PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

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08b - Kleene's fixed point theorem

Partial functions

Comparing functions

given two functions $f, g : A \to B$, when can we say $f = g$?

$$\forall a \in A \,.\, f(a) = g(a)$$

if we see functions as relations

$$\{\ (a, f(a)) \mid a \in A\ \} \ \subseteq\ A \times B$$

we can use set equality

Example

$$f(n) = n!$$

$$f(n) = n(n+1)/2$$

n $f(n)$

$$f = \{ \quad (0,1), (1,1), (2,2), (3,6), \ldots, (k, k!), \ldots \}$$

$$f = \{ \quad (0,0), (1,1), (2,3), (3,6), \ldots, (k, T_k), \ldots \}$$

Partial functions

let $f\colon A \rightharpoonup B$, or equivalently $f\colon A \to B \cup \{ \perp \}$

the function f can be undefined on some inputs

we can still see partial functions as relations

$$\{ (a, f(a)) \mid a \in A, f(a) \neq \perp \} \subseteq A \times B$$

omit pairs where $f(a)$ is undefined

Partial functions

$D = (A \rightharpoonup B) = \mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$\begin{aligned} & \quad\;\; {\color{red} n \quad f(n)} \\ f = \{ \quad & (0, 0), \\ & (2, 1), \\ & (4, 2), \\ & (6, 3), \\ & \ldots \\ & (2k, k), \\ & \ldots \qquad\qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0), (1,2), \\ & (2,1), (3,6), \\ & (4,2), (5,10), \\ & (6,3), (7,14), \\ & \ldots \\ & (2k, k), (1+2k, 2+4k), \\ & \ldots \qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$g = \{ \begin{array}{l} (0,0),(1,2), \\ (2,1),(3,6), \\ (4,2),(5,10), \\ (6,3),(7,14), \\ \ldots \\ (2k,k),(1+2k,2+4k), \\ \ldots \end{array} \}$$

$$f = \{ \begin{array}{l} (0,0), \\ (2,1), \\ (4,2), \\ (6,3), \\ \ldots \\ (2k,k), \\ \ldots \end{array} \}$$

$f \sqsubseteq g?$ ✓

$g \sqsubseteq f?$ ✗

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,0) } $\sqsubseteq$ { (0,0), (1,1) } $\sqsubseteq \cdots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,0) } $\sqsubseteq$ { (0,0), (1,1) } $\sqsubseteq$ { (0,0), (1,1), (2,2) } $\sqsubseteq$ ...

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \{ (0,0), (1,1), (2,2) \} \sqsubseteq \{ (0,0), (1,1), (2,2), (3,3) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

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which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \left\{ \begin{array}{l} (0,1), \\ (1,1) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (0,1), \\ (1,1), \\ (2,2) \end{array} \right\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{(0,1)\} \sqsubseteq \{(0,1), (1,1)\} \sqsubseteq \{(0,1), (1,1), (2,2)\} \sqsubseteq \{(0,1), (1,1), (2,2), (3,6)\} \sqsubseteq \{(0,1), (1,1), (2,2), (3,6), (4,24)\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (3,6), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (1,1) \} \sqsubseteq \{ (1,1), (2,4) \} \sqsubseteq \{ (1,1), (2,4), (3,81) \} \sqsubseteq \{ (1,1), (2,4), (3,81), (4,256) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (4,2) \} \sqsubseteq \{ (4,2), (6,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3), (10,5) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (1,6) \} \sqsubseteq \{ (1,6), (2,28) \} \sqsubseteq \{ (1,6), (2,28), (3,496) \} \sqsubseteq \{ (1,6), (2,28), (3,496), (4,8128) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Functional property

$\mathbf{Pf}(A,B) = \{f : A \rightharpoonup B\}$ partial functions

$$\mathbf{Pf}(A,B) = \{f \subseteq A \times B \mid \forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2\}$$

functional property

$f(a)\downarrow \triangleq \exists b \in B.\ (a,b) \in f$ function f is defined on a

$$f \sqsubseteq g \Leftrightarrow (\forall a \in A. \quad f(a)\downarrow \ \Rightarrow \ (g(a)\downarrow \wedge f(a) = g(a)))$$
$$\Leftrightarrow f \subseteq g$$

$(\mathbf{Pf}(A,B), \sqsubseteq)$ is a PO with bottom

what is bottom?

is it complete?

the empty relation
(the function always undefined)

Is Pf complete?

$(\mathbf{Pf}(A,B),\sqsubseteq)$

complete?

Given a chain $\{f_i\}_{i\in\mathbb{N}}$ let us consider $\bigcup_{i\in\mathbb{N}} f_i \subseteq A\times B$

we want to prove that $\bigcup_{i\in\mathbb{N}} f_i \in \mathbf{Pf}(A,B)$

i.e. that $f=\bigcup_{i\in\mathbb{N}} f_i$ satisfies the functional property

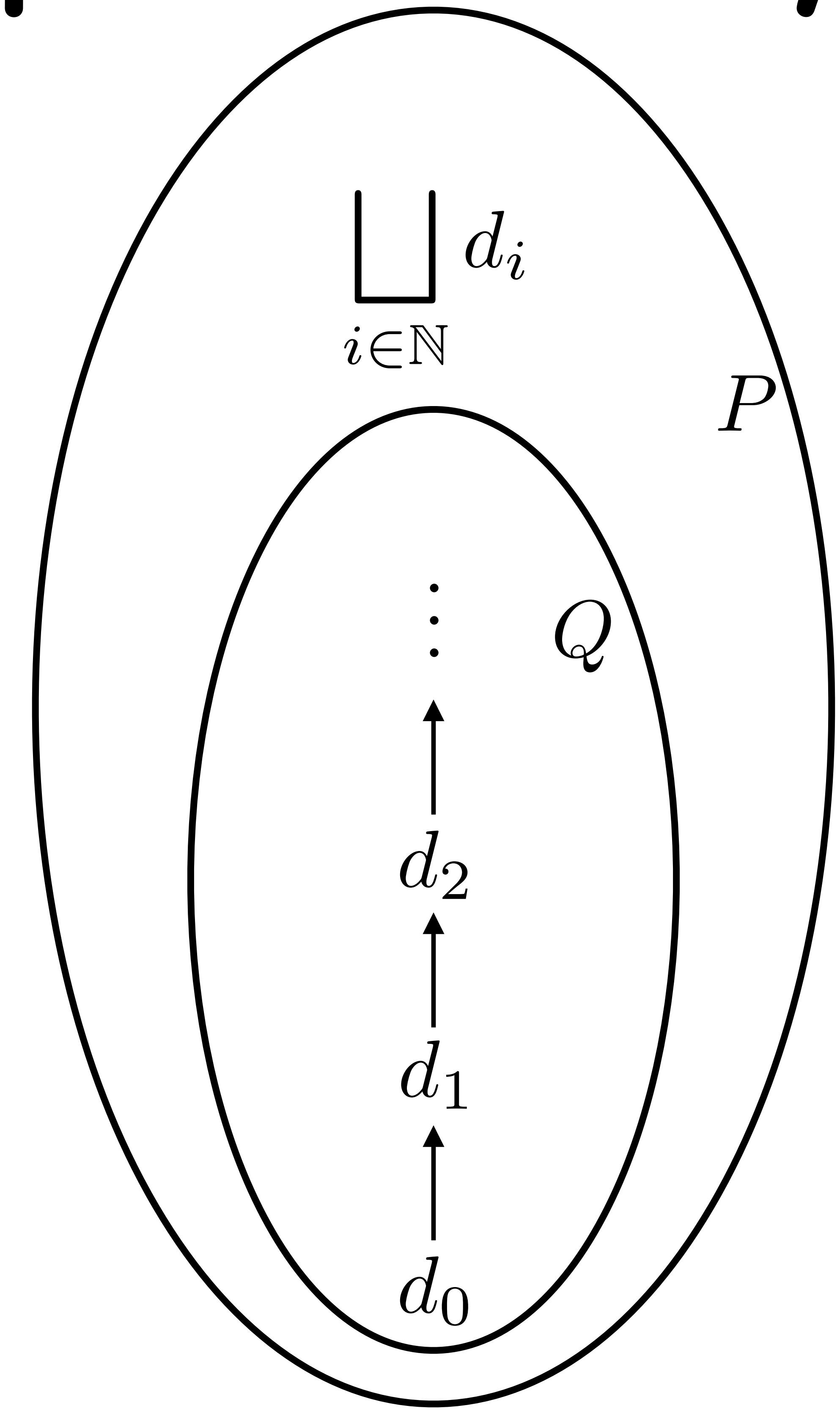
we know that each f_i is functional

$$\forall i\in\mathbb{N}.\forall a\in A.\forall b_1,b_2\in B.(a,b_1)\in f_i \wedge (a,b_2)\in f_i \Rightarrow b_1=b_2$$

we need to prove f is functional

$$\forall a\in A.\forall b_1,b_2\in B.(a,b_1)\in f \wedge (a,b_2)\in f \Rightarrow b_1=b_2$$

pictorially



is the limit in Q ?

