Logistics

Lecture notes*

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Chapter 5

Basic network flow problems

This chapter aims at providing an introduction to some basic transportation (or routing) problems. Specifically, the minimum cost flow problem and some special cases, i.e. the shortest path problem and the transportation problem, will be shortly overviewed.

5.1 The minimum cost flow problem

In order to introduce the problem, let us consider the following example. Bavarian Motor Company (BMC) manufactures luxury cars in Germany and exports them in the U.S.; they are currently holding 200 cars available at the port in Newark and 300 cars available at the port in Jacksonville. From there, the cars are transported (by rail or truck) to five distributors having a specific requirement of cars (see Figure 5.1).

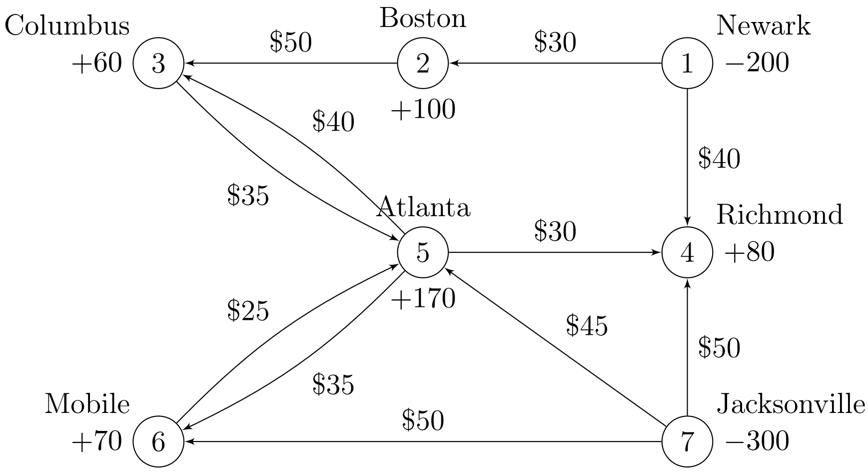

Figure 5.1: Logistics network of BMC

In the network, Newark and Jacksonville are *supply nodes* (or origins): negative numbers (e.g. −200) represent their supply; Boston, Columbus, Atlanta, Richmond and Mobile are *demand nodes* (or destinations): positive numbers (e.g. +100) represent their demand. There can also be *transshipment nodes* (with value 0), not in the picture.

The problem is to determine how to transport (flowing) cars along the arcs of the network to satisfy the demands at a minimum cost.

In order to formulate the problem, let us introduce the decision variables x_{ij}, where x_{ij} indicates the number of cars shipped (or *flowing*) from i to j along (i,j), $\forall(i,j)$. So, in the example of BMC we have 11 flow variables.

We need to write the so-called flow conservation constraints. For instance, for node 1 (Newark), which is a supply node:

$$x_{12} + x_{14} \leq 200, \text{ i.e. } -x_{12} - x_{14} \geq -200,$$

and for node 2 (Boston), which is a demand node:

$$\underbrace{x_{12}}_{\text{entering flow}} - \overbrace{x_{23}}^{\text{leaving flow}} = 100.$$

The LP model of the BMC problem is showed in Model 5.1. It is a special minimum cost flow model, because in the classical minimum cost flow model the flow conservation constraints are all equalities, i.e. the total supply equals the total demand, which is a necessary condition to have feasible solutions. The optimal solution (minimum cost flow) is depicted in Figure 5.2.

In order to implement the BMC model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Chapter 5.1.

$$\begin{aligned}
&\min\ 30x_{12} + 40x_{14} + 50x_{23} + 35x_{35} + 40x_{53} + 30x_{54} + \\
&\quad + 35x_{56} + 25x_{65} + 50x_{74} + 45x_{75} + 50x_{76} \\
&-x_{12} - x_{14} \geq -200 \\
&x_{12} - x_{23} = 100 \\
&x_{23} + x_{53} - x_{35} = 60 \\
&x_{14} + x_{54} + x_{74} = 80 \\
&x_{35} + x_{65} + x_{75} - x_{53} - x_{54} - x_{56} = 170 \\
&x_{56} + x_{76} - x_{65} = 70 \\
&-x_{74} - x_{75} - x_{76} \geq -300 \\
&x_{12}, x_{14}, \ldots \geq 0
\end{aligned}$$

