PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni
http://www.di.unipi.it/~bruni/

http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/start

08a - Complete Partial Orders

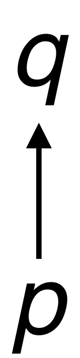
Partial orders

Partially ordered set (Poset or just PO)

a set

$(P, \sqsubseteq)$ — a binary relation $\sqsubseteq \subseteq P \times P$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
antisymmetric	$\forall p, q \in P.$	$p \sqsubseteq q \;\wedge\; q \sqsubseteq p \;\Rightarrow\; p = q$
transitive	$\forall p, q, r \in P.$	$p \sqsubseteq q \;\wedge\; q \sqsubseteq r \;\Rightarrow\; p \sqsubseteq r$



$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

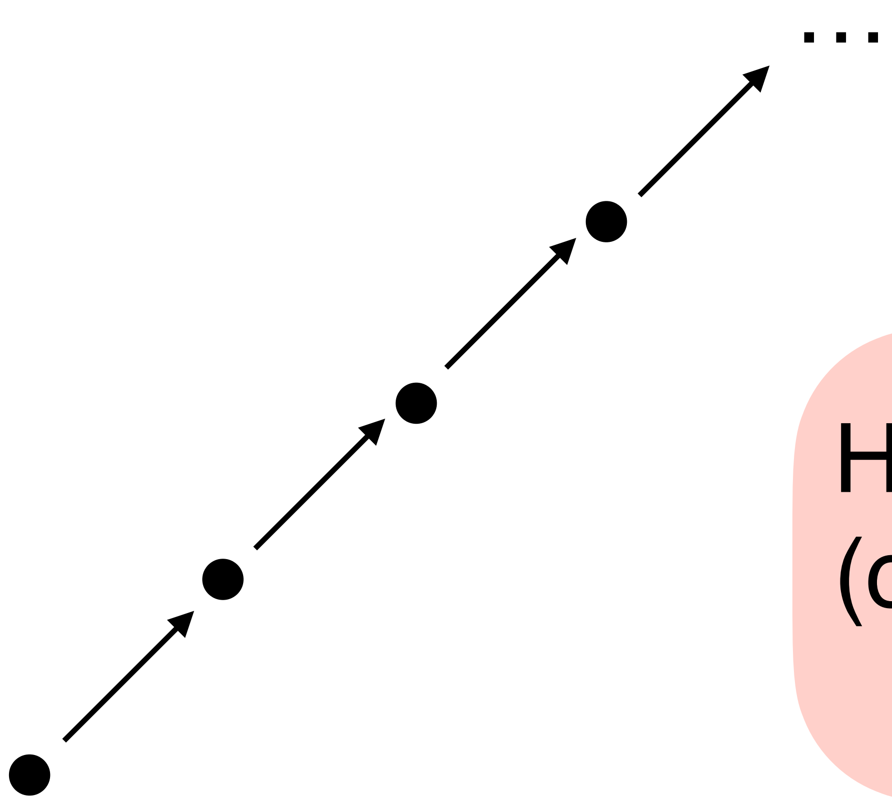
$p \sqsubset q$ means $p \sqsubseteq q \;\wedge\; p \neq q$

Total orders

$(P, \sqsubseteq)$ PO

total $\quad \forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

a PO where any two elements are **comparable**



Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

each element is **comparable** only to itself

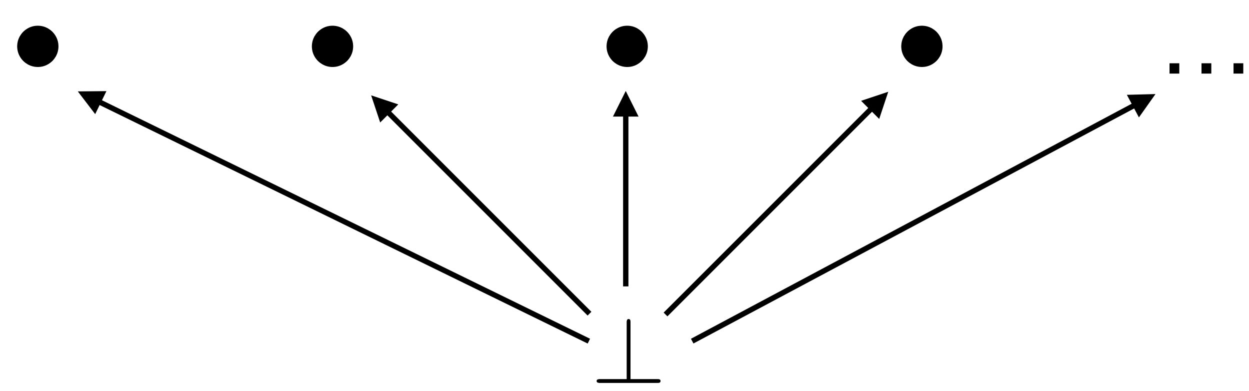
• • • • …

Flat orders

$(P, \sqsubseteq)$ PO

flat $\quad \forall p, q \in P. \quad p \sqsubseteq q \;\Leftrightarrow\; p = q \;\vee\; p = \bot$

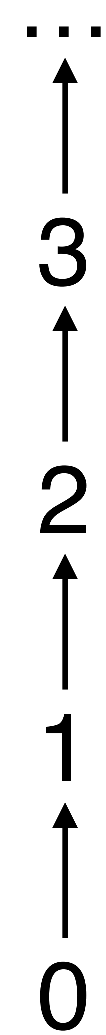
each element is **comparable** only to itself
and with a distinguished (smaller) element $\bot$