Pf is complete

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

Take $a \in A, b_1, b_2 \in B$ such that $(a, b_1) \in f \wedge (a, b_2) \in f$

we need to prove $b_1 = b_2$

$$(a, b_1) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists k \in \mathbb{N}. (a, b_1) \in f_k$$

$$m \triangleq \max\{k, h\}$$

$$(a, b_2) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists h \in \mathbb{N}. (a, b_2) \in f_h$$

Clearly $f_k \subseteq f_m$ $\quad$ $f_h \subseteq f_m$ $\quad$ f_m is functional

$$(a, b_1) \in f_m \qquad (a, b_2) \in f_m \qquad \Rightarrow \qquad b_1 = b_2$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$
\begin{array}{rll}
f_0 \quad \emptyset & \subseteq & \{(0,1)\} \; f_1 \\
& \subseteq & \{(0,1),(1,1)\} \; f_2 \\
& \subseteq & \{(0,1),(1,1),(2,2)\} \; f_3 \\
& \subseteq & \{(0,1),(1,1),(2,2),(3,6)\} \; f_4 \\
& \subseteq & \{(0,1),(1,1),(2,2),(3,6),(4,24)\} \; f_5 \\
& \subseteq & \cdots
\end{array}
$$

$\bigcup_{i \in \mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\perp)$

$B_\perp \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_\perp} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\perp} g(a)$$

PO? immediate to check

bottom? $f_\perp(a) = \perp$ for any $a \in A$

complete? we will prove it later
(as an instance of a more general result)

$$(\bigsqcup_{i \in \mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a)$$ (flat order, limit exists)

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO $\quad (E, \sqsubseteq_E)$ PO $\quad f : D \to E$

f is monotone if $\forall d_1, d_2 \in D.\ d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

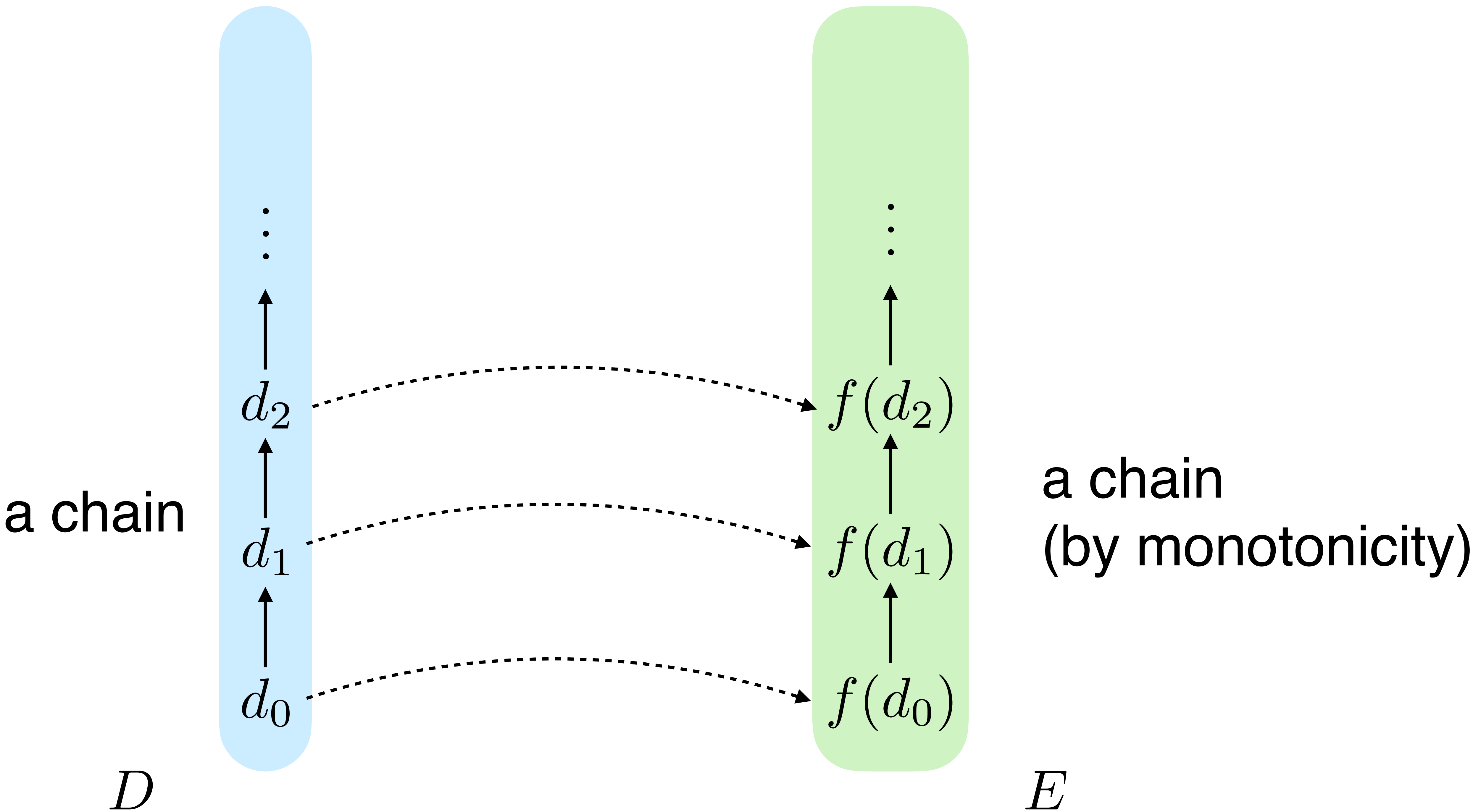
$\{d_i\}_{i \in \mathbb{N}}$ a chain in D

f monotone

$\Rightarrow \{f(d_i)\}_{i \in \mathbb{N}}$ a chain in E

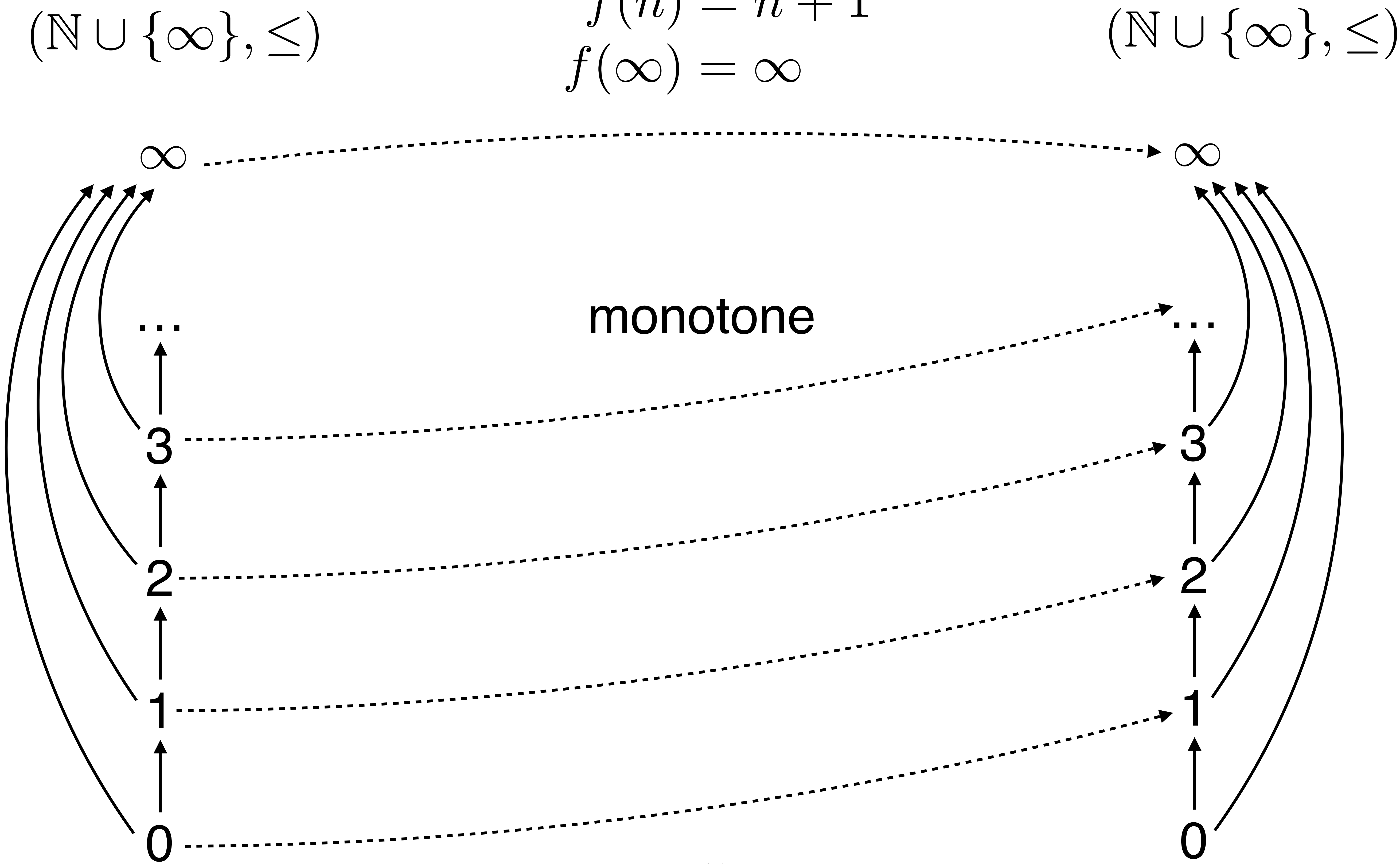
When $D = E$ we say $f : D \to D$ is a function on D