Model 5.1: BMC example

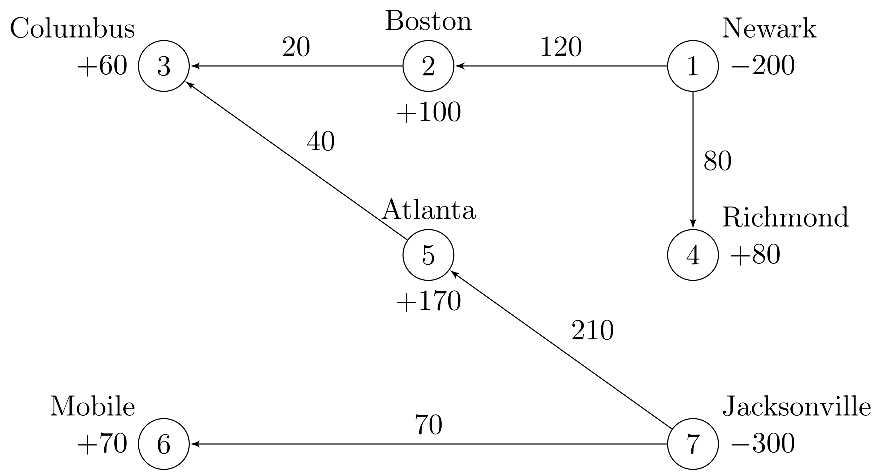

Figure 5.2: Optimal solution to the BMC problem

5.2 The shortest path problem

Consider the following example. American Car Association (ACA) provides various services to its members, including travel route planning, e.g. identifying routes that minimize the travel times. Given the logistic network in Figure 5.3, we want to determine a shortest (i.e. minimum travel time) path from Birmingham to Virginia Beach (i.e. from node 1 to node 11 in Figure 5.3).

The shortest path problem can be formulated as a special minimum cost flow problem, by sending one unit of flow from the origin of the path (node 1) to the destination (node 11). The corresponding LP model is shown in Model 5.2. Within the model, the flow conservation constraints relative to the origin and destination nodes have balance respectively -1 and $+1$, while all others have balance 0, being relative to transshipment nodes.

The optimal solution (*shortest path*), with 11.5 hours travel time (and score 15), is depicted in Figure 5.3. If we change the objective function, and maximize with respect to scores, then the *longest path* has score 35 (but its total travel time is 16).

For minimum cost flow problems, the so-called integrality property holds: if supplies and demands are integer, then there exists an integer optimal solution. Being a special case of minimum cost flow, the shortest path problem has this property, too.

Note that, if we add some side constraints, e.g. we want to find the most scenic path "within 14 hours of travel time", then the LP model does not necessarily return an integer optimal solution (i.e. a path): integrality constraints have to be added to the model (making it an ILP one).

In order to implement the ACA model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Chapter 5.2.

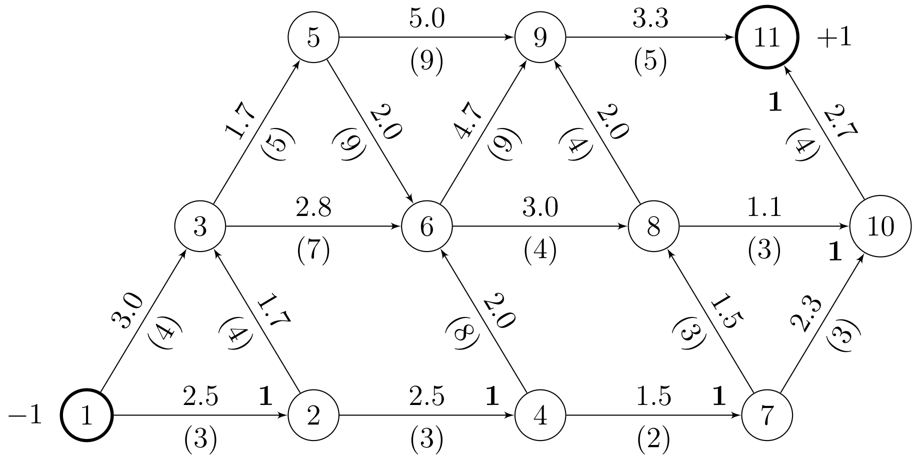

Figure 5.3: Network of the ACA example. Labels on edges represent the estimated travel times in hours and, in parentheses, scores for scenic quality of the link (secondary objective). The optimal solution is in bold.