Exercise

$(\mathbb{N}, \leq)$

PO?	Total?	Discrete?	Flat?
✓	✓	✗	✗



Exercise

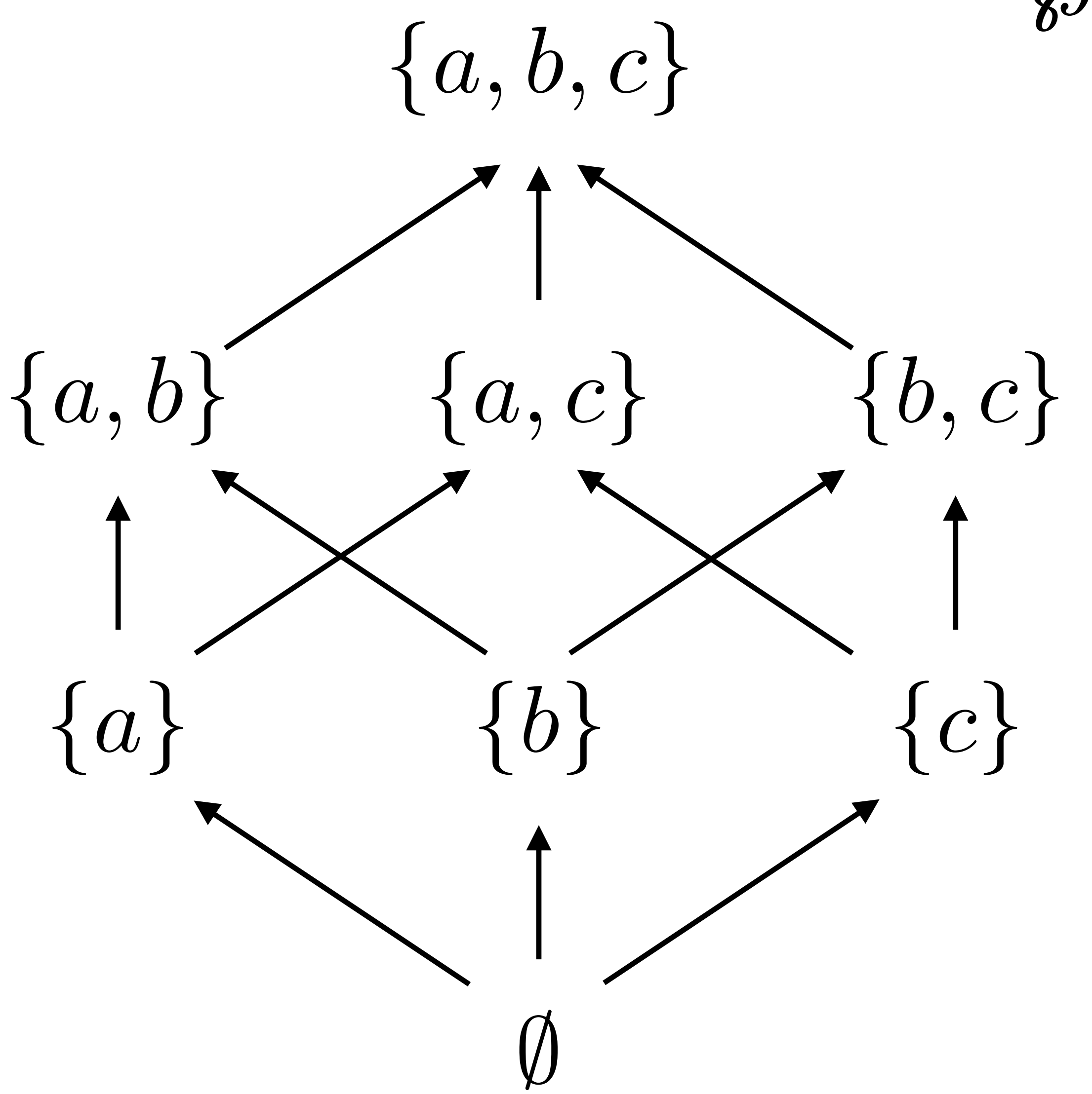
$(\wp(S), \subseteq)$

	PO?	Total?	Discrete?	Flat?
	✓	$\lvert S\rvert < 2$	$S = \emptyset$	$\lvert S\rvert < 2$

$\wp(\emptyset) = \{\emptyset\}$

$\wp(\{a\}) = \{\emptyset, \{a\}\}$

example: $S = \{a, b, c\}$



$\{a, b\} \not\subseteq \{b, c\}$, $\{a, b\} \not\supseteq \{b, c\}$

$\{a\} \not\subseteq \{b\}$, $\{a\} \not\supseteq \{b\}$

Exercise

$(\mathbb{N}, =)$

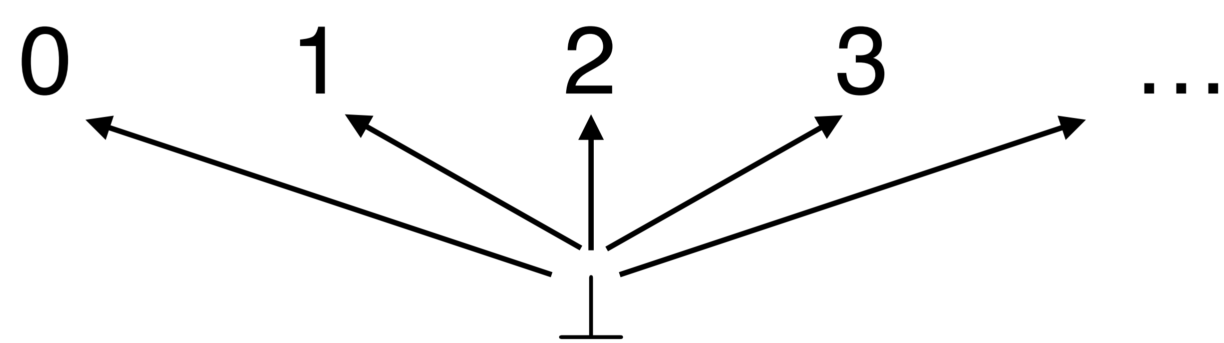
PO?	Total?	Discrete?	Flat?
✓	✗	✓	✗

0 1 2 3 …

Exercise

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$$

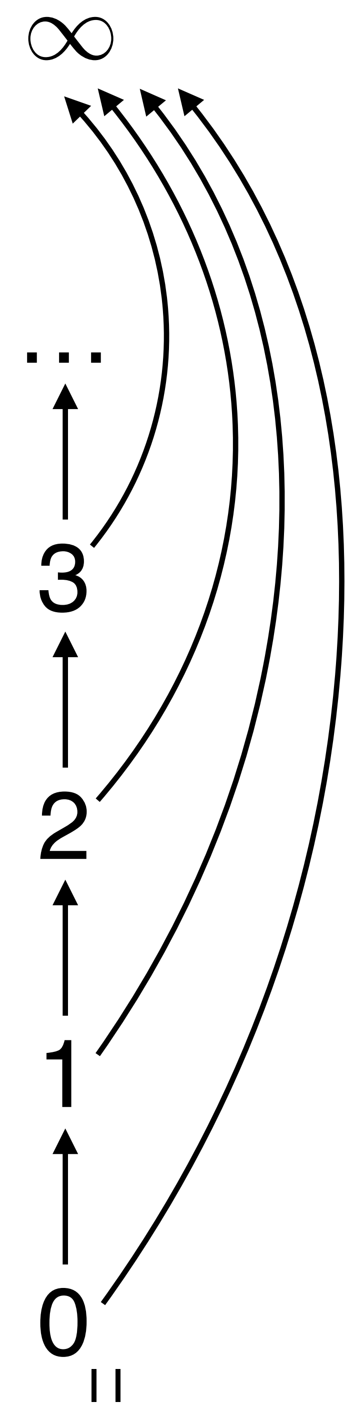
PO?	Total?	Discrete?	Flat?
✓	✗	✗	✓



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO? Total? Discrete? Flat?



Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	✗
$(T_\Sigma, \prec)$	✗	✗	✗	✗
$(\mathbb{N}, \neq)$	✗	✗	✗	✗

