Chapter 3
ZMP and Dynamics

The main topic of this section is the physics of the robot while that of the foregoing chapter being the geometry.

We first show a method for measuring the ZMP which is an important physical quantity for humanoid robots. Then we show a method for calculating the ZMP for a given motion of a humanoid robot. Lastly, we explain a certain mistake on the ZMP and cases which cannot be handled by using the ZMP.

3.1 ZMP and Ground Reaction Forces

The base of an industrial robot is fixed to the ground while the sole of a humanoid robot is not fixed and just contacts with the ground. Because of this, although the industrial robots can move freely within the joint movable range, the humanoid robot has to move with keeping the difficult condition of maintaining contact between the sole and the ground. Here, given a motion of a humanoid robot, we need to judge whether or not the contact can be maintained between the sole and the ground. Also, we need to plan a motion of a humanoid robot maintaining contact between the sole and the ground. We usually use the ZMP for these kinds of purposes.

3.1.1 ZMP Overview

1 Definition of ZMP

In 1972, Vukobratović and Stepanenko defined the **Zero-Moment Point (ZMP)** at the beginning of the paper on control of humanoid robots[1]. Everything of the argument regarding the ZMP starts from here.

[1] We can find the same definition in the book [81]. Later, some delicate aspects of the ZMP definition were discussed by Vukobratović and Borovac [88].

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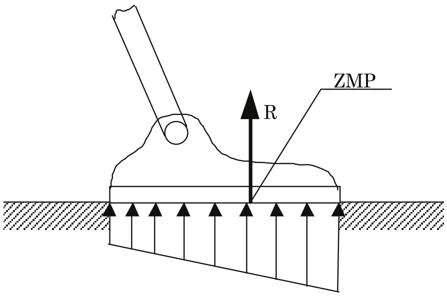
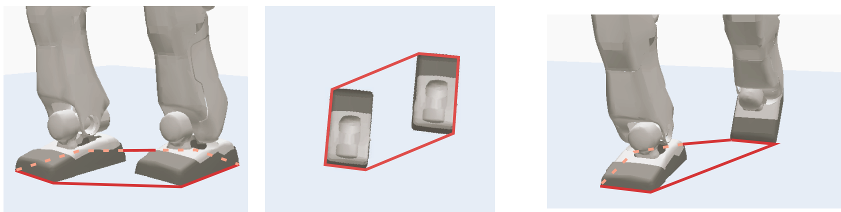


Fig. 3.1 Definition of Zero-Moment Point (ZMP) [90]

In Fig. 3.1 an example of force distribution across the foot is given. As the load has the same sign all over the surface, it can be reduced to the resultant force R, the point of attack of which will be in the boundaries of the foot. Let the point on the surface of the foot, where the resultant R passed, be denoted as the zero-moment point, or ZMP in short.

2 ZMP and Support Polygon



(a) Full contact of both feet (b) Partial contact

Fig. 3.2 Support Polygon

We explain the support polygon which is another important concept related to the ZMP. As shown in Fig. 3.2, let us consider the region formed by enclosing all the contact points between the robot and the ground by using an elastic cord braid. We call this region as the **support polygon**. Mathematically the support polygon is defined as a convex hull, which is the smallest convex set including all contact points. Definitions of the convex set and the convex hull are explained in the appendix of this chapter.

Rather than detailed discussions, we first show a simple and important relationship between the ZMP and the support polygon, i.e.,

The ZMP always exists inside of the support polygon.

Here, Vukovratović originally stated "the point which exists inside the boundary of the foot." In order to illustrate this more concretely, consider the images in Fig.3.3 illustrating the relationship among the center of mass (CoM), ZMP and the support polygon while a human stands on the ground. We call **the ground projection of CoM** the point where the gravity line from the CoM intersects the ground. As shown in Fig. 3.3(a), when a human stands on the ground, the ZMP coincides with the ground projection of CoM. In such a case, a human can keep balance if the ground projection of CoM is included strictly inside of the support polygon. On the other hand, when a human moves dynamically as shown in Fig. 3.3(b), the ground projection of CoM may exist outside the support polygon. However, the ZMP never exists outside the support polygon. In the following, we will explain the reason why the ZMP is always included in the support polygon.