Monotonicity illustrated



Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

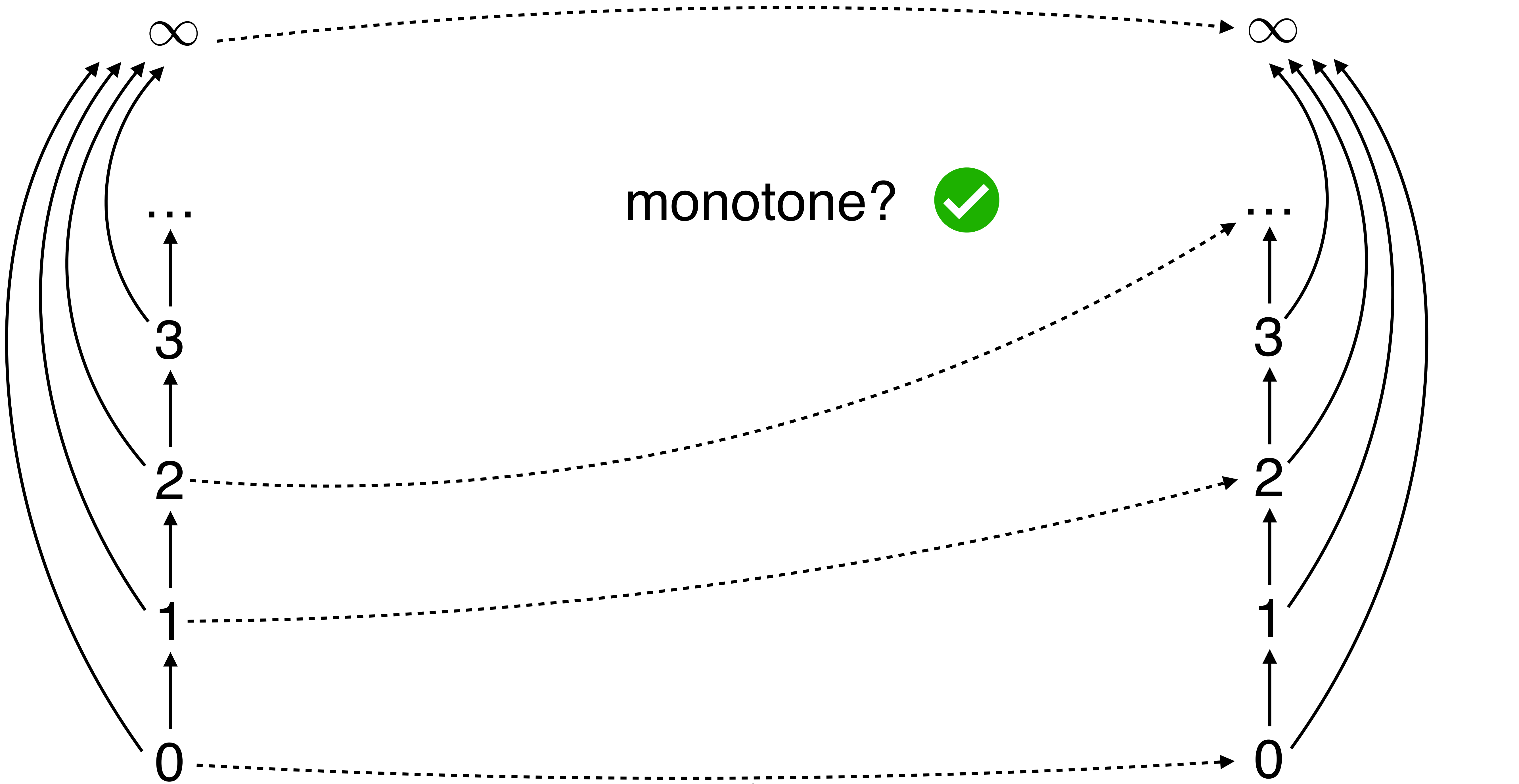


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

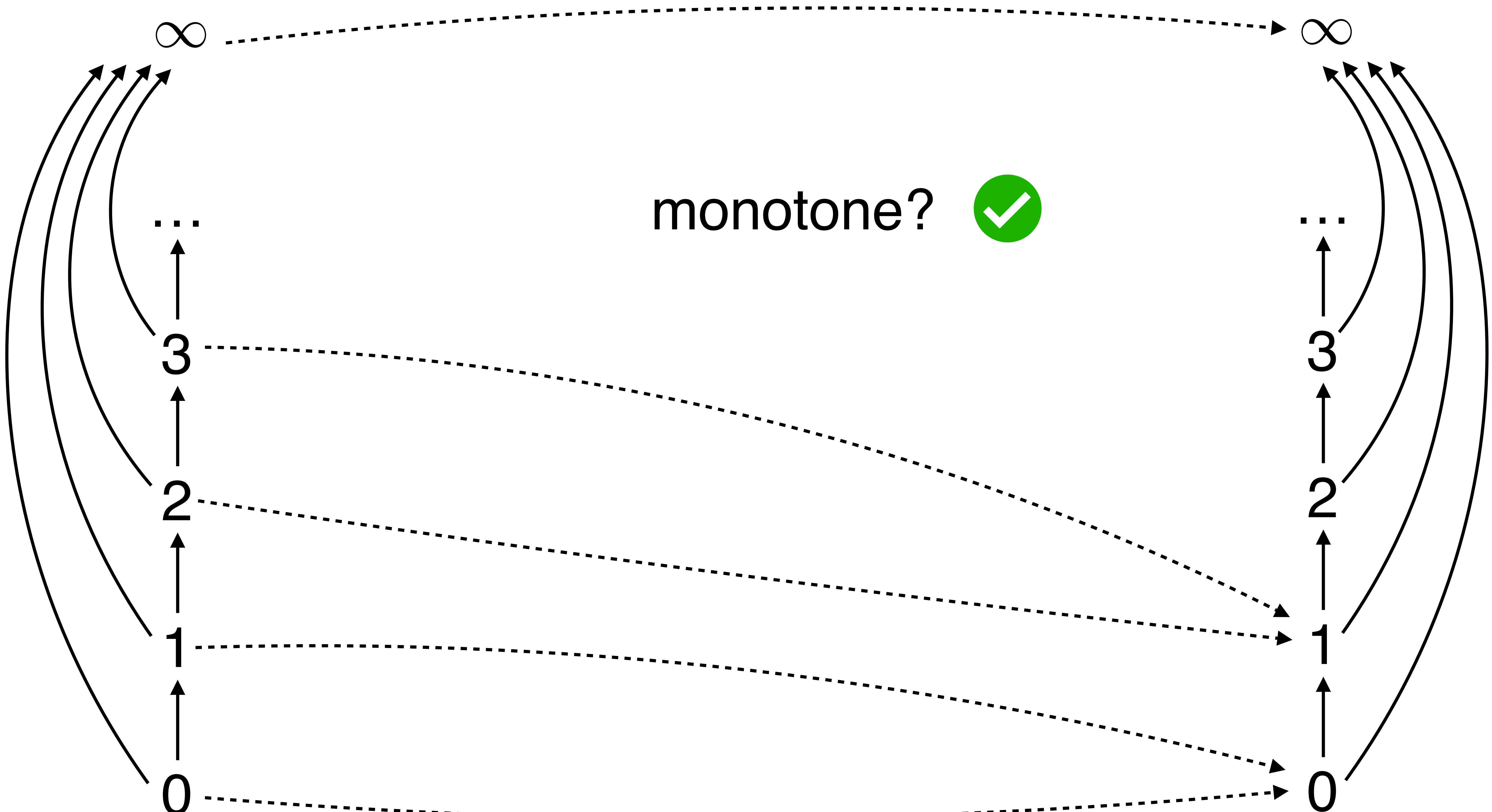
$(\mathbb{N} \cup \{\infty\}, \leq)$



Exercise

$$f(n) = n/2$$
$$f(\infty) = \infty$$

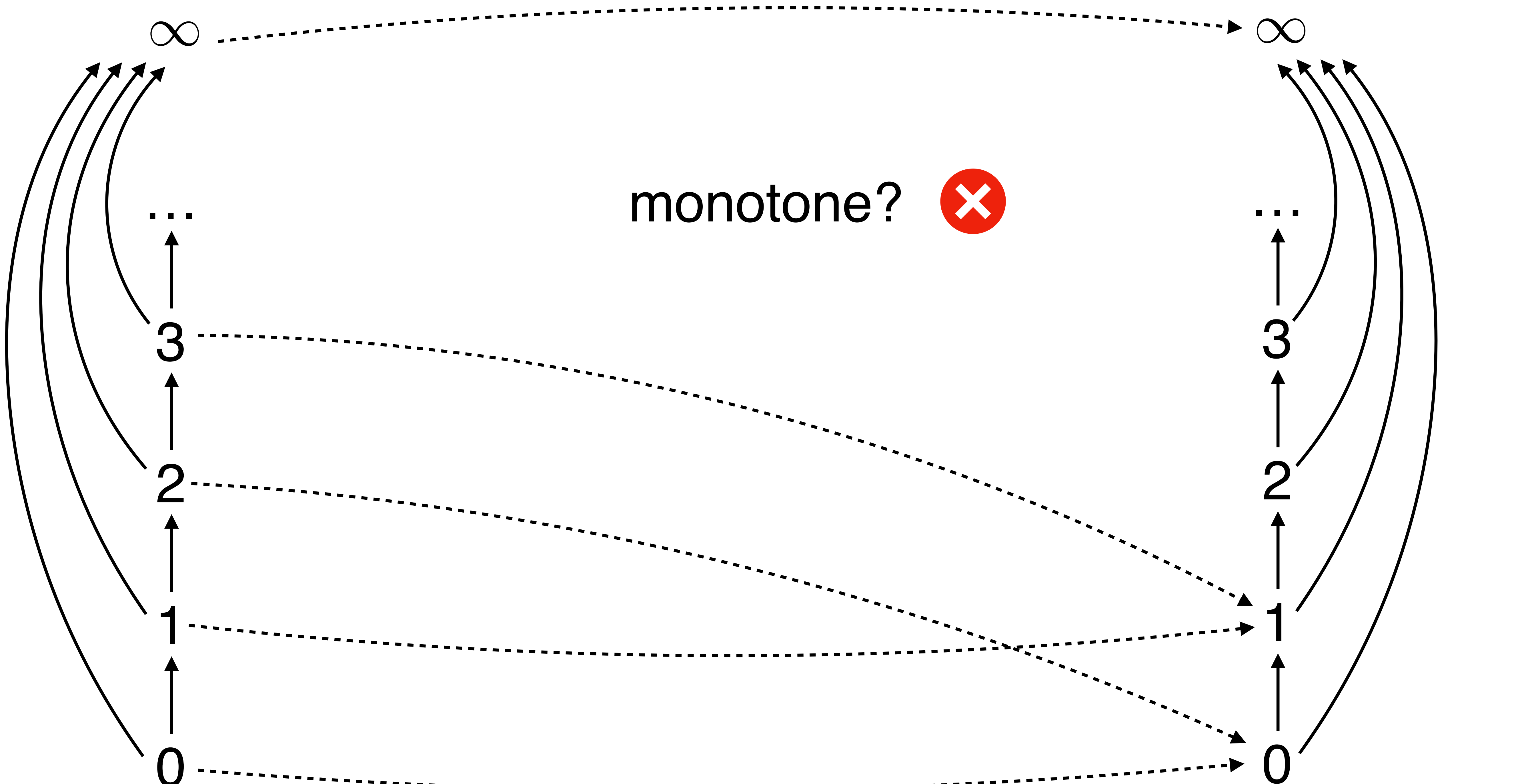