$$
\begin{aligned}
&\min\ 2.5x_{12} + 3x_{13} + 1.7x_{23} + 2.5x_{24} + 1.7x_{35} + 2.8x_{36} + \\
&\quad + 2x_{46} + 1.5x_{47} + 2x_{56} + 5x_{59} + 3x_{68} + 4.7x_{69} + \\
&\quad + 1.5x_{78} + 2.3x_{7,10} + 2x_{89} + 1.1x_{8,10} + 3.3x_{9,11} + 2.7x_{10,11} \\
&-x_{12} - x_{13} = -1 \\
&x_{12} - x_{23} - x_{24} = 0 \\
&x_{13} + x_{23} - x_{35} - x_{36} = 0 \\
&x_{24} - x_{46} - x_{47} = 0 \\
&x_{35} - x_{56} - x_{59} = 0 \\
&x_{36} + x_{46} + x_{56} - x_{68} - x_{69} = 0 \\
&x_{47} - x_{78} - x_{7,10} = 0 \\
&x_{68} + x_{78} - x_{89} - x_{8,10} = 0 \\
&x_{59} + x_{69} + x_{89} - x_{9,11} = 0 \\
&x_{7,10} + x_{8,10} - x_{10,11} = 0 \\
&x_{9,11} + x_{10,11} = 1 \\
&x_{12}, x_{13}, \ldots \geq 0
\end{aligned}
$$

Model 5.2: The ACA example

5.3 The transportation problem

The transportation problem is the special case of the minimum cost flow problem on a bipartite graph. In order to introduce it, consider the following example.

Tropicsun is a distributor of fresh citrus products. Its productive assets are as follows:

- three large citrus groves:
  - Dora currently has 275,000 bushels of citrus;
  - Eustis currently has 400,000 bushels of citrus;
  - Clermont currently has 300,000 bushels of citrus;
- three citrus processing plants:
  - Ocala has the processing capacities to handle 200,000 bushels;
  - Orlando has the processing capacities to handle 600,000 bushels;
  - Leesburg has the processing capacities to handle 225,000 bushels.

Tropicsun contracts with a local trucking company to transport the fruit from its groves to its processing plants; the company charges a flat rate for each mile a bushel will be transported ("bushel–miles"). Distances (in miles) between groves and processing plants are listed in Table 5.1.

	Ocala	Orlando	Leesburg
Dora	21	50	40
Eustis	35	30	22
Clermont	55	20	25

Table 5.1: Distances in miles between groves and processing plants of Tropicsun

The problem is to determine how to ship bushels from each grove to each plant in order to minimize the total number of bushel–miles, i.e. the total cost. Each grove must send all the current bushels. The logistics network of Tropicsun is depicted in Figure 5.4.

The decision variables x_{ij} are defined as the number of bushels to ship from i to j, for each (i, j) in the network. The LP model is presented in Model 5.3. Note that, for instance, $x_{14} + x_{15} + x_{16} = 275{,}000$ is equivalent to $-x_{14} - x_{15} - x_{16} = -275{,}000$, where the negative balance stands for a supply.

The optimal solution is depicted in Figure 5.5 (recall that the integrality property holds true).

In order to implement the Tropicsun model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Chapter 3.10.

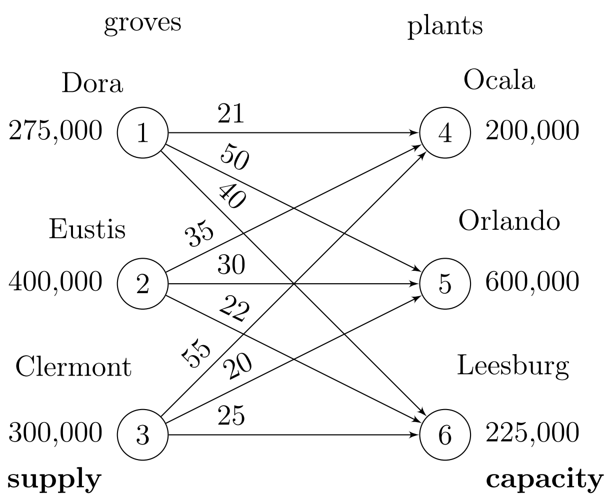

Figure 5.4: Logistics network of Tropicsun

$$
\begin{aligned}
\min\ & 21x_{14} + 50x_{15} + 40x_{16} + 35x_{24} + 30x_{25} + 22x_{26} + 55x_{34} + 20x_{35} + 25x_{36} \\
& \left.\begin{aligned}
x_{14} + x_{15} + x_{16} &= 275{,}000 \\
x_{24} + x_{25} + x_{26} &= 400{,}000 \\
x_{34} + x_{35} + x_{36} &= 300{,}000
\end{aligned}\right\} \quad \text{groves} \\
& \left.\begin{aligned}
x_{14} + x_{24} + x_{34} &\le 200{,}000 \\
x_{15} + x_{25} + x_{35} &\le 600{,}000 \\
x_{16} + x_{26} + x_{36} &\le 225{,}000
\end{aligned}\right\} \quad \text{plants} \\
& x_{14}\,, x_{15}\,, \ldots \ge 0
\end{aligned}
$$