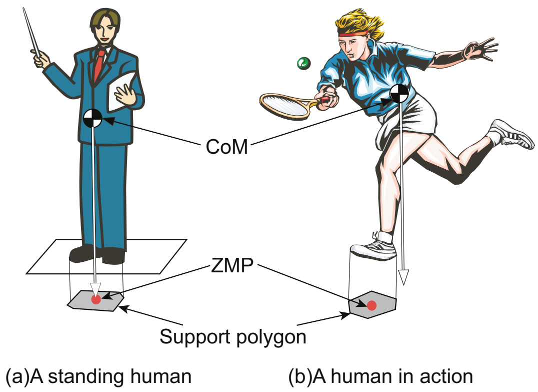


Fig. 3.3 CoG, ZMP, and Support Polygon

3.1.2 2D Analysis

1 ZMP in 2D

In Fig. 3.1, although only the vertical component of the ground reaction force is shown, the horizontal component of it also exists due to the friction between the ground and the soles of the feet.

In Fig. 3.4(a) and (b), we separately show the vertical component $\rho(\xi)$ and the horizontal component $\sigma(\xi)$ of the ground reaction force per unit length of the sole. These forces simultaneously act on a humanoid robot.

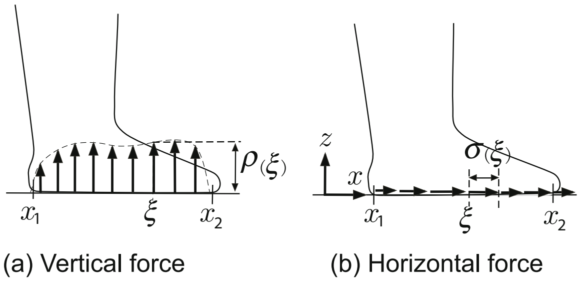


Fig. 3.4 Ground Reaction Force of 2D Model

Let us replace the forces distributed over the sole by a equivalent force and moment at a certain point in the sole. The force (f_x and f_z) and the moment ($\tau(p_x)$) at the point p_x in the sole can be expressed by

$$f_x = \int_{x_1}^{x_2} \sigma(\xi) d\xi \tag{3.1}$$

$$f_z = \int_{x_1}^{x_2} \rho(\xi) d\xi \tag{3.2}$$

$$\tau(p_x) = -\int_{x_1}^{x_2} (\xi - p_x)\rho(\xi) d\xi. \tag{3.3}$$

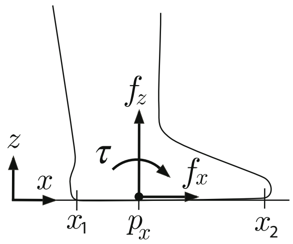


Fig. 3.5 Ground Reaction Forces and their Equivalent Force and Moment

Focusing on (3.3) with respect to the moment, let us consider the point p_x where moment becomes zero. Considering $\tau(p_x) = 0$ for (3.3), p_x can be obtained as follows:

$$p_x = \frac{\int_{x_1}^{x_2} \xi \rho(\xi) d\xi}{\int_{x_1}^{x_2} \rho(\xi) d\xi}. \tag{3.4}$$

Here, $\rho(\xi)$ is equivalent to the pressure since it is the vertical component of forces per unit length. Thus, p_x defined in (3.4) is the **center of pressure** and is the ZMP defined in the previous section. For 2D cases, since the ZMP is a point where **the moment of the ground reaction force becomes zero**, it has become the origin of the name.

2 Region of ZMP in 2D

Generally speaking, since the vertical component of the ground reaction force does not become negative unless magnets or suction cups are attached at the sole,

$$\rho(\xi) \geq 0.$$

Substituting this relationship into (3.4), we obtain

$$x_1 \leq p_x \leq x_2. \tag{3.5}$$

Equation (3.5) means that the ZMP is included in the segment of contact between the sole and the ground and does not exist outside it.