$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



Exercise

$$f(n) = n \ \% \ 2$$
$$f(\infty) = \infty$$

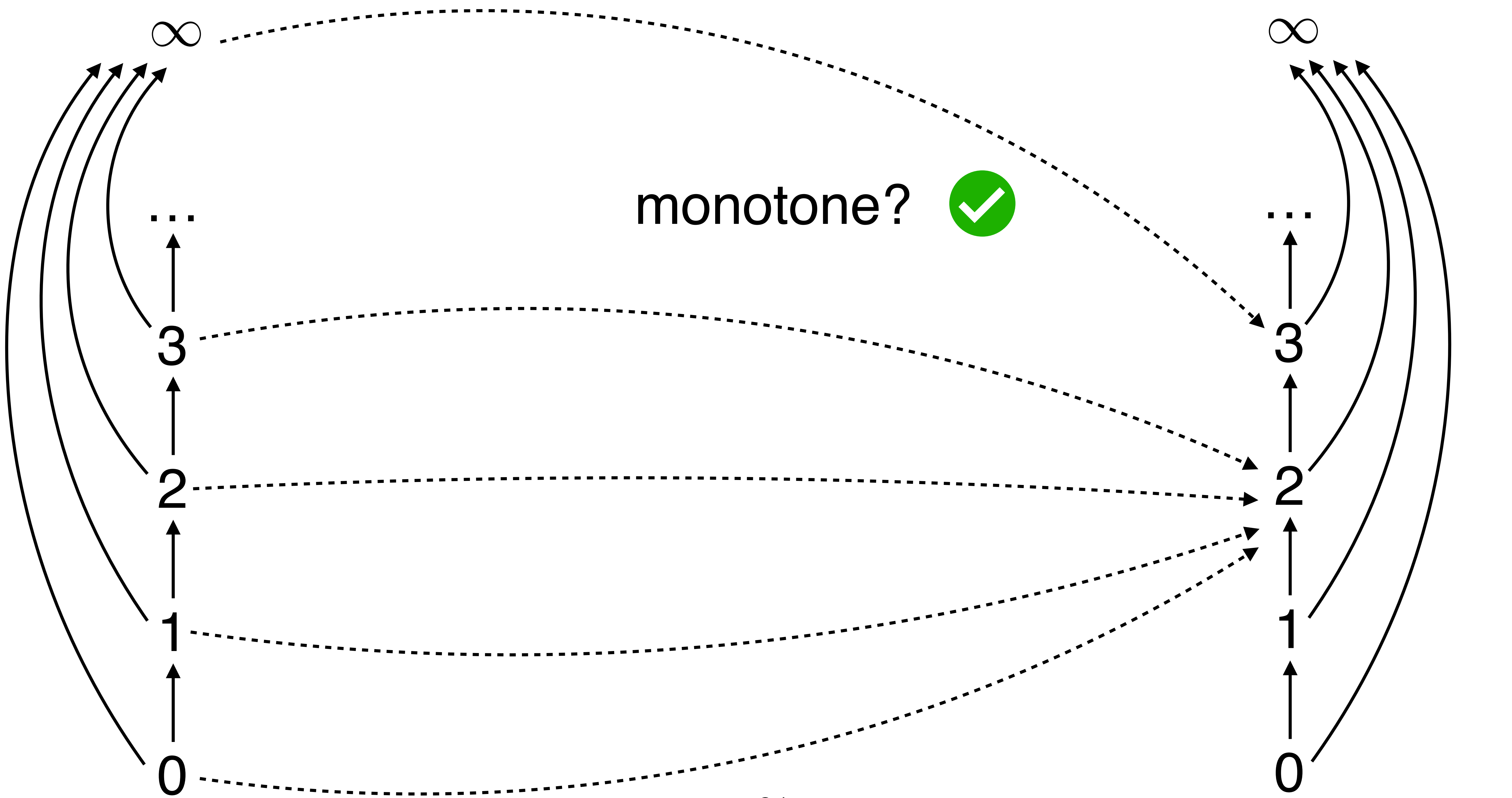
$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



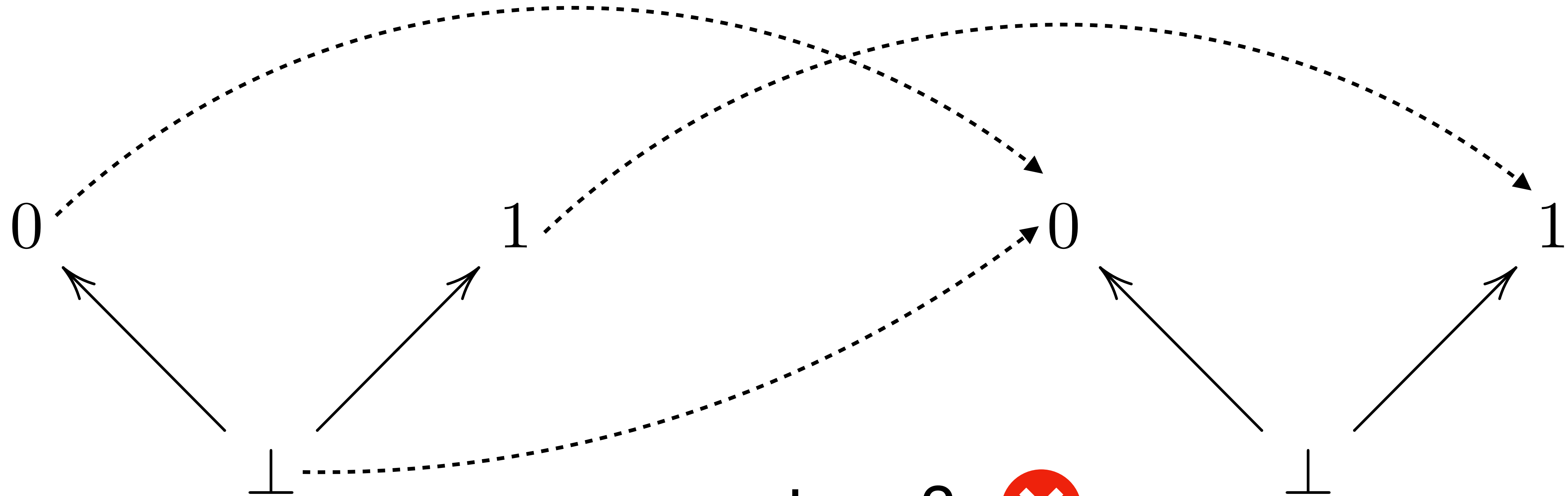
Exercise

$$f(n) = 2$$
$$f(\infty) = 3$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