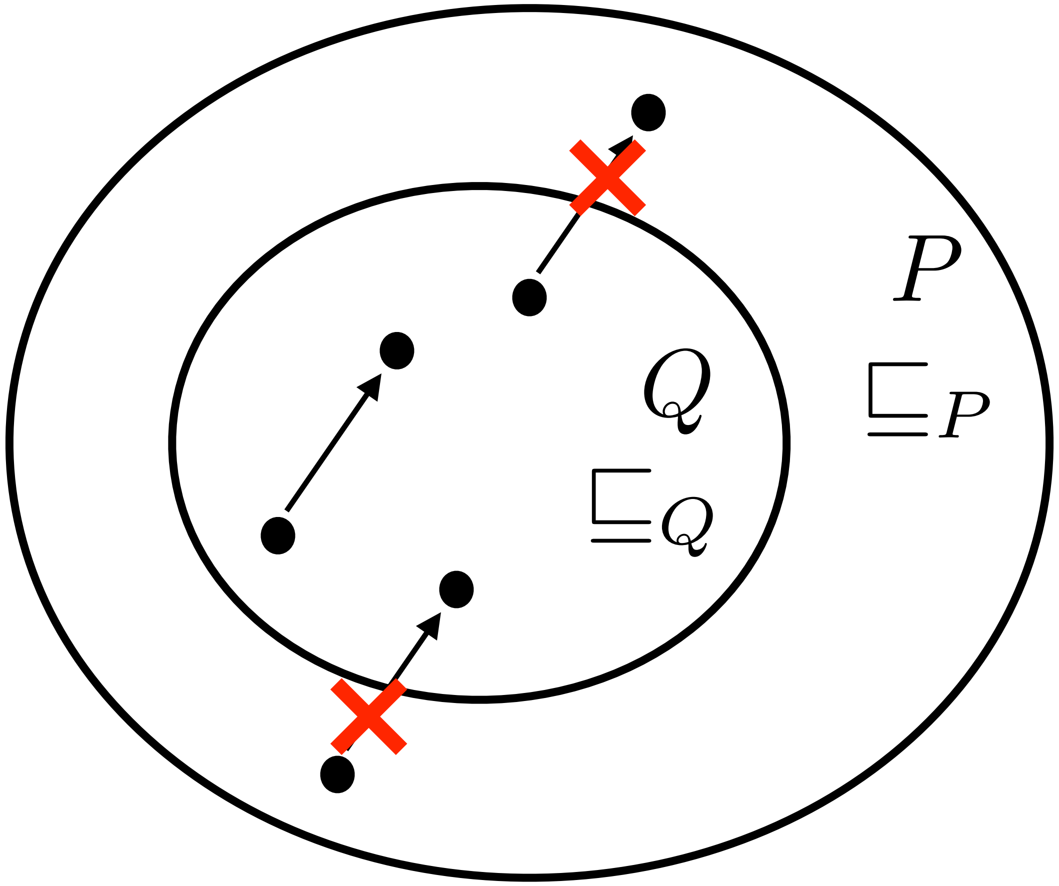
Subset of a PO

$(P, \sqsubseteq_P)$ PO $\qquad Q \subseteq P$

let $\quad \sqsubseteq_Q \triangleq \sqsubseteq_P \cap (Q \times Q)$

TH. $(Q, \sqsubseteq_Q)$ is a PO

TH. if $(P, \sqsubseteq_P)$ is total, then $(Q, \sqsubseteq_Q)$ is total



PO $\sqsubseteq$	w.f. $\prec$
reflexive	not reflexive (otherwise cycle!)
antisymmetric	antisymmetric (otherwise cycle!) $p \prec q \wedge q \prec p$ is always false
transitive	can be transitive ($\prec^+$ w.f.)
has infinite descending chains (if nonempty)	no infinite descending chain
	$\prec^*$ is always a PO
$\sqsubset$ can be w.f.	

Element properties
(least, minimal, …)

Least element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell \in Q$

ℓ is a least element of Q if $\forall q \in Q. \quad \ell \sqsubseteq q$

TH. (uniqueness of least element)

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell_1, \ell_2$ least elements of Q implies $\ell_1 = \ell_2$

$$\left.\begin{array}{lll} \ell_1 \text{ least element of } Q & \Rightarrow & \ell_1 \sqsubseteq \ell_2 \\ \ell_2 \text{ least element of } Q & \Rightarrow & \ell_2 \sqsubseteq \ell_1 \end{array}\right\} \Rightarrow \ell_1 = \ell_2$$

by antisymmetry

Bottom

$(P, \sqsubseteq)$ PO — the least element of P (if it exists) is called bottom and denoted $\bot$

sometimes written $\bot_P$

Examples

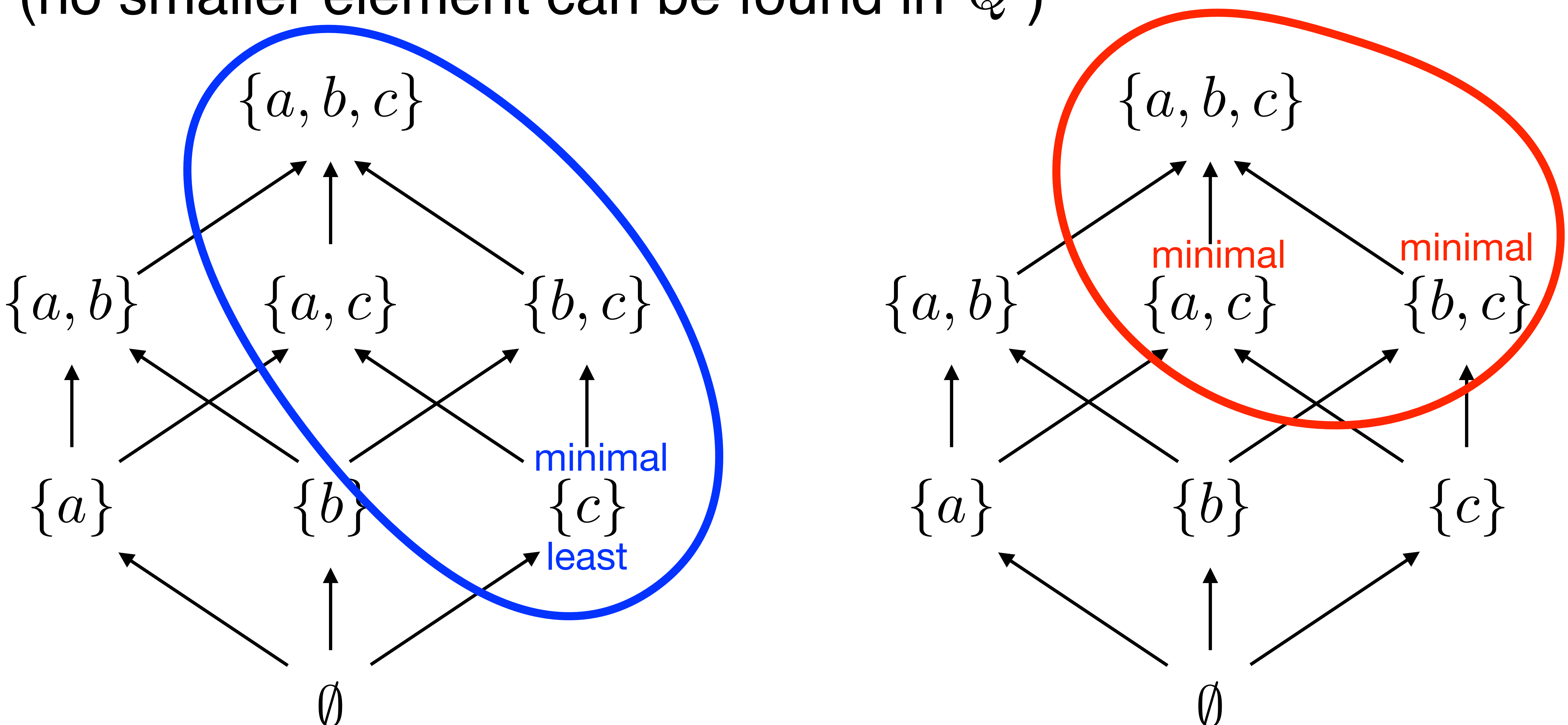
PO	bottom?
$(\mathbb{N} \cup \{\infty\}, \leq)$	0
$(\wp(S), \subseteq)$	$\varnothing$
$(\mathbb{Z}, \leq)$	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	$-\infty$

Minimal element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad m \in Q$

m is a minimal element of Q if $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$

(no smaller element can be found in Q)



Least vs minimal

least $\forall q \in Q. \quad \ell \sqsubseteq q$	minimal $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$
unique	not necessarily unique
minimal	
	not necessarily least can be least