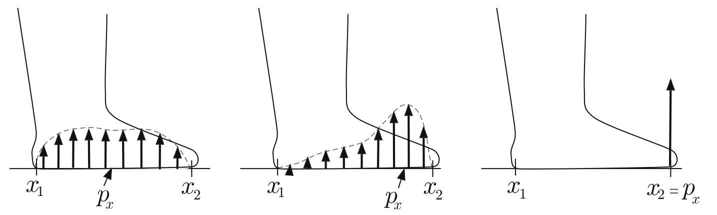


Fig. 3.6 ZMP and Pressure Distribution

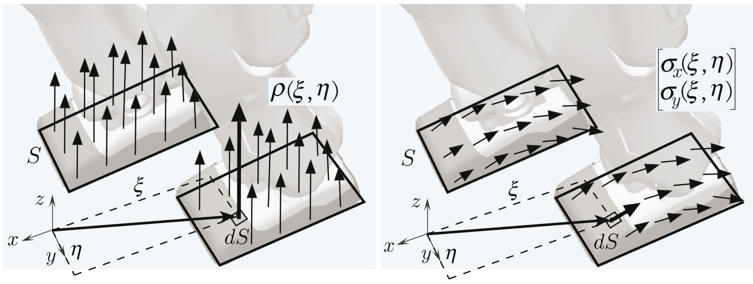
Figure 3.6 shows the relationship between the pressure distribution and the position of the ZMP. As shown in (a), when the reaction force is distributed over the sole almost equally, the ZMP exists at the center of the sole. On the other hand, as shown in (b), when the distribution is inclined to the front part of the sole, the ZMP exists at the front part of the sole. Furthermore, as shown in (c), when a point at the toe supports all the reaction forces, the ZMP also exists at the toe. In this case, since the surface contact between the sole and ground is not guaranteed any longer, the foot begins to rotate about the toe by only a slight external disturbance applied to the robot. To reduce the danger of falling down when a humanoid robot moves, it is desirable to have the ZMP located inside of the support polygon while maintaining a certain margin from the end.

3.1.3 3D Analysis

We now extend the concept of the ZMP to 3D cases.

1 Ground Reaction Force in 3D

Let us consider the ground reaction force applied to the robot moving in 3D space from the flat ground. The horizontal component and the vertical component of the ground reaction forces are shown in Figs. 3.7(a) and (b), respectively. In actual situations, the sum of these two components is applied to the robot at the same time.



(a) Vertical reaction forces (b) Horizontal reaction forces

Fig. 3.7 Ground Reaction Force in 3D

Let $\boldsymbol{r} = [\xi\ \eta\ 0]^T$ be the position vector defined on the ground. Also, let $\rho(\xi, \eta)$ be the vertical component of the ground reaction force applied at a unit area. The sum of the vertical component of ground reaction force is expressed as

$$f_z = \int_S \rho(\xi, \eta) dS, \tag{3.6}$$

where $\int_S$ denotes the area integration at the contact between the sole and the ground. The moment $\boldsymbol{\tau}_n(\boldsymbol{p})$ of the ground reaction force about a point $\boldsymbol{p} = [p_x\ p_y\ 0]^T$ can be calculated as

$$\boldsymbol{\tau}_n(\boldsymbol{p}) \equiv [\tau_{nx}\ \tau_{ny}\ \tau_{nz}]^T \tag{3.7}$$

$$\tau_{nx} = \int_S (\eta - p_y)\rho(\xi, \eta) dS \tag{3.8}$$

$$\tau_{ny} = -\int_S (\xi - p_x)\rho(\xi, \eta) dS \tag{3.9}$$

$$\tau_{nz} = 0.$$

As well as the 2D cases, assuming

$$\tau_{nx} = 0 \tag{3.10}$$

$$\tau_{ny} = 0 \tag{3.11}$$

for (3.8) and (3.9), the point where the moment of the vertical component of the ground reaction force becomes zero can be expressed as

$$p_x = \frac{\int_S \xi\rho(\xi,\eta)dS}{\int_S \rho(\xi,\eta)dS} \tag{3.12}$$

$$p_y = \frac{\int_S \eta\rho(\xi,\eta)dS}{\int_S \rho(\xi,\eta)dS}. \tag{3.13}$$

Since $\rho(\xi,\eta)$ is equivalent to the pressure over the surface of the sole, the point $\boldsymbol{p}$ is the **center of perssure** or in other word, ZMP.