Exercise



monotone? ✗

D $\qquad f : D \to D \qquad$ D

$$f(\bot) = f(0) = 0$$
$$f(1) = 1$$

$$\bot \sqsubseteq 1 \qquad\qquad f(\bot) = 0 \not\sqsubseteq 1 = f(1)$$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \qquad f(S) = \{\, m \in \mathbb{N} \mid \exists n \in S, m \leq n \,\} \qquad (\wp(\mathbb{N}), \subseteq)$$

monotone? ✓

Exercise

$$(\wp(\mathbb{N}), \subseteq) \qquad f(S) = \{\, m \in \mathbb{N} \mid \forall n \in S, n < m \,\} \qquad (\wp(\mathbb{N}), \subseteq)$$

monotone? ✗

Composition

TH. Any composition of monotone functions is monotone

$(D, \sqsubseteq_D)$ PO
$(E, \sqsubseteq_E)$ PO
$(F, \sqsubseteq_F)$ PO

$f : D \to E$ monotone
$g : E \to F$ monotone

$\Rightarrow$ $h = g \circ f : D \to F$ monotone

proof. we need to prove $\forall x, y \in D.\ x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

take $x \sqsubseteq_D y$

we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

then $g(f(x)) \sqsubseteq_F g(f(y))$ because g is monotone

$$\begin{array}{ccc} g(f(x)) & \sqsubseteq_F & g(f(y)) \\ = & & = \\ h(x) & & h(y) \end{array}$$

Continuous functions

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \to E$ monotone

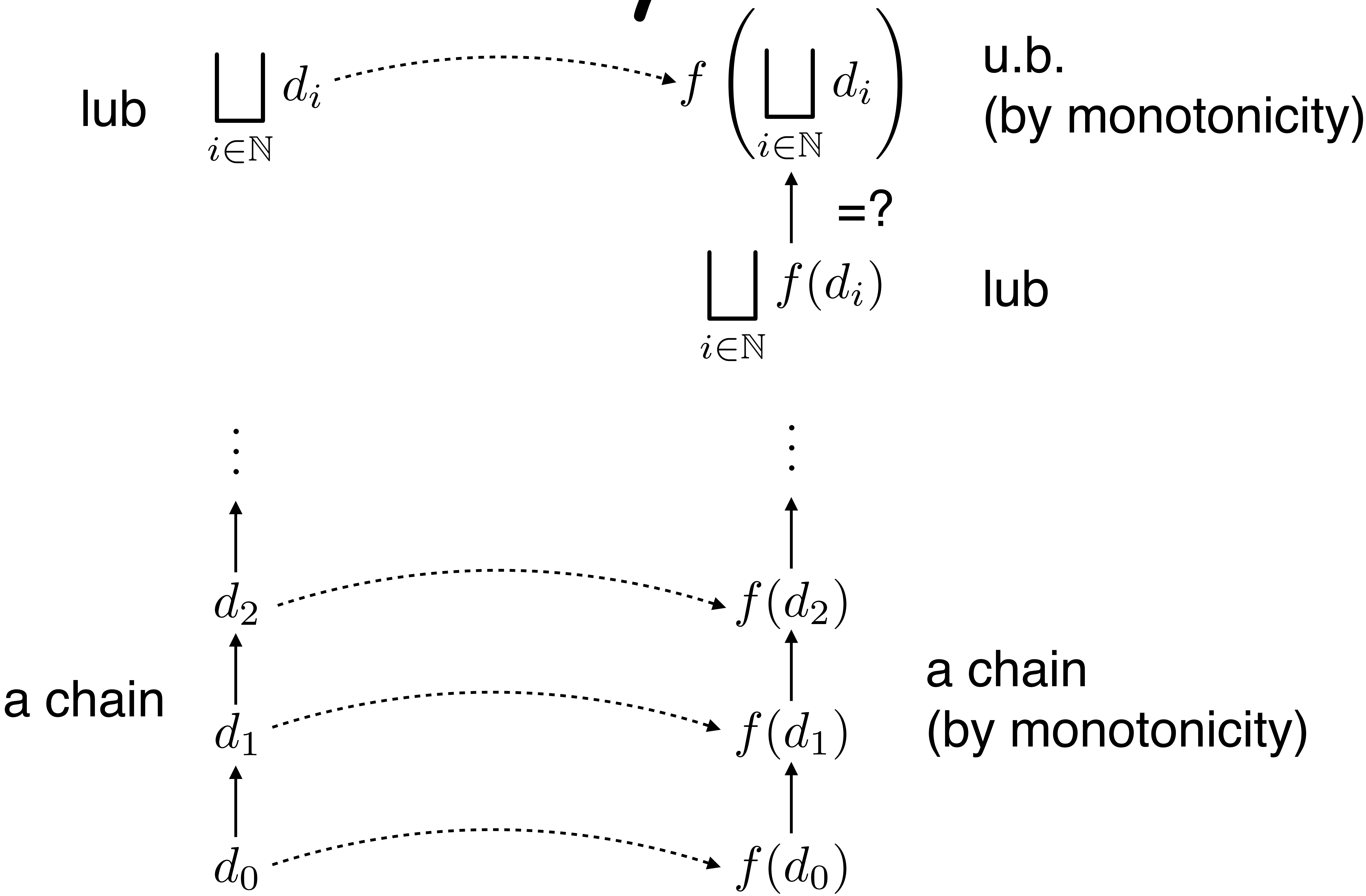
f is continuous if $\forall \{d_i\}_{i \in \mathbb{N}}.\ f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