Model 5.3: The Tropicsun problem

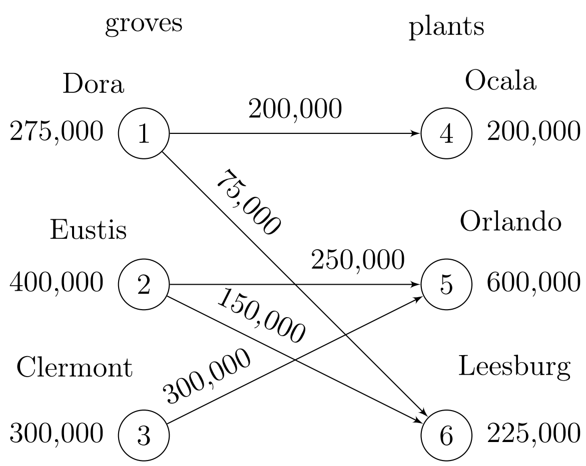

Figure 5.5: Optimal solution of the Tropicsun problem

5.4 LP model to the minimum cost flow problem

By generalizing the previous examples, we can state the *minimum cost flow problem* in a more formal way. Let $G = (N, A)$ be a given directed graph, b_i be the *balance* of node i, $\forall i \in N$, u_{ij} the *upper capacity* of link (i, j), $\forall (i, j) \in A$, and c_{ij} be the *unit transportation cost* along (i, j), $\forall (i, j) \in A$.

Let us introduce the decision (flow) variables x_{ij}, where x_{ij} is the amount (flow) to be pushed along (i, j), $\forall (i, j) \in A$. Then we can state the following Linear Programming model:

$$
\begin{aligned}
&\min \sum_{(i,j)\in A} c_{ij}\, x_{ij} \\
&\sum_{(j,i)\in \mathrm{BS}(i)} x_{ji} - \sum_{(i,j)\in \mathrm{FS}(i)} x_{ij} = b_i \quad \forall i \in N \qquad ① \\
&0 \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in A \qquad ②
\end{aligned}
\qquad \text{(mincostflow)}
$$

where

① are the flow conservation constraints:

- BS(i) is the *backward star* of node i (see Figure 5.6);
- FS(i) is the *forward star* of node i (see Figure 5.6);
- b_i is $\begin{cases} < 0 & \text{if } i \text{ is a supply node} \\ > 0 & \text{if } i \text{ is a demand node} \\ 0 & \text{if } i \text{ is a transshipment node} \end{cases}$;

② are the capacity constraints.

BS(i) FS(i)

Figure 5.6: Backward and forward star of node i

In order for feasible solutions to exist, the sum of all node balances must be zero, i.e. $\sum_{i\in N} b_i = 0$.

For this kind of problem, the integrality property holds: if b_i are integer $\forall i \in N$, and u_{ij} are integer $\forall (i, j) \in A$, then there exists an integer minimum cost flow (solving the (mincostflow) model we get an optimum integer solution).

5.4.1 Special cases

By considering the special minimum cost flow problems previously introduced, then the model to the shortest path problem from $s \in N$ to $t \in N$ (e.g. ACA, section 5.2) can be obtained from the minimum cost flow model by setting $b_i = \begin{cases} -1 & \text{if } i = s \\ +1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$ and $u_{ij} = +\infty, \ \forall (i,j) \in A$ (uncapacitated problem).

On the other hand, the transportation problem (e.g. Tropicsun, section 5.3) is a special minimum cost flow case where:

- G is a bipartite graph, i.e. $G = (\underbrace{N_1, N_2}_{N}, A)$, e.g. the one in Figure 5.4;
- $b_i < 0, \ \forall i \in N_1$ (all nodes in N_1 are supply nodes);
- $b_i > 0, \ \forall i \in N_2$ (all nodes in N_2 are demand nodes).

Textbooks

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