On the other hand, let us consider the effect of the horizontal component of the ground reaction force. Let $\sigma_x(\xi,\eta)$ and $\sigma_y(\xi,\eta)$ be the x and y components, respectively, of the horizontal ground reaction forces. The sum of them can be expressed as

$$f_x = \int_S \sigma_x(\xi,\eta)dS \tag{3.14}$$

$$f_y = \int_S \sigma_y(\xi,\eta)dS. \tag{3.15}$$

The moment $\boldsymbol{\tau}_t(\boldsymbol{p})$ of the horizontal ground reaction force about a point $\boldsymbol{p}$ on the ground surface is expressed as

$$\begin{aligned}\boldsymbol{\tau}_t(\boldsymbol{p}) &\equiv [\tau_{tx}\ \tau_{ty}\ \tau_{tz}]^T \\ \tau_{tx} &= 0 \\ \tau_{ty} &= 0 \\ \tau_{tz} &= \int_S \{(\xi - p_x)\sigma_y(\xi,\eta) - (\eta - p_y)\sigma_x(\xi,\eta)\}dS.\end{aligned} \tag{3.16}$$

These equations mean that the horizontal ground reaction forces generate the vertical component of the moment.

From the above discussions, we can see that, as shown in Fig. 3.8, the ground reaction forces distributed over the surface of the sole can be replaced by the force

$$\boldsymbol{f} = [f_x\ f_y\ f_z]^T,$$

and the moment

$$\begin{aligned}\boldsymbol{\tau}_p &= \boldsymbol{\tau}_n(\boldsymbol{p}) + \boldsymbol{\tau}_t(\boldsymbol{p}) \\ &= [0\ 0\ \tau_{tz}]^T,\end{aligned}$$

about the ZMP $\boldsymbol{p}$. When a robot moves, $\tau_{tz} = 0$ is not generally satisfied. Thus, the ZMP is not a point where all components of the moment becomes zero for 3D cases. The ZMP is defined as **the point where the horizontal component of the moment of the ground reaction forces becomes zero** for 3D cases.

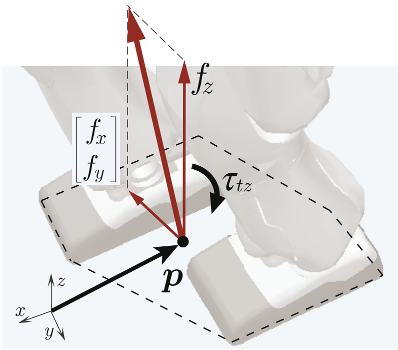


Fig. 3.8 Ground Reaction Forces and Equivalent Force and Moment for 3D Models

2 Region of ZMP in 3D

Let us define the region of the ZMP for 3D cases. For simplicity, we consider the ground reaction forces $\boldsymbol{f}_i = [f_{ix}\ f_{iy}\ f_{iz}]^T$ acting at the discretized points $\boldsymbol{p}_i \in S\ (i = 1, \cdots, N)$ as shown in Fig. 3.9. This approximation becomes more exact as the number of discretized points increases.

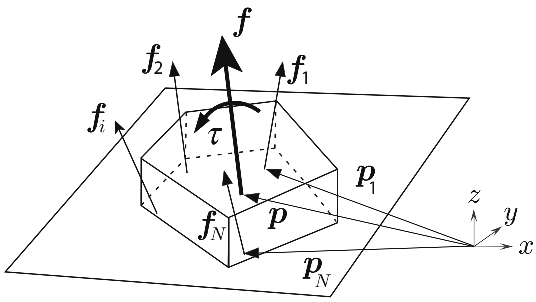


Fig. 3.9 Force/Moment at the ZMP Expressed by Forces at Discretized Points

Next, distributed N force vectors are replaced by a force and a moment vectors acting at the point $\boldsymbol{p}$ as

$$\boldsymbol{f} = \sum_{i=1}^{N} \boldsymbol{f}_i \tag{3.17}$$

$$\boldsymbol{\tau}(\boldsymbol{p}) = \sum_{i=1}^{N} (\boldsymbol{p}_i - \boldsymbol{p}) \times \boldsymbol{f}_i. \tag{3.18}$$

The position of the ZMP can be obtained by setting the first and the second elements of (3.18) be zero. This yields

$$\boldsymbol{p} = \frac{\sum_{i=1}^{N} \boldsymbol{p}_i f_{iz}}{\sum_{i=1}^{N} f_{iz}}. \tag{3.19}$$