chain

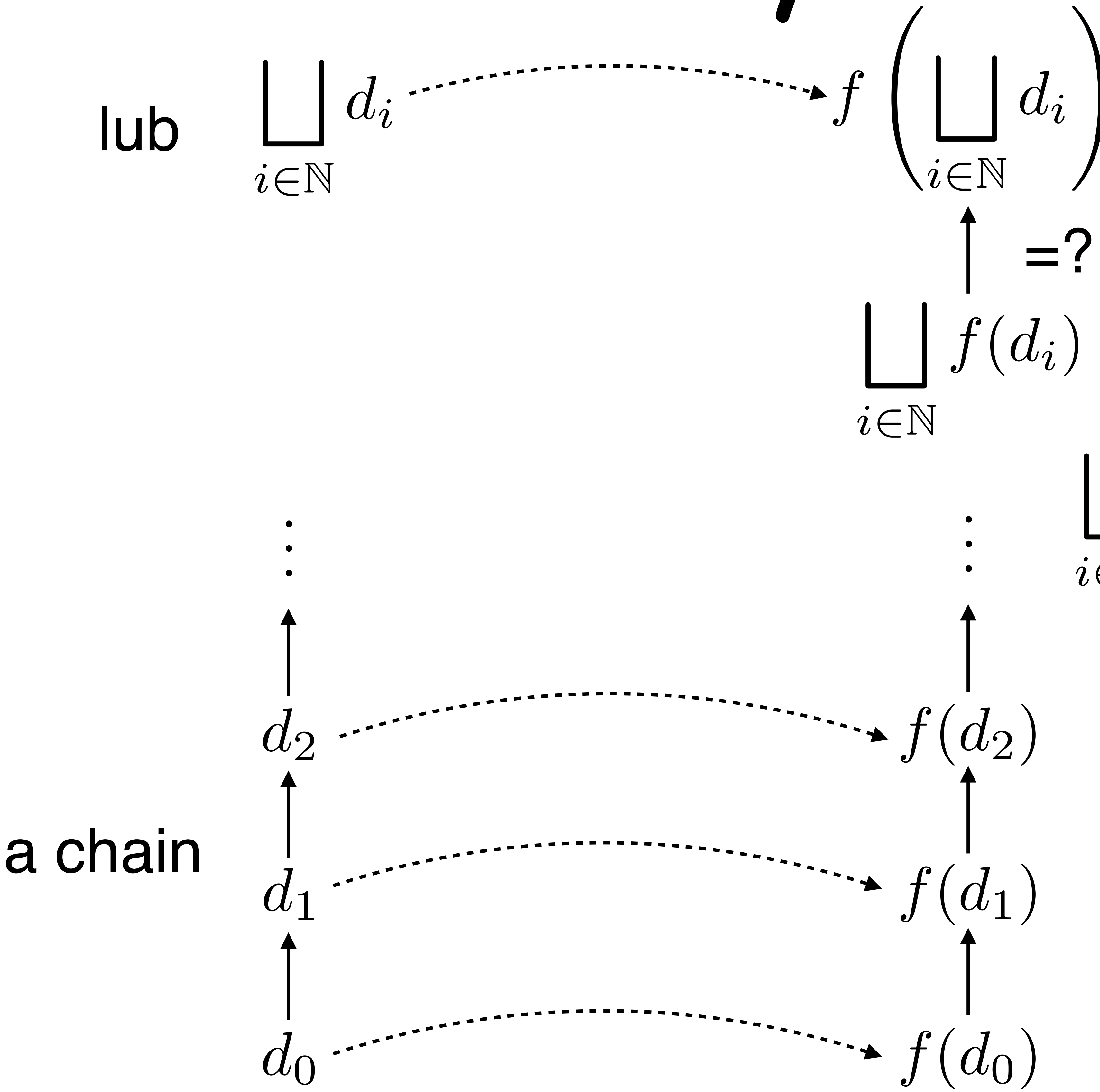
limit in D limit in E

Continuous = limit preserving

Continuity illustrated



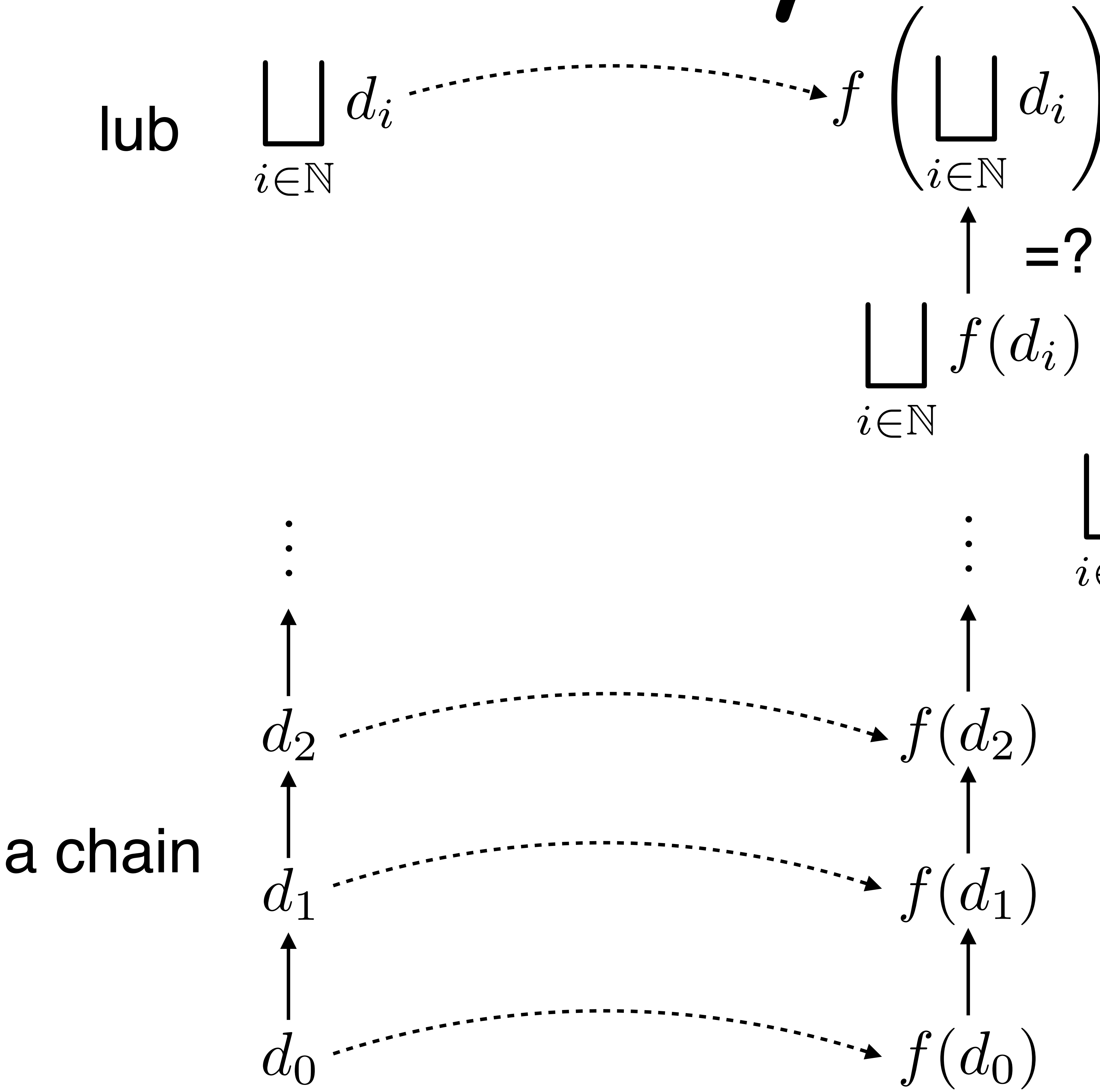
Continuity illustrated



$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \sqsubseteq_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

follows from monotonicity (and CPO)

Continuity illustrated

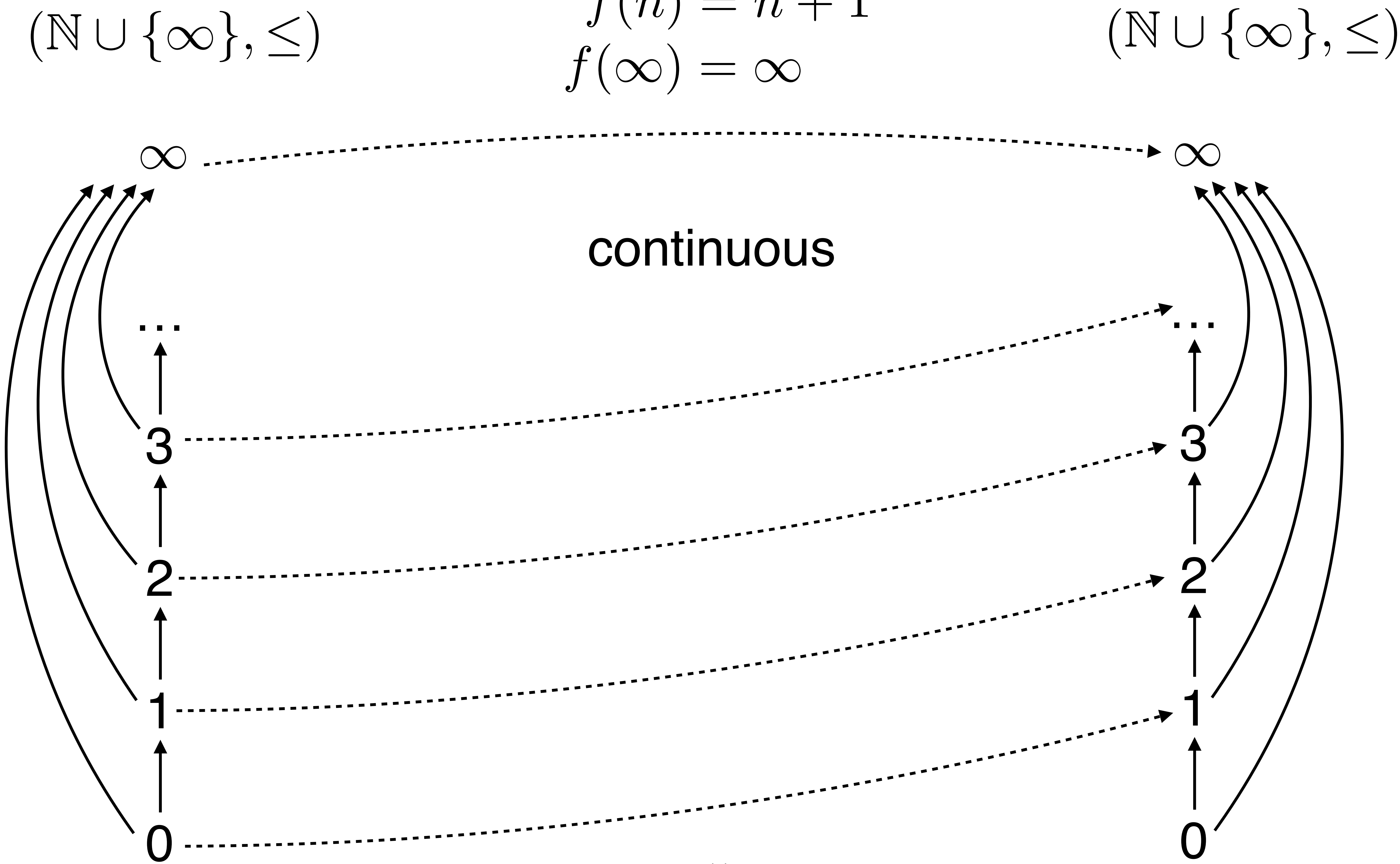


$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \overset{?}{\sqsupseteq}_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

