$\neg\neg\psi \equiv \psi$$

plus usual De Morgan's laws

mu-calculus: semantics

$$[\![\cdot]\!] : \mathcal{F} \to (P \cup X \to \wp(V)) \to \wp(V)$$

defined by structural induction

$$[\![\mathbf{tt}]\!]\rho \triangleq V \qquad [\![\psi_0 \wedge \psi_1]\!]\rho \triangleq [\![\psi_0]\!]\rho \cap [\![\psi_1]\!]\rho$$

$$[\![\mathbf{ff}]\!]\rho \triangleq \emptyset \qquad [\![\psi_0 \vee \psi_1]\!]\rho \triangleq [\![\psi_0]\!]\rho \cup [\![\psi_1]\!]\rho$$

$$[\![p]\!]\rho \triangleq \rho(p) \qquad [\![\Diamond\psi]\!]\rho \triangleq \{v \mid \exists w \in [\![\psi]\!]\rho.\ v \to w\}$$

$$[\![\neg p]\!]\rho \triangleq V \setminus \rho(p) \qquad [\![\Box\psi]\!]\rho \triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![\psi]\!]\rho\}$$

$$[\![\mu x.\ \psi]\!]\rho \triangleq \mathit{fix}\ \lambda S.[\![\psi]\!]\rho[{}^{S}/_{x}]$$

$$[\![x]\!]\rho \triangleq \rho(x)$$

$$[\![\nu x.\ \psi]\!]\rho \triangleq \mathit{FIX}\ \lambda S.[\![\psi]\!]\rho[{}^{S}/_{x}]$$

mu-calculus: fixpoint

V finite

$f : \wp(V) \to \wp(V)$ monotone (hence continuous)

we can compute the least fixpoint by

$$fix\ f = \bigcup_{n \in \mathbb{N}} f^n(\emptyset)$$

we can compute the greatest fixpoint by

$$FIX\ f = \bigcap_{n \in \mathbb{N}} f^n(V)$$

Examples

$$[\![\Diamond \mathbf{tt}]\!]\rho \triangleq \{v \mid \exists w \in [\![\mathbf{tt}]\!]\rho.\ v \to w\}$$

$$= \{v \mid \exists w \in V.\ v \to w\}$$

$\Diamond \mathbf{tt}$

non deadlocked states

$$[\![\Box \mathbf{ff}]\!]\rho \triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![\mathbf{ff}]\!]\rho\}$$

$$= \{v \mid \forall w.\ v \to w \Rightarrow w \in \emptyset\}$$

$$= \{v \mid v \not\to\}$$

$\Box \mathbf{ff}$

deadlocks

Examples

$$[\![\Diamond \mathbf{ff}]\!]\rho \triangleq \{v \mid \exists w \in [\![\mathbf{ff}]\!]\rho.\ v \to w\}$$

$$= \{v \mid \exists w \in \emptyset.\ v \to w\}$$

$$= \emptyset$$

$\Diamond \mathbf{ff}$

false

$$[\![\Box \mathbf{tt}]\!]\rho \triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![\mathbf{tt}]\!]\rho\}$$

$$= \{v \mid \forall w.\ v \to w \Rightarrow w \in V\}$$

$$= V$$

$\Box \mathbf{tt}$

true

Examples

$$[\![\mu x.\ x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![x]\!]\rho[{}^{S}/_{x}]$$
$$= \mathit{fix}\ \lambda S.\ S$$
$$= \emptyset$$

$\mu x.\ x$

false

$$[\![\nu x.\ x]\!]\rho \triangleq \mathit{FIX}\ \lambda S.\ [\![x]\!]\rho[{}^{S}/_{x}]$$
$$= \mathit{FIX}\ \lambda S.\ S$$
$$= V$$

$\nu x.\ x$

true

Examples

$$[\![\mu x.\ \Diamond x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![\Diamond x]\!]\rho[^S/_x]$$
$$= \mathit{fix}\ \lambda S.\ \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\}$$
$$= \mathit{fix}\ \lambda S.\ \{v \mid \exists w \in S.\ v \to w\}$$

$$S_0 = \emptyset$$
$$S_1 = \{v \mid \exists w \in S_0.\ v \to w\}$$
$$= \{v \mid \exists w \in \emptyset.\ v \to w\}$$
$$= \emptyset$$

$\mu x.\ \Diamond x$

false

Examples

how to represent?