Reverse order

TH. $(P, \sqsubseteq)$ PO implies $(P, \sqsupseteq)$ PO

note:
$\sqsupseteq \; = \; \sqsubseteq^{-1}$

proof. it is immediate to check that $\sqsupseteq$

is reflexive
is antisymmetric
is transitive

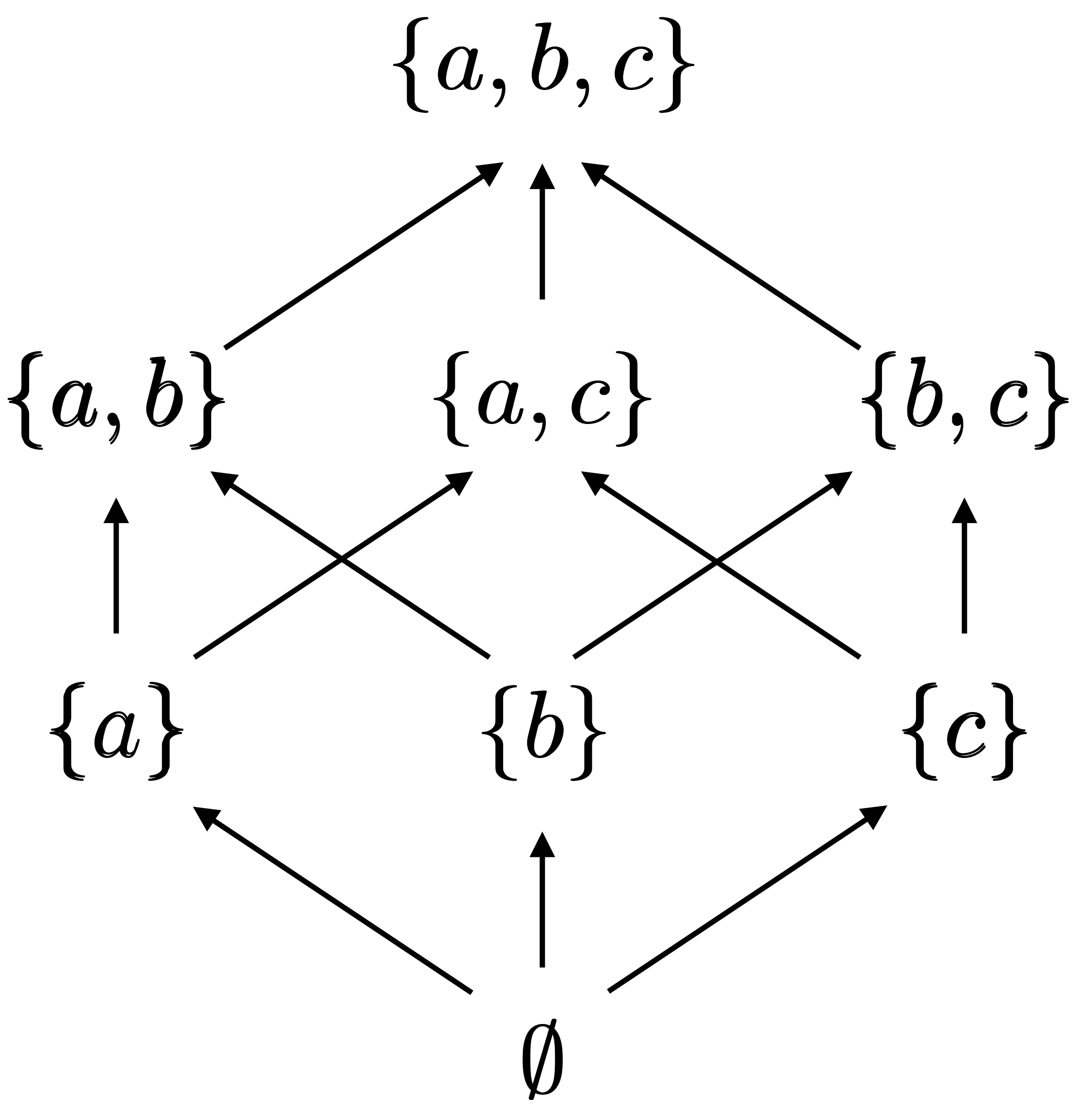
$(P, \sqsubseteq)$ PO $\quad Q \subseteq P$

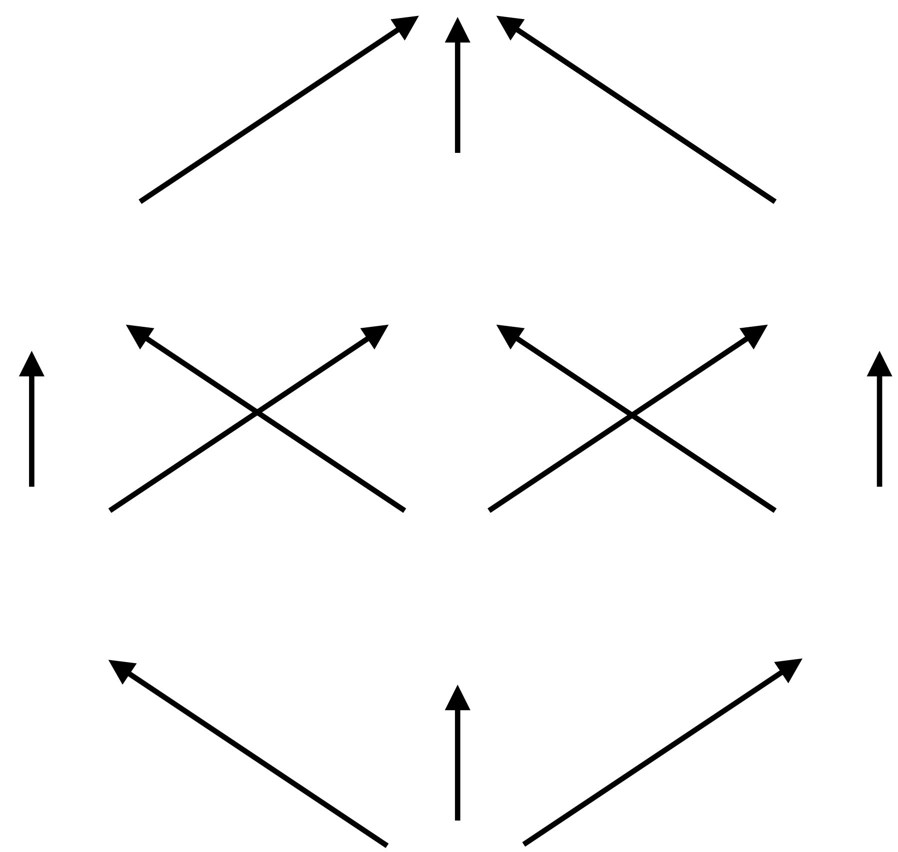
greatest element: least element of Q w.r.t. $(P, \sqsupseteq)$

top element: $\top$ greatest element of P (if it exists)