For ordinary humanoid robots without magnets or suction cups at the sole, the vertical component of the ground reaction forces becomes zero for all discretized points, i.e.,

$$f_{iz} \geq 0 \qquad (i = 1, \cdots, N). \tag{3.20}$$

Here, introducing the new variables $\alpha_i = f_{iz}/\sum_{j=1}^{N} f_{jz}$, we obtain

$$\begin{cases} \alpha_i \geq 0 & (i = 1, \cdots, N) \\ \sum_{i=1}^{N} \alpha_i = 1. \end{cases} \tag{3.21}$$

Rewriting (3.19) by using α_i, the region of the ZMP can be expressed as

$$\boldsymbol{p} \in \left\{ \sum_{i=1}^{N} \alpha_i \boldsymbol{p}_i \,\middle|\, \boldsymbol{p}_i \in S \ (i = 1, \cdots, N) \right\}. \tag{3.22}$$

By comparing (3.21) and (3.22) with the definition of convex hull (3.90) in section 3.6, we can see that the ZMP is included in the convex hull of the set S, in other words, the support polygon.

3.2 Measurement of ZMP

This section explains methods for measuring the position of the ZMP by using several sensors attached at the feet of a humanoid robot. For a biped walking robots to measure the position of the ZMP, we should consider two cases, i.e., (1) **the ZMP of each foot** considering the reaction force between either one of the feet and the ground, and (2) the ZMP considering the reaction force between both feet and the ground. During the double support phase, these two ZMPs becomes different.

3.2.1 General Discussion

Let us consider the model shown in Fig. 3.10. In this model, there are two rigid bodies contacting each other where one of them also contacts the ground. The forces and moments applied by one rigid body to the other are measured at multiple points. This model imitates the foot of a humanoid robot. When the robot moves and the foot is forced on the ground, the output of the force/torque sensor at the foot is generated. By using this sensor information, the position of the ZMP is measured.

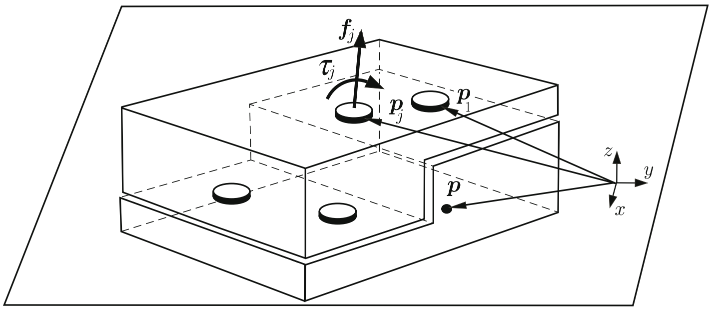


Fig. 3.10 Definition of Variables with respect to the Position and the Output of Force/Torque Sensors

Let us assume that, at the points $\boldsymbol{p}_j$ $(j = 1, \cdots, N)$ with respect to the reference coordinate system, the forces $\boldsymbol{f}_j$ and moments $\boldsymbol{\tau}_j$ are measured. Here, the moment about the point $\boldsymbol{p} = [p_x \ p_y \ p_z]^T$ is

$$\boldsymbol{\tau}(\boldsymbol{p}) = \sum_{j=1}^{N} (\boldsymbol{p}_j - \boldsymbol{p}) \times \boldsymbol{f}_j + \boldsymbol{\tau}_j. \tag{3.23}$$

The position of the ZMP can be obtained by setting the x and y components of (3.23) be zero and by solving for p_x and p_y as

$$p_x = \frac{\sum_{j=1}^{N} \{-\tau_{jy} - (p_{jz} - p_z) f_{jx} + p_{jx} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \tag{3.24}$$

$$p_y = \frac{\sum_{j=1}^{N} \{\tau_{jx} - (p_{jz} - p_z) f_{jy} + p_{jy} f_{jz}\}}{\sum_{j=1}^{N} f_{jz}} \tag{3.25}$$

where

$$\begin{aligned}
\boldsymbol{f}_j &= [f_{jx} \ f_{jy} \ f_{jz}]^T \\
\boldsymbol{\tau}_j &= [\tau_{jx} \ \tau_{jy} \ \tau_{jz}]^T \\
\boldsymbol{p}_j &= [p_{jx} \ p_{jy} \ p_{jz}]^T.
\end{aligned}$$

Equations (3.24) and (3.25) are the basis for measuring the position of the ZMP[2].