EO ψ $\Diamond\psi$

AO ψ $\Box\psi$

EF p

$$p \vee \ldots$$
$$p \vee \Diamond(p \vee \cdots)$$
$$p \vee \Diamond(p \vee \Diamond(p \vee \cdots))$$
$$x = p \vee \Diamond x$$
$$\mu x.\ p \vee \Diamond x$$
$$\nu x.\ p \vee \Diamond x$$

?

EF p ?

Example

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cup [\![\Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &= V \\
S_1 &= \rho(p) \cup \{v \mid \exists w \in S_0.\ v \to w\} \\
&= \rho(p) \cup \{v \mid \exists w \in V.\ v \to w\} \\
&= \rho(p) \cup \{v \text{ can move}\}
\end{aligned}
$$

EF p ?

Example

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cup [\![\Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$S_0 = V$$

$$S_1 = \rho(p) \cup \{v \text{ can move}\}$$

$$
\begin{aligned}
S_2 &= \rho(p) \cup \{v \mid \exists w \in S_1.\ v \to w\} \\
&= \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\} \cup \{v \text{ can make 2 moves}\}
\end{aligned}
$$

EF p ?

Examples

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[{}^{S}/_{x}] \cup [\![\Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[{}^{S}/_{x}].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$S_0 = V$

$S_1 = \rho(p) \cup \{v$ can move$\}$

$S_2 = \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\} \cup \{v$ can make 2 moves$\}$

$S_n = \{v$ can reach a state in $\rho(p)$ in less than n moves$\}$

$\cup\{v$ can make n moves$\}$

EF p ?

Example

$$\llbracket \nu x.\ p \vee \Diamond x \rrbracket \rho \triangleq FIX\ \ \lambda S.\ \llbracket p \vee \Diamond x \rrbracket \rho[^S/_x]$$
$$= FIX\ \ \lambda S.\ \llbracket p \rrbracket \rho[^S/_x] \cup \llbracket \Diamond x \rrbracket \rho[^S/_x]$$
$$= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in \llbracket x \rrbracket \rho[^S/_x].\ v \to w\}$$
$$= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$S_n = \{v$ can reach a state in $\rho(p)$ in less than n moves$\}$

$\cup\{v$ can make n moves$\}$

$\bigcap_{n \in \mathbb{N}} S_n = \{v$ can reach a state in $\rho(p)$ or has an infinite path$\}$

EF p ?

Example

$$[\![\mu x.\ p \vee \Diamond x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[^S/_x]$$
$$= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$$S_0 = \emptyset$$

$$S_1 = \rho(p) \cup \{v \mid \exists w \in S_0.\ v \to w\}$$
$$= \rho(p) \cup \{v \mid \exists w \in \emptyset.\ v \to w\}$$
$$= \rho(p)$$

EF p ?

Example

$$\llbracket \mu x.\ p \vee \Diamond x \rrbracket \rho \triangleq \mathit{fix}\ \lambda S.\ \llbracket p \vee \Diamond x \rrbracket \rho[^S/_x]$$

$$= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$$S_0 = \emptyset$$

$$S_1 = \rho(p)$$

$$S_2 = \rho(p) \cup \{v \mid \exists w \in S_1.\ v \to w\}$$

$$= \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\}$$

$$= \{v \text{ can reach a state in } \rho(p) \text{ in less than 2 moves}\}$$

EF p ?

Example

$$[\![\mu x.\ p \vee \Diamond x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}]$$
$$= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$S_0 = \emptyset$

$S_1 = \rho(p)$

$S_2 = \{v$ can reach a state in $\rho(p)$ in less than 2 moves$\}$

$S_n = \{v$ can reach a state in $\rho(p)$ in less than n moves$\}$

$\bigcup_{n \in \mathbb{N}} S_n = \{v$ can reach a state in $\rho(p)\}$

$\mu x.\ p \vee \Diamond x$

EF p

Example

which formula for
"some deadlock is reachable"?