maximal element: minimal element of Q w.r.t. $(P, \sqsupseteq)$

Reversed powerset



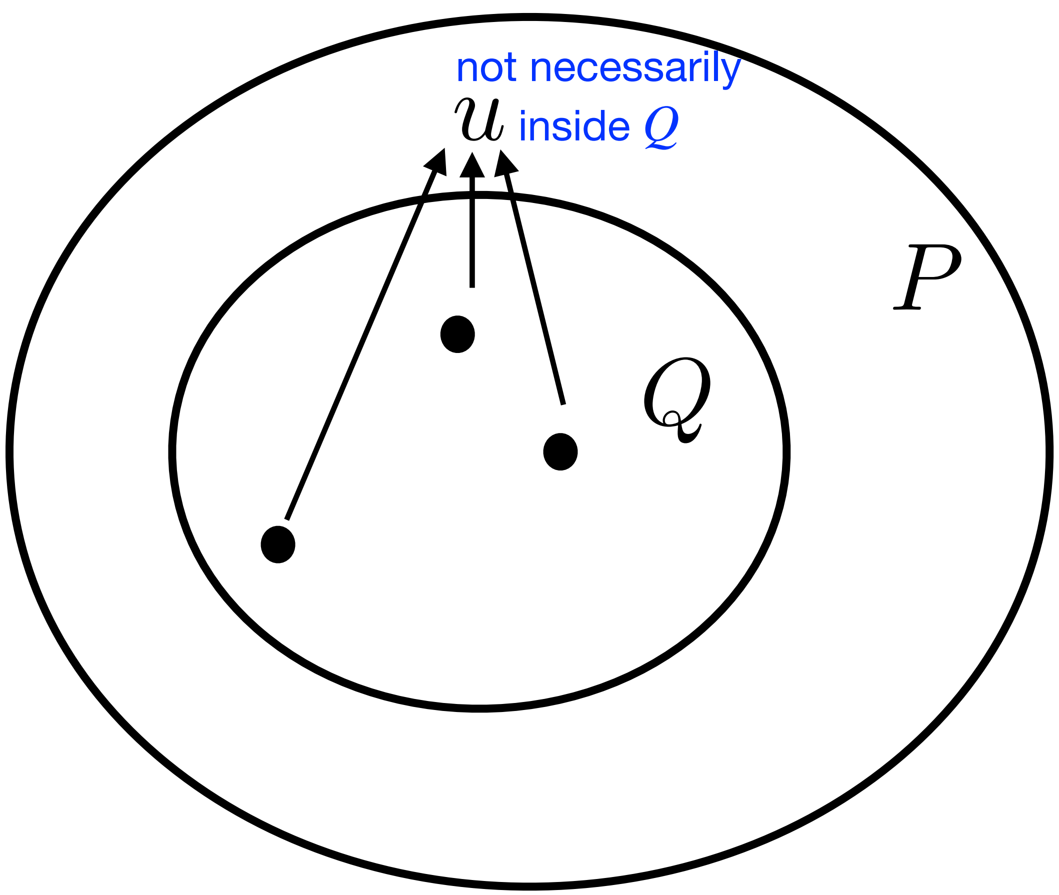


Upper bound

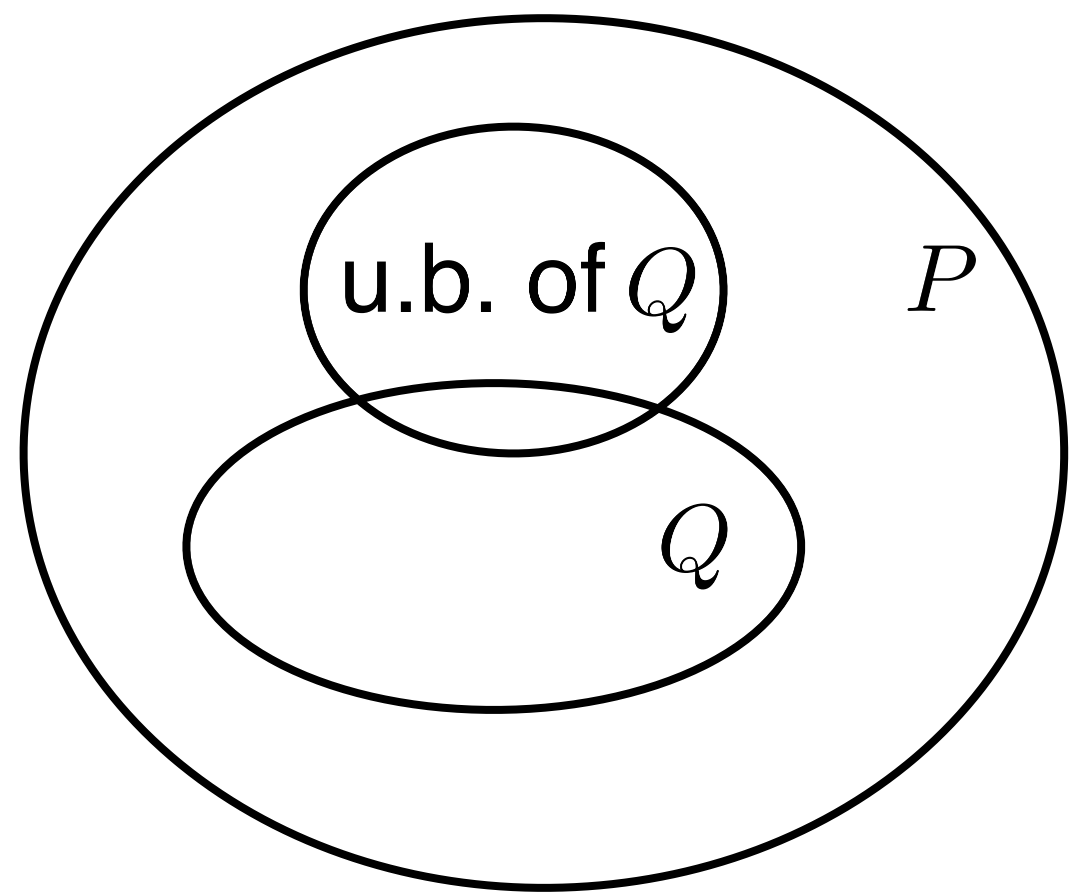
$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad u \in P$

u is an upper bound of Q if $\forall q \in Q.\ \ q \sqsubseteq u$

(all the elements of Q are smaller than u)



Q may have many upper bounds



Least upper bound

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad p \in P$

p is the least upper bound (lub) of Q if

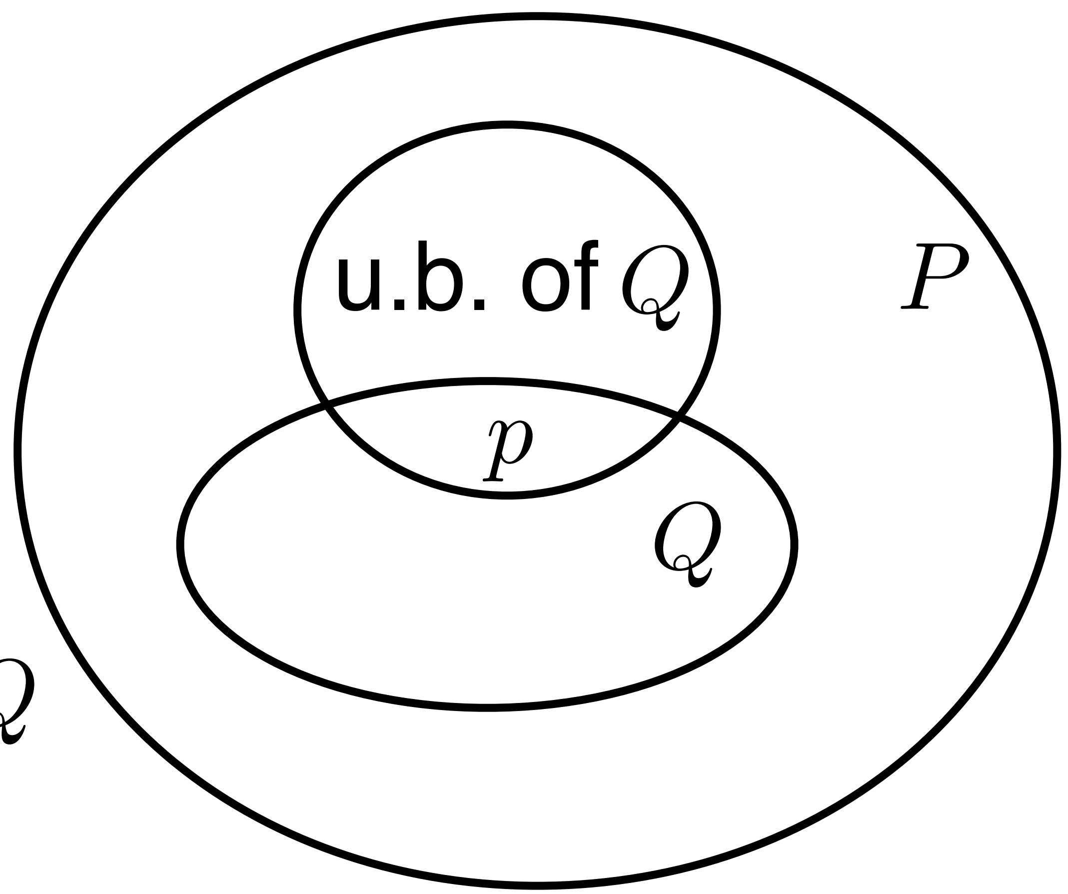
1. it is an upper bound of Q $\quad \forall q \in Q. \quad q \sqsubseteq p$
2. it is smaller than any other upper bound of Q

$$\forall u \in P. \quad (\forall q \in Q. q \sqsubseteq u) \Rightarrow p \sqsubseteq u$$