[2] When a foot does not contact the ground, the ZMP position cannot be determined since the denominators of (3.24) and (3.25) become zero. Therefore, when measuring the ZMP, we have to introduce a threshold and set $p_x = p_y = 0$ when the denominator is less than the threshold.

3.2.2 ZMP of Each Foot

First, focusing on the contact between one foot and the ground, we measure the ZMP.

1 Measurement Using 6 Axis Force/Torque Sensor

Figure 3.11 shows the foot of the humanoid robot HRP-2 [65]. The ground reaction force applied to the sole is transmitted to the sensor mount through rubber bushes and dampers. A 6 axis force/torque sensor is attached at the sensor mount, and the force is transmitted to the ankle of the robot through this sensor. The rubber bushes and the dampers are positioned to prevent large impulse forces from being transmitted to the robot. Since the displacement of them is small, we do not consider the displacement when calculating the ZMP.

A 6 axis force/torque sensor is coordinated to simultaneously measure the force $\boldsymbol{f} = [f_x,\ f_y,\ f_z]$ and the moment $\boldsymbol{\tau} = [\tau_x\ \tau_y\ \tau_z]$ applied from outside the robot. This sensor is mainly used for measuring the force at the end effector of industrial robots. An example of 6 axis force/torque sensor is shown in Fig. 3.12. To measure the ZMP of a humanoid robot, the force/torque sensor must be light and must be strong enough to accept the large impulsive force applied to the sensor.

To obtain the ZMP from the measured data of 6 axis force/torque sensor, we set $N = 1$ in (3.24) and (3.25).

Let the position of the ZMP in the right and the left foot be $\boldsymbol{p}_R$ and $\boldsymbol{p}_L$, respectively, as shown in Fig. 3.13. Especially when the center of measurement

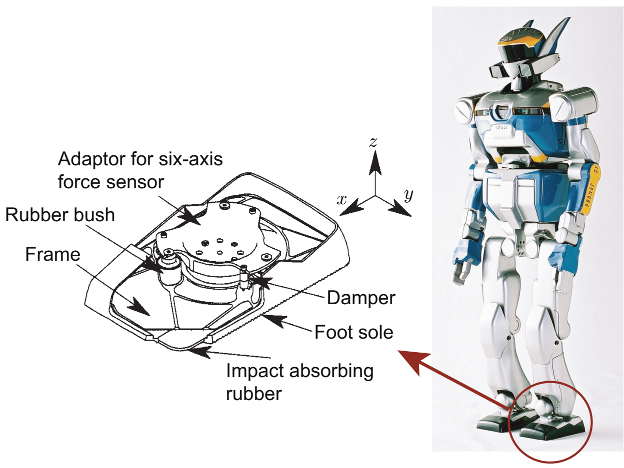


Fig. 3.11 Foot of HRP-2 [65].

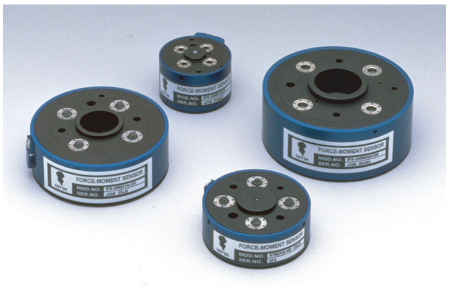


Fig. 3.12 An Example of 6 Axis Force/Torque Sensor (courtesy of Nitta Corp.)

