Principles of software composition 2017/18
Mid-term exam – April 5, 2018

[**Ex. 1**] Let us extend the syntax of arithmetic expressions with the term $a^\times$, whose operational semantics is defined by the rules

$$\frac{\langle a, \sigma\rangle \to n}{\langle a^\times, \sigma\rangle \to n} \qquad \frac{\langle a, \sigma\rangle \to n \quad \langle a^\times, \sigma\rangle \to m}{\langle a^\times, \sigma\rangle \to n \underline{\times} m}$$

1. Prove termination of extended expressions by structural induction.
2. Prove by rule induction that $\forall \sigma, n.\ P(\langle 1^\times, \sigma\rangle \to n)$, where

$$P(\langle 1^\times, \sigma\rangle \to n) \stackrel{\mathsf{def}}{=} n = 1$$

[**Ex. 2**] Let w be the command:

$$w \stackrel{\mathsf{def}}{=} \textbf{while } x \times x = y \textbf{ do } (\, x := x \times x \,;\, y := x \times y\,)$$

Find the set of memories S such that $\forall \sigma \in S.\ \langle w, \sigma\rangle \not\to$.

[**Ex. 3**] Let $(D, \preceq)$ be the CPO with bottom such that $D = \mathbb{N} \cup \{\infty_1, \infty_2\}$ and $\preceq \cap (\mathbb{N} \times \mathbb{N}) = \leq$, ∞_2 is the top element and $x \preceq \infty_1$ iff $x \neq \infty_2$.

1. Consider the function $succ : D \to D$ such that $\forall n \in \mathbb{N}. succ(n) = n + 1$ and $succ(\infty_1) = succ(\infty_2) = \infty_2$.
   Prove that the function $succ$ is monotone but not continuous.
2. [**Optional**] Let $\{d_i\}_{i \in \mathbb{N}}$ be a chain.
   Prove that if $\bigsqcup_{i \in \mathbb{N}} d_i = \infty_2$ then the chain is finite.
   Hint: Note that if ∞_1 or ∞_2 belong to the chain then it is finite.

[**Ex. 4**] Let **Pf** be the domain of partial functions over positive natural numbers (ordered as usual and whose bottom element $\bot_{\mathbf{Pf}}$ is the always undefined function). Let $\Gamma : \mathbf{Pf} \to \mathbf{Pf}$ the continuous function defined by

$$\Gamma \stackrel{\mathsf{def}}{=} \lambda \varphi.\ \lambda m.\ (m = 1) \to 2\,,\, 2m + \varphi(m - 1).$$

Take $\varphi_n \stackrel{\mathsf{def}}{=} \Gamma^n(\bot_{\mathbf{Pf}})$ and $f \stackrel{\mathsf{def}}{=} \mathsf{fix}\ \Gamma = \bigsqcup_{n \in \mathbb{N}} \varphi_n$.
Prove that $\forall n > 0.\ \varphi_n(n) = n(n+1)$ to conclude that $\forall n > 0.\ f(n) = n(n+1)$.

[**Ex. 5**] Consider the Haskell types `Arc` $a = (a, a)$, `Graph` $a =$ [`Arc` a] and `Nodes` $a = [a]$. Implement a function

$$\texttt{nodes} :: (\texttt{Eq}\ a) \Rightarrow \texttt{Graph}\ a \to \texttt{Nodes}\ a$$

that returns the list of nodes of a graph, without repetead elements.