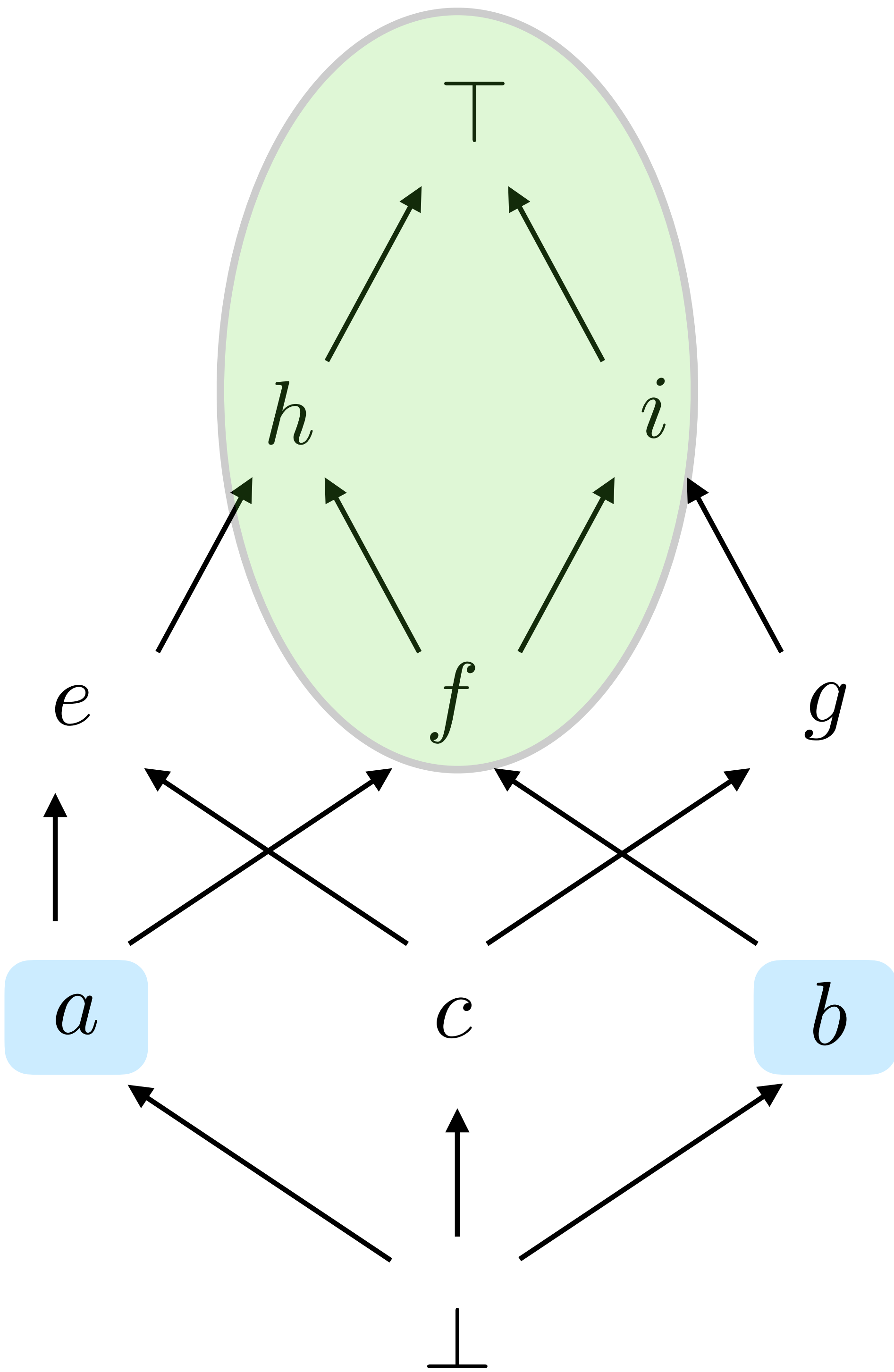
we write $p = lub\ Q$

intuitively, it is the least element that represents all of Q

p not necessarily an element of Q



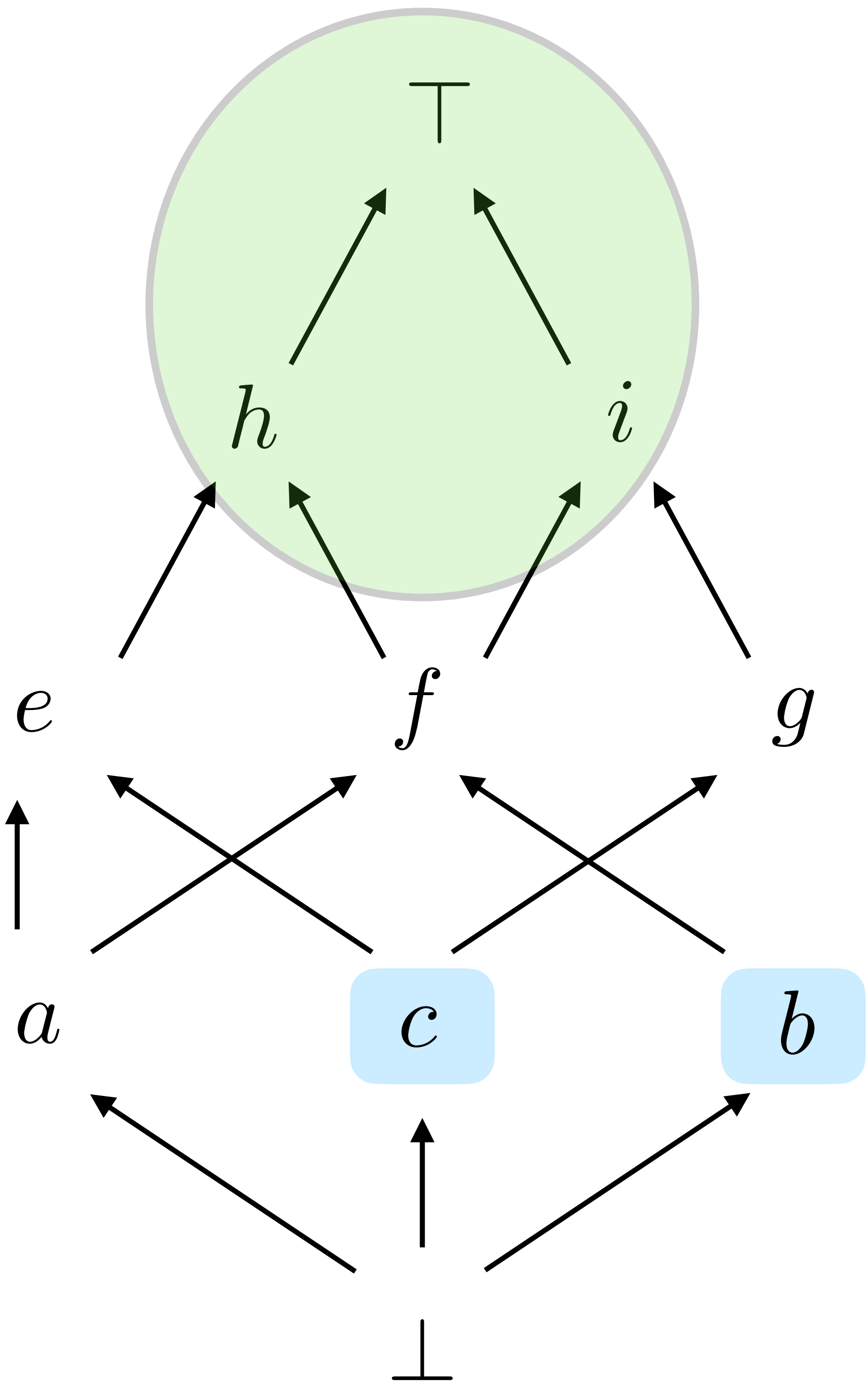
Exercise



Upper bounds of $\{a, b\}$? $\{f, h, i, \top\}$

lub? f

Exercise



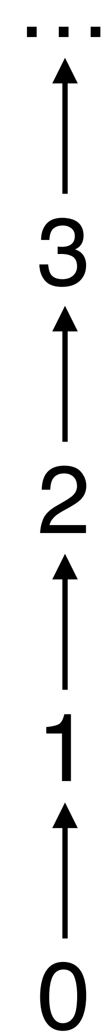
Upper bounds of $\{b, c\}$? $\{h, i, \top\}$

lub?   no lub!