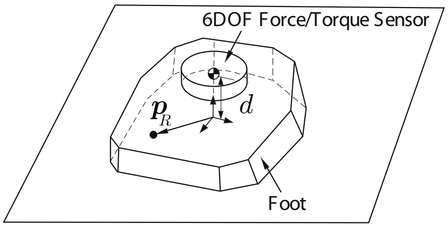


Fig. 3.13 Definition of Variables for Calculation of ZMP by 6 Axis Force/Torque Sensor

of the sensor lies on the z axis of the reference coordinate system, the position of the ZMP of each foot can be obtained very simply. For the right foot,

$$p_{Rx} = (-\tau_{1y} - f_{1x}d)/f_{1z} \tag{3.26}$$
$$p_{Ry} = (\tau_{1x} - f_{1y}d)/f_{1z} \tag{3.27}$$

where

$$\boldsymbol{p}_R = [p_{Rx}\ p_{Ry}\ p_{Rz}]^T$$
$$\boldsymbol{p}_1 = [0\ \ 0\ \ d]^T .$$

2 Measurement of ZMP by Multiple Force Sensors

Next, we explain the method to measure the ZMP by using multiple force sensors. Fig. 3.14 shows the humanoid robot H5 [70]. To make the foot light, the ZMP is measured by using twelve **force sensing registers: FSR** and sorbothane sandwiched by two aluminum planes (Fig. 3.14(b)). Since the electric resistance changes according to the applied force, the FSR can be used as a **one dimensional force sensor** to measure the vertical component of ground reaction force.

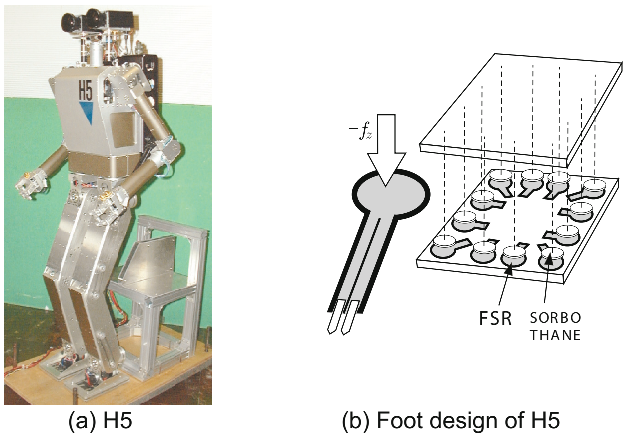


Fig. 3.14 Humanoid Robot H5 and its Foot
(Courtesy of Dept. of Mechano-Informatics, The Univ. of Tokyo)

To measure the ZMP, the x and y components of the force are set to be 0 in (3.24) and (3.25). As shown in Fig.3.15, when there are N one-dimensional force sensors, the ZMP can be obtained by

$$p_x = \frac{\sum_{j=1}^{N} p_{jx} f_{jz}}{\sum_{j=1}^{N} f_{jz}} \tag{3.28}$$

$$p_y = \frac{\sum_{j=1}^{N} p_{jy} f_{jz}}{\sum_{j=1}^{N} f_{jz}}. \tag{3.29}$$

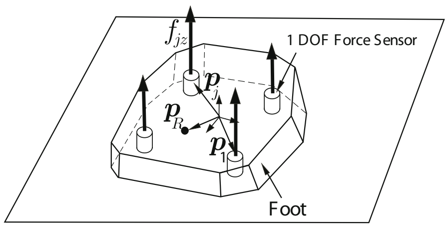


Fig. 3.15 Definition of Variables for Calculation of ZMP by Multiple 1 Axis Force Sensor

Figure 3.16 shows the humanoid robot Morph3 and its foot [129, 120]. Morph3 measures the ZMP by using four **3 axis force sensors** attached at each foot (Fig. 3.16(b)). The 3 axis force sensor measures the 3 dimensional forces applied to the sole split into four parts. By using this measuring system, we can obtain measurement on the point of contact. To calculate the ZMP

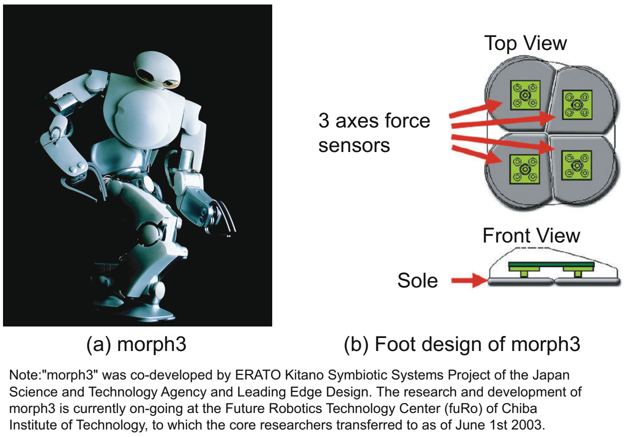


Fig. 3.16 Humanoid Robot Morph3 and its Foot

of each foot, the element of moment, τ_{jx} and τ_{jy}, in (3.24) and (3.25) are set to be zero.