$\Box\mathbf{ff}$

deadlocks

$\mu x.\ \Box\mathbf{ff} \vee \Diamond x$

$\mu x.\ p \vee \Diamond x$

EF p

which formula for
"deadlock free"?

$$\neg(\mu x.\ \Box\mathbf{ff} \vee \Diamond x) = \nu x.\ \neg(\Box\mathbf{ff} \vee \Diamond\neg x)$$

$$= \nu x.\ \neg(\Box\mathbf{ff}) \wedge \neg(\Diamond\neg x)$$

$$= \nu x.\ \Diamond\mathbf{tt} \wedge \Box x$$

Example

$$[\![\nu x.\ p \wedge \Box x]\!]\rho \triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\}$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}$$

$$S_0 = V$$

$$\begin{aligned} S_1 &= \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_0\} \\ &= \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in V\} \\ &= \rho(p) \cap V \\ &= \rho(p) \end{aligned}$$

Example

$$
\begin{aligned}
[\![\nu x.\ p \wedge \Box x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}
\end{aligned}
$$

$$S_0 = V$$

$$S_1 = \rho(p)$$

$$
\begin{aligned}
S_2 &= \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_1\} \\
&= \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \rho(p)\} \\
&= \{v \text{ s.t. all nodes reachable in less than 2 moves are in } \rho(p)\}
\end{aligned}
$$

Example

$$[\![\nu x.\ p \wedge \Box x]\!]\rho \triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\}$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in S\}$$

$S_0 = V$

$\nu x.\ p \wedge \Box x$

AG p

$S_1 = \rho(p)$

$S_2 = \{v$ s.t. all nodes reachable in less than 2 moves are in $\rho(p)\}$

$S_n = \{v$ s.t. all nodes reachable in less than n moves are in $\rho(p)\}$

$\bigcap_{n \in \mathbb{N}} S_n = \{v$ can only reach states in $\rho(p)\}$

Invariants & possibly

Invariants $Inv(\psi) \triangleq \nu x.\ \psi \wedge \Box x$

Possibly $Pos(\psi) \triangleq \mu x.\ \psi \vee \Diamond x$

Example

which temporal formula?

$\mu x.\ q \vee (p \wedge \Diamond x)$ E$(p$ U $q)$ (CTL)

$\mu x.\ q \vee (p \wedge \Diamond x \wedge \Box x)$ A$(p$ U $q)$ (LTL / CTL)

$\nu x.\ \underbrace{\mu y.\ (p \wedge \Diamond x) \vee \Diamond y}$ E G F p (CTL*)

after a finite number of steps you reach a state where

1) *p* holds

2) there is a next step where the property holds recursively

Alternation depth

Fixed points *alternate* if a least fixed point influences the greatest fixed point, and vice-versa.

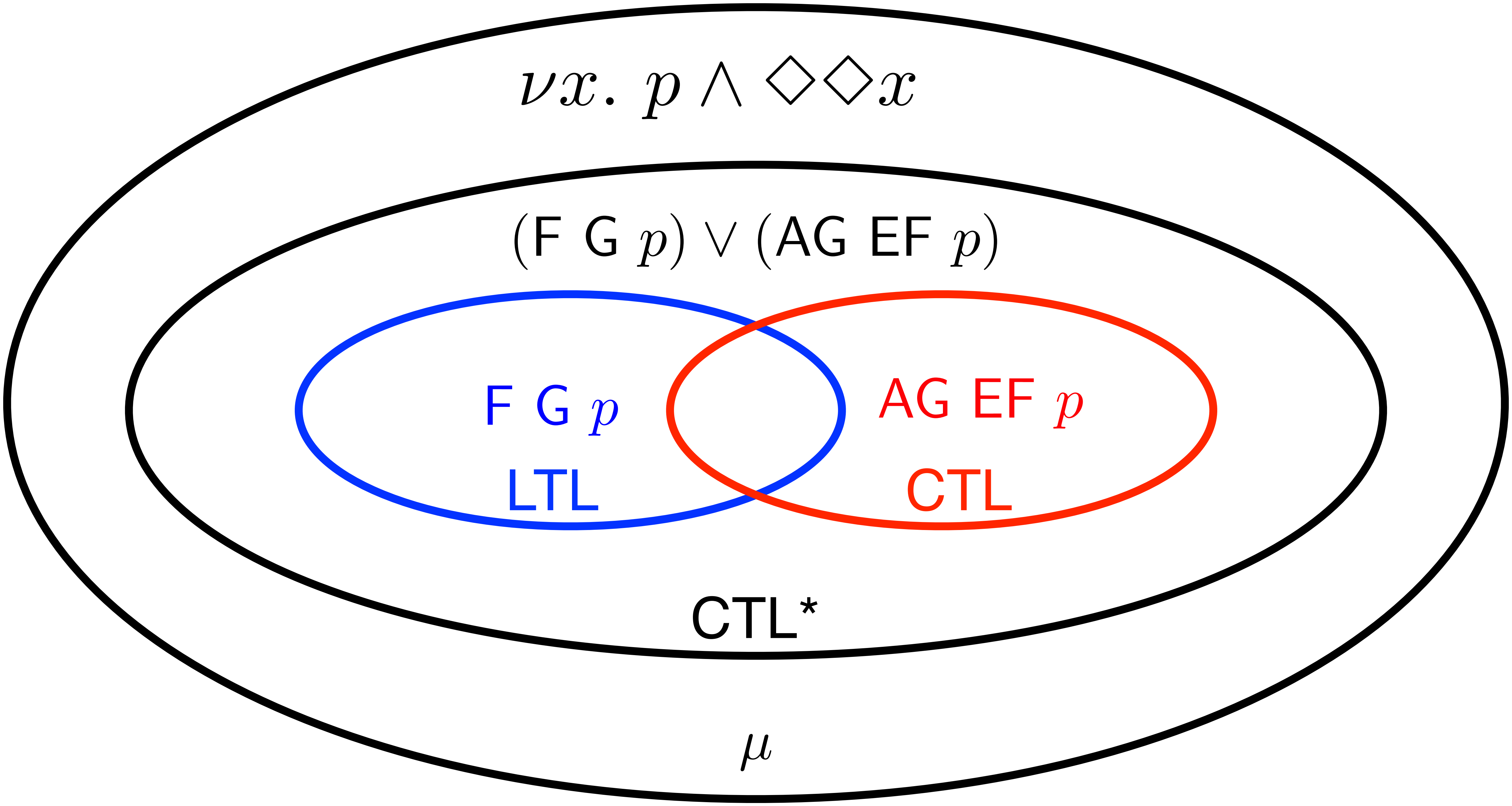
The *alternation depth* counts the number of alternations.