continuity

Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

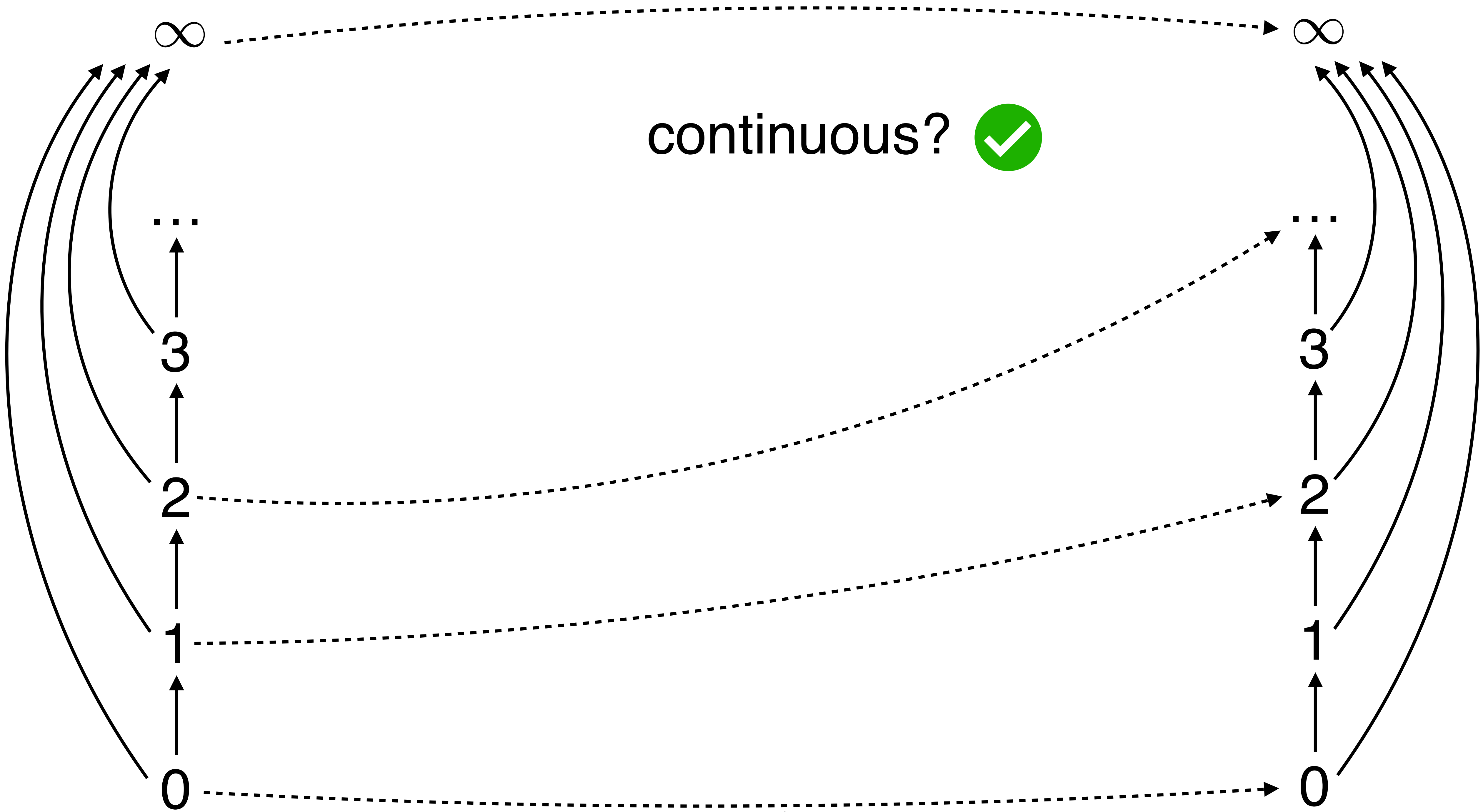


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



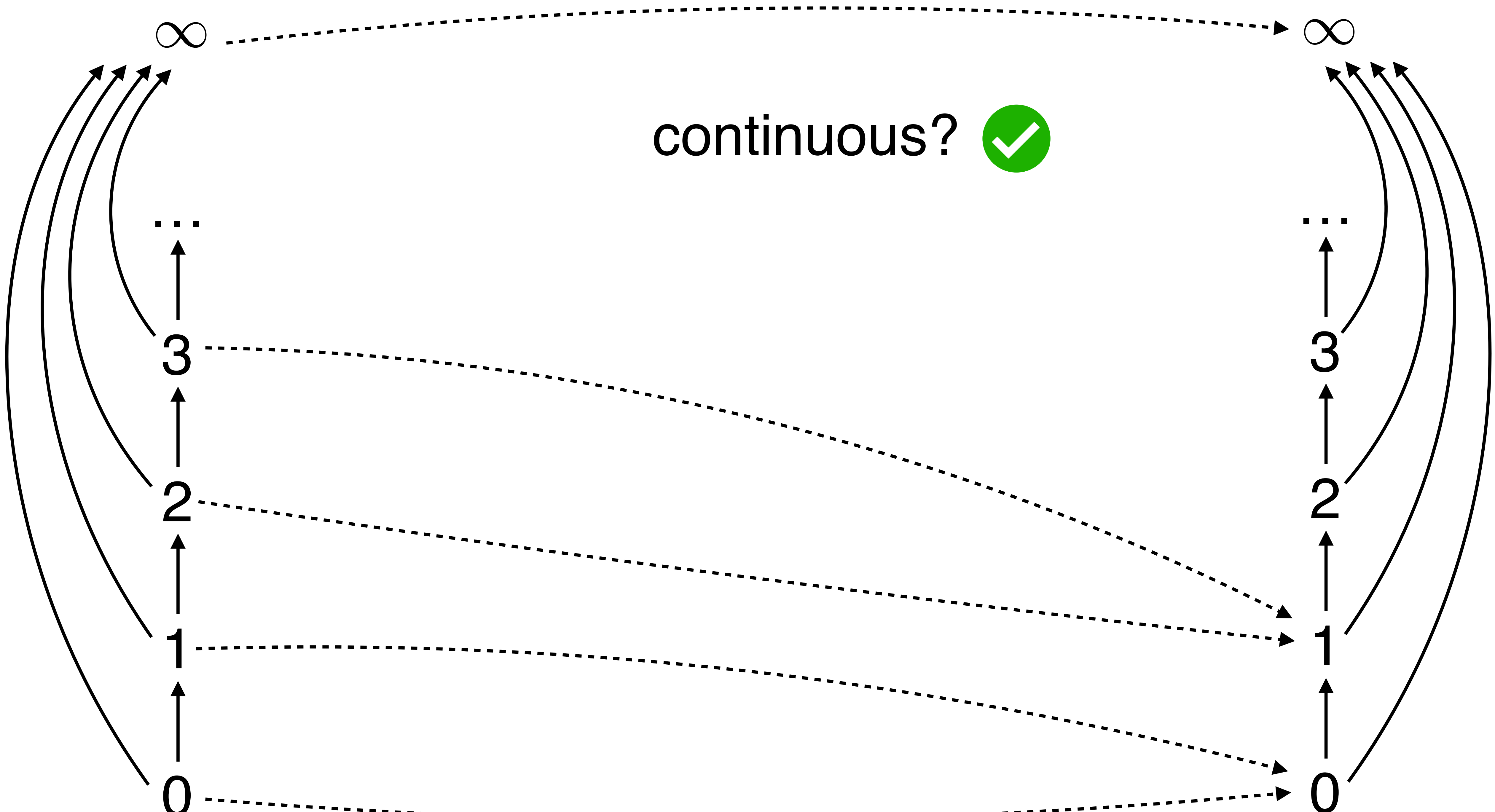
Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