Exercise

$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub? if Q finite $lub\ Q = \max Q$

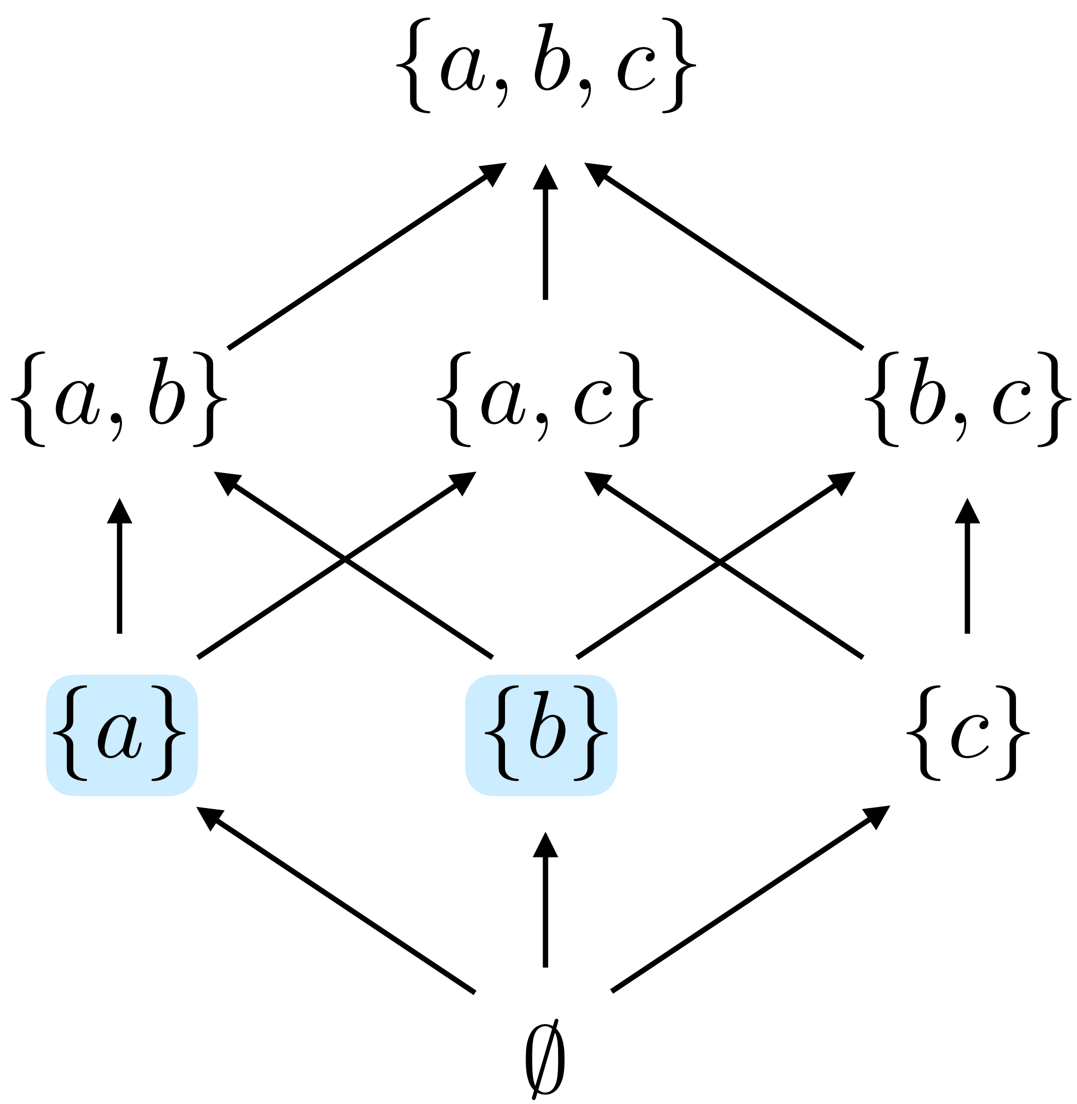
otherwise no lub



Exercise

$(\wp(S), \subseteq)$ $\quad Q \subseteq \wp(S)$ lub?

$$lub\ Q = \bigcup_{T \in Q} T$$



$$lub\ \{\{a\}, \{b\}\} = \{a, b\}$$

Complete partial orders (CPO)

Completeness: the idea

D a domain

$\sqsubseteq$ a way to compare element

$x \sqsubseteq y$ x is a (less precise) approximation of y

x and y are consistent,
but y is more accurate than x

PO

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$$

(n+1)th approximation

third approximation

second approximation

first approximation

does any sequence of approximations tend to some limit?

Chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ is a chain if $\forall i \in \mathbb{N}.\ d_i \sqsubseteq d_{i+1}$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

any chain is an infinite list

finite chain: there are only finitely many distinct elements

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_{i+1}$$

or equivalently

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_k$$

Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$ is an infinite chain

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$ is a finite chain

any chain has infinite length

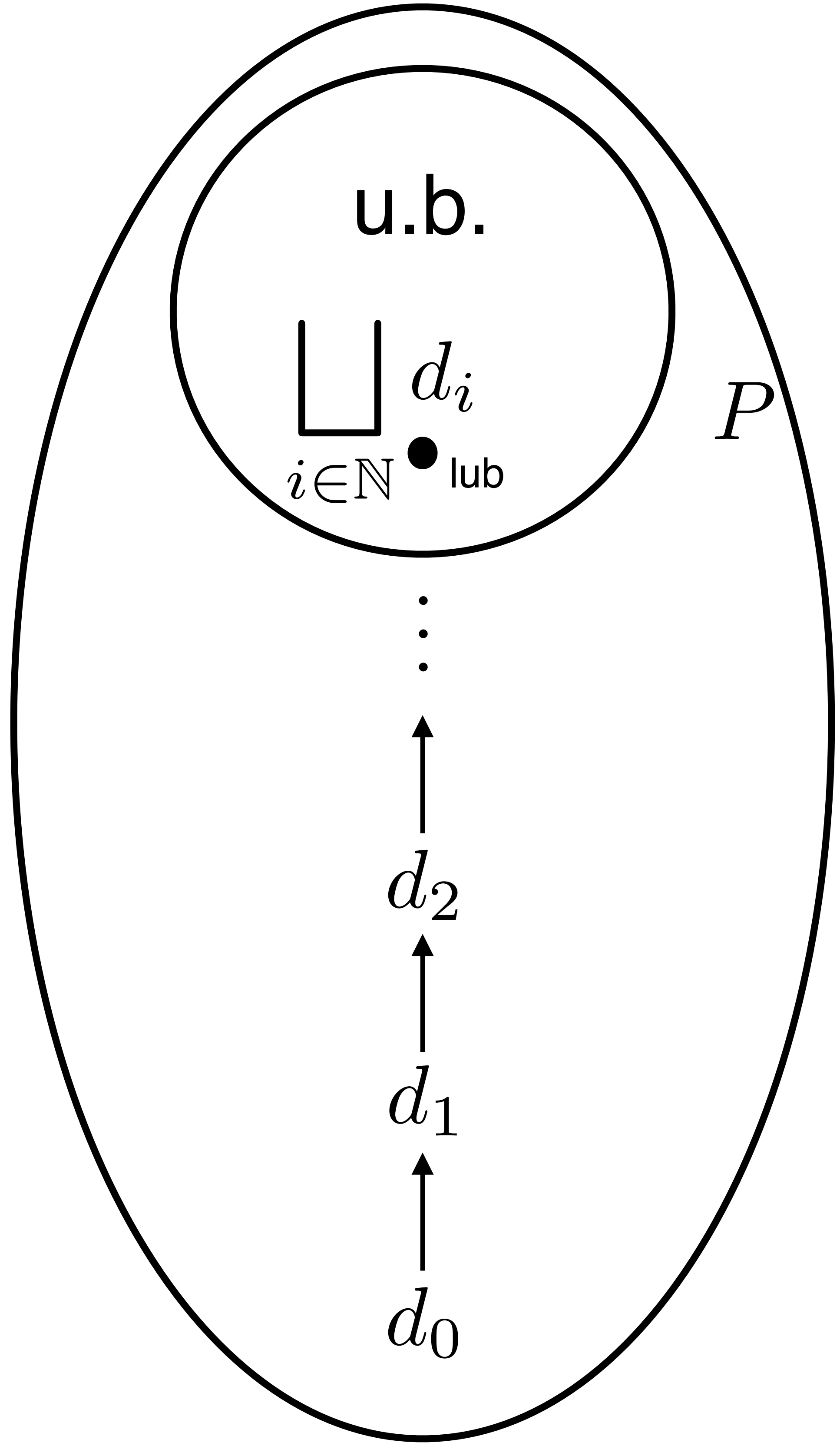
Limit of a chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

we denote by $\bigsqcup_{i \in \mathbb{N}} d_i$ the lub of $\{d_i\}_{i \in \mathbb{N}}$ if it exists

and call it the limit of the chain

Limit illustrated



Example

$(\mathbb{N}, \leq)$

$$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$$

has no lub
(empty set of upper bounds)

$$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$$

has lub 5
(which upper bounds?)