The alternation hierarchy of the μ-calculus is strict: strictly more properties are expressible as the alternation depth increases.

CTL can be encoded in μ-calculus with alternation depth 1

CTL* and LTL can be encoded in μ-calculus with alternation depth at most 2

Expressiveness



mu-calculus with labels

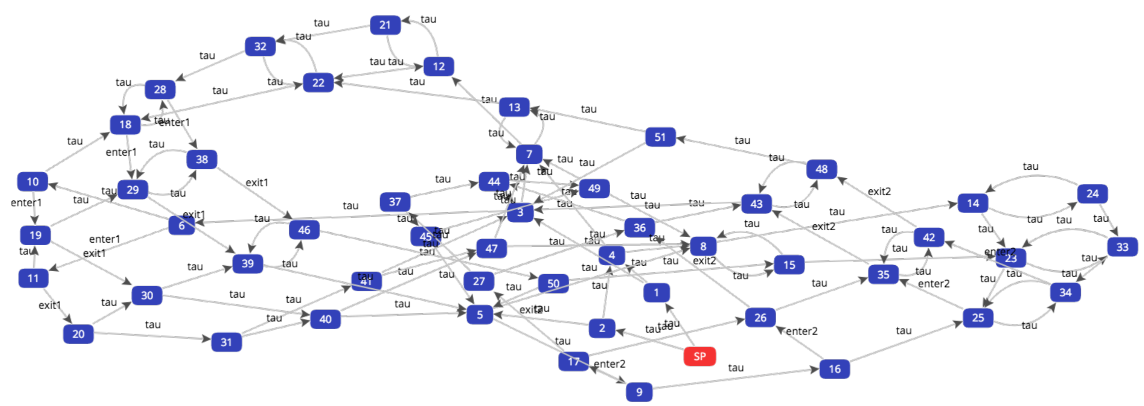
$$\psi \quad ::= \quad \ldots \quad | \quad \Diamond_L \psi \quad | \quad \Box_L \psi$$

set of labels

$$[\![\Diamond_L \psi]\!]\rho \triangleq \{v \mid \exists \mu \in L.\ \exists w \in [\![\psi]\!]\rho.\ v \xrightarrow{\mu} w\}$$

$$[\![\Box_L \psi]\!]\rho \triangleq \{v \mid \forall \mu \in L.\ \forall w.\ v \xrightarrow{\mu} w \Rightarrow w \in [\![\psi]\!]\rho\}$$

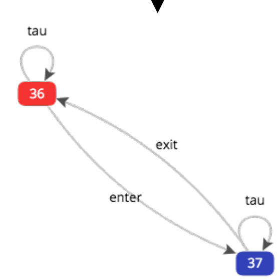
Space reduction



$$\overset{?}{\models} \psi$$

identify
bisimilar
states

$$\Updownarrow$$



$$\overset{?}{\models} \psi$$