3.2.3 ZMP for Both Feet Contact

Until the previous section, the position of the ZMP of each foot can be obtained as $\boldsymbol{p}_R$ and $\boldsymbol{p}_L$. The ground reaction forces $\boldsymbol{f}_R$ and $\boldsymbol{f}_L$ are also obtained from the sensor information. By using this information, we calculate the ZMP in the case where both feet are in contact with the ground. By using (3.24) and (3.25), the ZMP can be obtained as

$$p_x = \frac{p_{Rx} f_{Rz} + p_{Lx} f_{Lz}}{f_{Rz} + f_{Lz}} \tag{3.30}$$

$$p_y = \frac{p_{Ry} f_{Rz} + p_{Ly} f_{Lz}}{f_{Rz} + f_{Lz}} \tag{3.31}$$

where

$$\begin{aligned}
\boldsymbol{f}_R &= [f_{Rx}\ f_{Ry}\ f_{Rz}]^T \\
\boldsymbol{f}_L &= [f_{Lx}\ f_{Ly}\ f_{Lz}]^T \\
\boldsymbol{p}_R &= [p_{Rx}\ p_{Ry}\ p_{Rz}]^T \\
\boldsymbol{p}_L &= [p_{Lx}\ p_{Ly}\ p_{Lz}]^T .
\end{aligned}$$

During the single support phase, since the vertical component of the ground reaction force becomes zero, the ZMP calculated using (3.30) and (3.31) coincides with the ZMP of the supporting foot. This yields

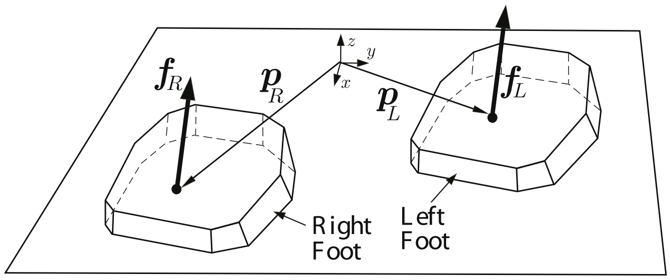


Fig. 3.17 Definition of Variables for ZMP for Both Feet Contact

$$[p_x\ p_y\ p_z]^T = \begin{cases} [p_{Rx}\ p_{Ry}\ p_{Rz}]^T & \text{for support of right foot} \\ [p_{Lx}\ p_{Ly}\ p_{Lz}]^T & \text{for support of left foot.} \end{cases} \tag{3.32}$$

We conclude this section stating that, when we consider the balance of a humanoid robot, we can use (3.30) and (3.31) taking the both feet into account regardless of the supporting foot.

3.3 Dynamics of Humanoid Robots

From the previous discussion, we can express the ground reaction force acting upon a humanoid robot by using the ZMP, the linear force, and the moment about a vertical line passing the ZMP. In this section, we discuss the relationship between the ground reaction force and the robot's motion. After showing basic equations, we explain the principle of it. Lastly, we show some calculation algorithms.

3.3.1 Humanoid Robot Motion and Ground Reaction Force

1 Basic Physical Parameters

Let us consider a humanoid robot with arbitrary structure. While it can be composed of metal, plastic, and ceramics etc., we assume that we can clearly identify between the robot and other things. We can define the following ten physical parameters classified into four groups:

Mass: Total Robot's mass. M [kg]
Center of Mass: Robot's center of mass. $\boldsymbol{c} \equiv [x\ y\ z]^T$ [m]
Momentum: Measure of an robot's translational motion[3].
$\mathcal{P} \equiv [\mathcal{P}_x\ \mathcal{P}_y\ \mathcal{P}_z]^T$ [Ns]

[3] We often call it the linear momentum to distinguish it from the angular momentum.