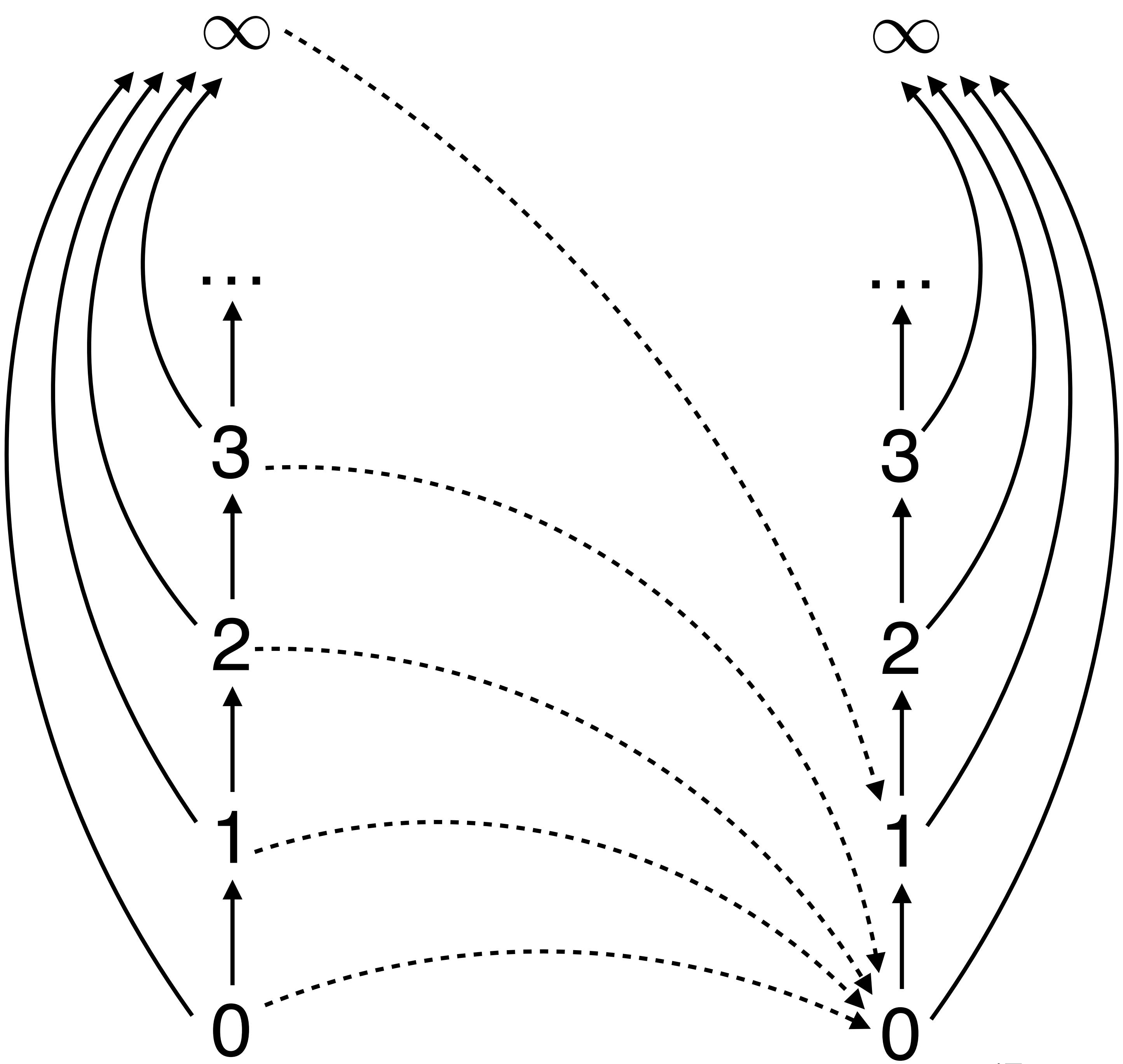
continuous? ✓



Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous



$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$

$$d_i = 2 \cdot i \qquad \bigsqcup_{i \in \mathbb{N}} d_i = \infty$$

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(\infty) = 1$$

$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \qquad f(S) = \{ m \in \mathbb{N} \mid \exists n \in S, m \leq n \} \qquad (\wp(\mathbb{N}), \subseteq)$$

continuous?

Exercise

$(\wp(\mathbb{N}), \subseteq)$ $\quad$ $f(S) = \begin{cases} S & \text{if } S \text{ finite} \\ \mathbb{N} & \text{otherwise} \end{cases}$ $\quad$ $(\wp(\mathbb{N}), \subseteq)$

continuous? ✖

Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \to E$
monotone $\Rightarrow$ f continuous

proof. Take a chain $\{d_i\}_{i \in \mathbb{N}}$

$\{d_i\}_{i \in \mathbb{N}}$ is finite $\Rightarrow$ $\exists k \in \mathbb{N}.\ \bigsqcup_{i \in \mathbb{N}} d_i = d_k$

$\Downarrow$

$\{f(d_i)\}_{i \in \mathbb{N}}$ is finite $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$

Composition

TH. Any composition of continuous functions is continuous

$(D, \sqsubseteq_D)$ CPO
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$ CPO

$f : D \to E$ continuous
$g : E \to F$ continuous

$\Rightarrow$ $h = g \circ f : D \to F$ continuous

proof. take a chain $\{d_i\}_{i \in \mathbb{N}}$

we need to prove $h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = g\left(f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)\right) = g\left(\bigsqcup_{i \in \mathbb{N}} f(d_i)\right) = \bigsqcup_{i \in \mathbb{N}} g(f(d_i))$$
$$= \bigsqcup_{i \in \mathbb{N}} h(d_i)$$

Kleene’s fixpoint theorem

Repeated application

$$f : D \to D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots(f}^{n \text{ times}}(d))\cdots)$$

$$f^n : D \to D$$

Lemma

$(D, \sqsubseteq)$ $\mathrm{PO}_{\bot}$ $f : D \to D$ monotone $\Rightarrow$ $\{f^n(\bot)\}_{n\in\mathbb{N}}$ is a chain

proof. we need to prove $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq f^{n+1}(\bot)$

by mathematical induction $P(n) \triangleq f^n(\bot) \sqsubseteq f^{n+1}(\bot)$

$P(0) \triangleq f^0(\bot) \sqsubseteq f^1(\bot)$ $\quad$ $f^0(\bot) = \bot \sqsubseteq f^1(\bot)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$ $\quad$ take a generic n

assume $P(n) \triangleq f^n(\bot) \sqsubseteq f^{n+1}(\bot)$

we want to prove $P(n+1) \triangleq f^{n+1}(\bot) \sqsubseteq f^{n+2}(\bot)$

$$f^n(\bot) \sqsubseteq f^{n+1}(\bot)$$

$$\Downarrow$$

$$f^{n+1}(\bot) = f(f^n(\bot)) \sqsubseteq f(f^{n+1}(\bot)) = f^{n+2}(\bot)$$

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\mathrm{CPO}_\perp$

then $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n\in\mathbb{N}}$
not necessarily a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $\quad f : D \rightarrow D$ monotone

fixpoint $\quad p \in D \quad f(p) = p$

pre-fixpoint $\quad p \in D \quad f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq)$ $\mathrm{CPO}_\perp$ $\quad f : D \to D$ continuous

let $\mathit{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\mathit{fix}(f)$ is a fix point of f

$$f(\mathit{fix}(f)) = \mathit{fix}(f)$$

2. $\mathit{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\mathit{fix}(f)$ is smaller than d

Kleene's theorem: 1

1. $f(\mathit{fix}(f)) = \mathit{fix}(f)$

proof.

$$f(\mathit{fix}(f)) = f\left(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)\right)$$ by def of fix

$$= \bigsqcup_{n\in\mathbb{N}} f(f^n(\bot))$$ by continuity

$$= \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot)$$ by def of f^n

$$= \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$$ by prefix independence of limits

$$= \mathit{fix}(f)$$ by def of fix

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\bot)\}_{n\in\mathbb{N}}$$

by definition $\mathit{fix}(f)$ is the lub of the same chain

and thus smaller than any other upper bound

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\bot) \sqsubseteq d$ by mathematical induction

$P(0) \triangleq f^0(\bot) \sqsubseteq d$ $\qquad f^0(\bot) = \bot \sqsubseteq d$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic n

assume $P(n) \triangleq f^n(\bot) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\bot) \sqsubseteq d$

$$f^n(\bot) \sqsubseteq d$$

(monot.) $\Downarrow$

$$f^{n+1}(\bot) \overset{\text{(by def)}}{=} f(f^n(\bot)) \sqsubseteq f(d) \overset{\text{(pre-fixpoint)}}{\sqsubseteq} d$$

Example

$n = 2 \cdot n$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\bot = 0$ $\mathrm{CPO}_\bot$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$n = n + 1$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\perp = 0$ $\text{CPO}_\perp$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$
$$f^1(0) = f(0) = 0 + 1 = 1$$
$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$
$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$ fixpoint

Example

$X = X \cap \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$X = \mathbb{N} \setminus X$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$X = X \cup \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$f(X) = X \cup \{1\}$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!

Badge exercise

Let D be a CPO

let $\{d_i\}_{i \in \mathbb{N}}$ be a chain in D

let $\{k_j\}_{j \in \mathbb{N}}$ be an infinite chain in $(\mathbb{N}, \leq)$

1. Prove that $\{d_{k_j}\}_{j \in \mathbb{N}}$ is a chain in D
2. Prove or disprove that $\bigsqcup_{j \in \mathbb{N}} d_{k_j} = \bigsqcup_{i \in \mathbb{N}} d_i$