Lemma on finite chains

Lemma (any finite chain has a limit)

$(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ a finite chain $\Rightarrow$ $\bigsqcup_{i\in\mathbb{N}} d_i$ exists

proof.

$\{d_i\}_{i\in\mathbb{N}}$ finite $\Rightarrow$ $\exists k.\ \forall i.\ d_{i+k} = d_k$

the elements of the chain are totally ordered

d_k is the greatest element of the chain

d_k is an upper bound $\forall i.\ d_i \sqsubseteq d_k$

d_k is the least upper bound

take u such that $\forall i.\ d_i \sqsubseteq u$ then $d_k \sqsubseteq u$

Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

if $\bigsqcup_{i \in \mathbb{N}} d_i$ exists $\Rightarrow \forall k. \bigsqcup_{i \in \mathbb{N}} d_{i+k} = \bigsqcup_{i \in \mathbb{N}} d_i$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i \in \mathbb{N}} d_i$$

=

$$d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i \in \mathbb{N}} d_{i+k}$$

Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ a chain

$$\text{if } \bigsqcup_{i\in\mathbb{N}} d_i \text{ exists } \Rightarrow \forall k.\ \bigsqcup_{i\in\mathbb{N}} d_{i+k} = \bigsqcup_{i\in\mathbb{N}} d_i$$

proof.

take a generic k

we prove that $\{d_i\}_{i\in\mathbb{N}}$ and $\{d_{i+k}\}_{i\in\mathbb{N}}$ have the same u.b.

(and thus the same lub)

1. if u is an u.b. of $\{d_i\}_{i\in\mathbb{N}}$ then is an u.b. of $\{d_{i+k}\}_{i\in\mathbb{N}}$
   because $\{d_{i+k}\}_{i\in\mathbb{N}} \subseteq \{d_i\}_{i\in\mathbb{N}}$
2. if u is an u.b. of $\{d_{i+k}\}_{i\in\mathbb{N}}$ we need to show $\forall j.\ d_j \sqsubseteq u$
   for $j \geq k$ it is obvious
   if $j < k$ then $d_j \sqsubseteq d_k \sqsubseteq u$ because $d_k \in \{d_{i+k}\}_{i\in\mathbb{N}}$

Complete partial order

$(P, \sqsubseteq)$ PO $\quad P$ is complete if each chain has a limit (lub)

TH. Any finite chain has a limit
(the last element in the sequence)

If P has only finite chains it is complete

If P is finite it is complete

Any discrete order is complete

Any flat order is complete

Example

$(\mathbb{N}, \leq)$ is not complete
(it is enough to exhibit a chain with no limit)

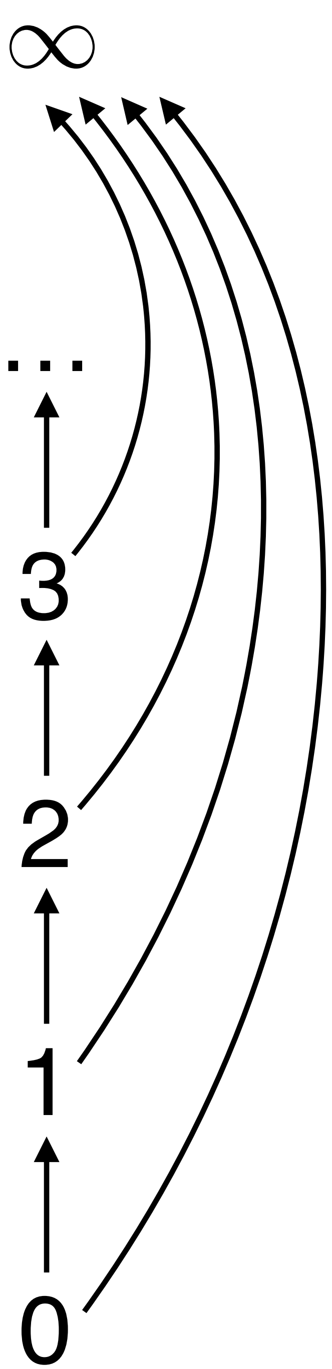
$$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$$

has no lub
(empty set of u.b.)

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

complete?



any infinite chain has limit ∞
(set of u.b. $\{\infty\}$)

Exercise

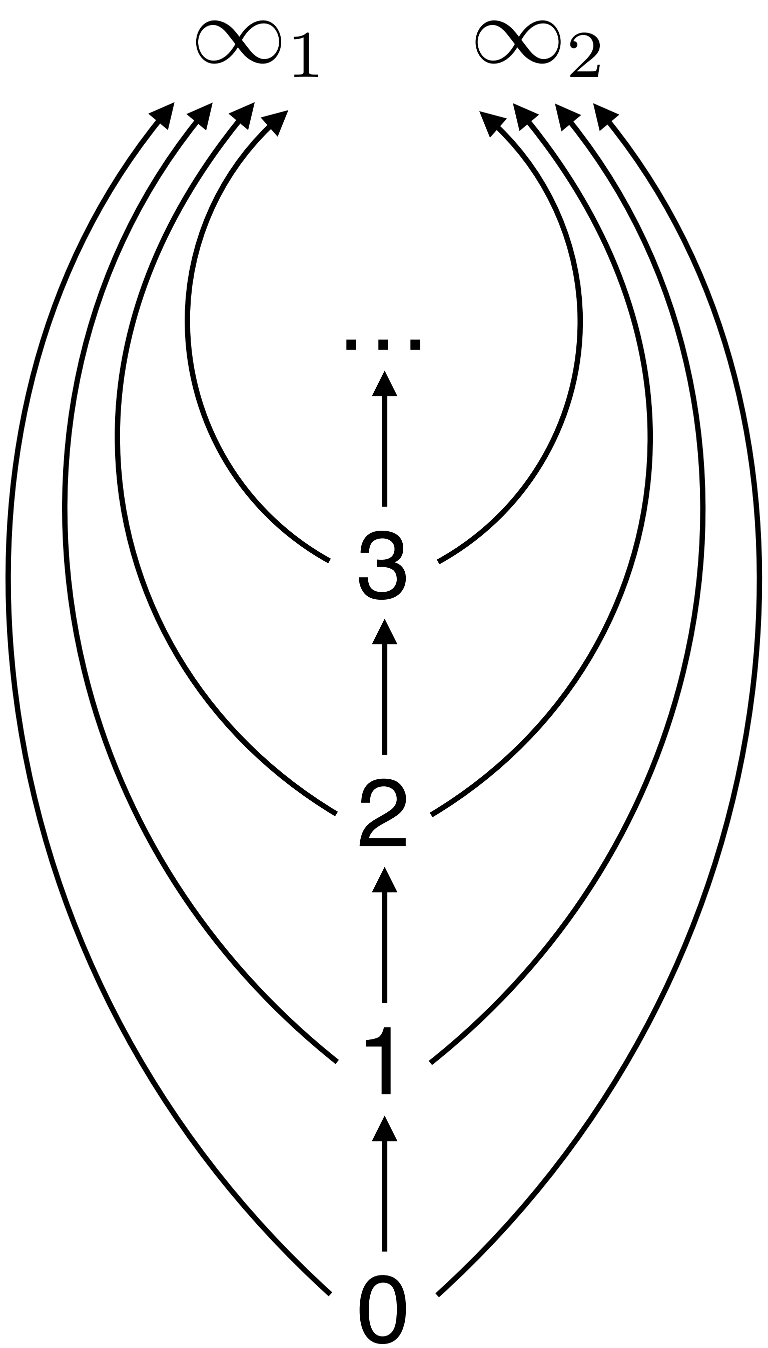
$(\wp(S), \subseteq)$ complete? ✓

$\{S_i\}_{i \in \mathbb{N}}$

$$\bigsqcup_{i \in \mathbb{N}} S_i = \bigcup_{i \in \mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$$

Exercise

$(\mathbb{N} \cup \{\infty_1, \infty_2\}, \leq)$ complete?



any infinite chain has no limit
(set of u.b. $\{\infty_1, \infty_2\}$)