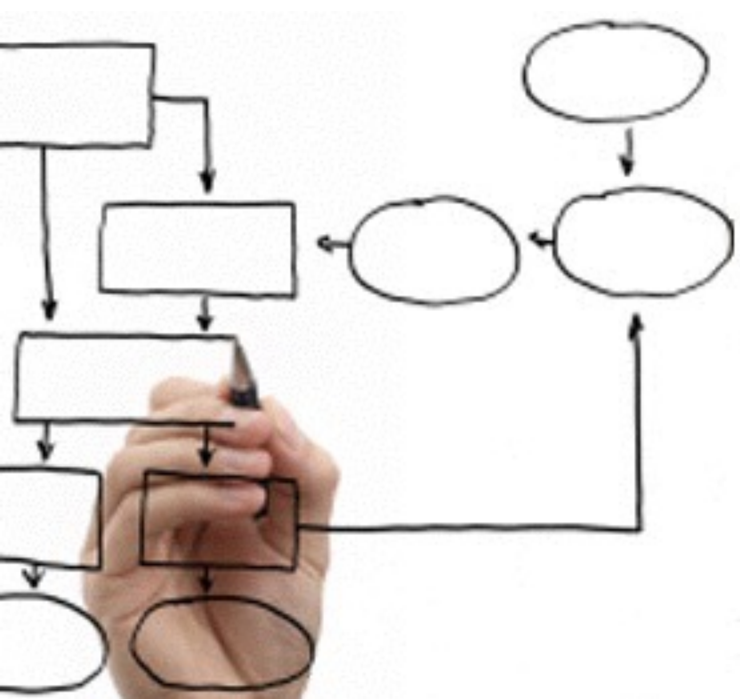
Methods for the specification and verification of business processes

MPB (6 cfu, 295AA)

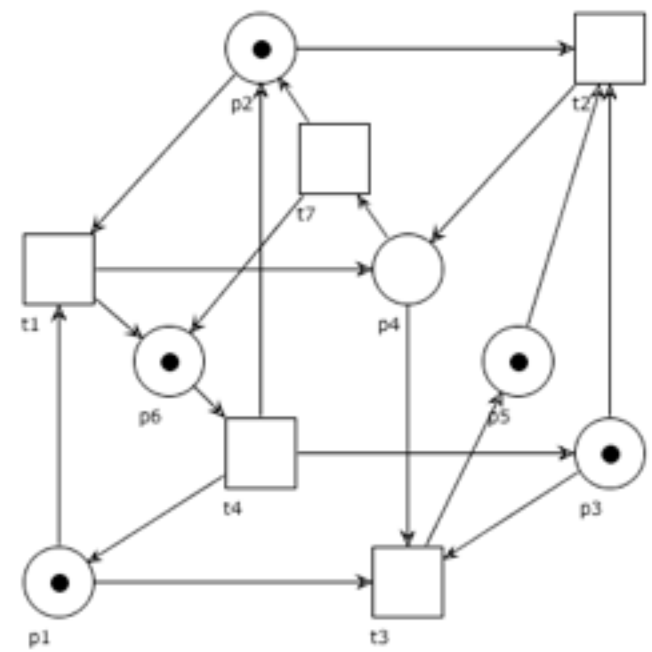
Roberto Bruni

http://www.di.unipi.it/~bruni



08 - Petri nets basics

Object



Formalization of the basic concepts of Petri nets

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Petri nets: basic definition

Carl Adam Petri

July 12, 1926 - July 2, 2010

http://www.informatik.uni-hamburg.de/TGI/mitarbeiter/profs/petri_eng.html

Introduced in 1962 (Petri's PhD thesis)
60's and 70's main focus on theory
80's focus on tools and applications
Now applied in several fields

Success due to simple and clean graphical and conceptual representation

K o m m u n i k a t i o n
m i t
A u t o m a t e n

Von der Fakultät für Mathematik und Physik
der Technischen Hochschule Darmstadt

zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer.nat.)

genehmigte
Dissertation

vorgelegt von
C a r l A d a m P e t r i
aus Leipzig

Referent: Prof.Dr.rer.techn.A.Walther
Korreferent: Prof.Dr.Ing.H.Unger

Tag der Einreichung: 27.7.1961
Tag der mündlichen Prüfung: 20.6.1962

D 17

Bonn 1962

Petri nets for us

Formal and abstract business process specification

Formal: the semantics of process instances becomes well defined and not ambiguous

Abstract: execution environment is disregarded

(Remind about separation of concerns)

Places

A place can stand for
a state
a medium
a buffer
a condition
a repository of resources
a type
...

Tokens

A token can stand for
a physical object
a piece of data
a resource
an activation mark
a message
a document
a case
...

Transitions

A transition can stand for
an event
an operation
a transformation
a transportation
a task
an activity
...

Notation: from sets…

Let S be a set.
Let $\wp(S)$ denote the set of sets over S.

Elements $A \in \wp(S)$ (i.e., $A \subseteq S$)
are in bijective correspondence with
functions $f : S \to \{0, 1\}$

$x \in A$ iff $f_A(x) = 1$

Notation: ... to multisets

Let $\mu(S)$ (or $S^{\oplus}$) denote the set of multisets over S.

Elements $B \in \mu(S)$ are in bijective correspondence with functions $M : S \rightarrow \mathbb{N}$

$M_B(x)$ is the number of instances of x in B
$x \in B$ iff $M_B(x) > 0$

Sets vs Multisets

Set

Multiset

Order of elements does not matter

Each element appears at most once

Order of elements does not matter

Each element can appear multiple times

Notation: sets

Empty set:
$\emptyset = \{ \ \}$ is such that $x \notin \emptyset$ for all $x \in S$

Set inclusion:
we write $A \subseteq B$ if $x \in A$ implies $x \in B$

Set strict inclusion:
we write $A \subset B$ if $A \subseteq B$ and $A \neq B$

Set union:
$A \cup B$ is the set s.t. $x \in (A \cup B)$ iff $x \in A$ or $x \in B$

Set difference:
$A - B$ is the set s.t. $x \in (A - B)$ iff $x \in A$ and $x \notin B$

Notation: multisets

Empty multiset:
$\emptyset$ is such that $\emptyset(x) = 0$ for all $x \in S$

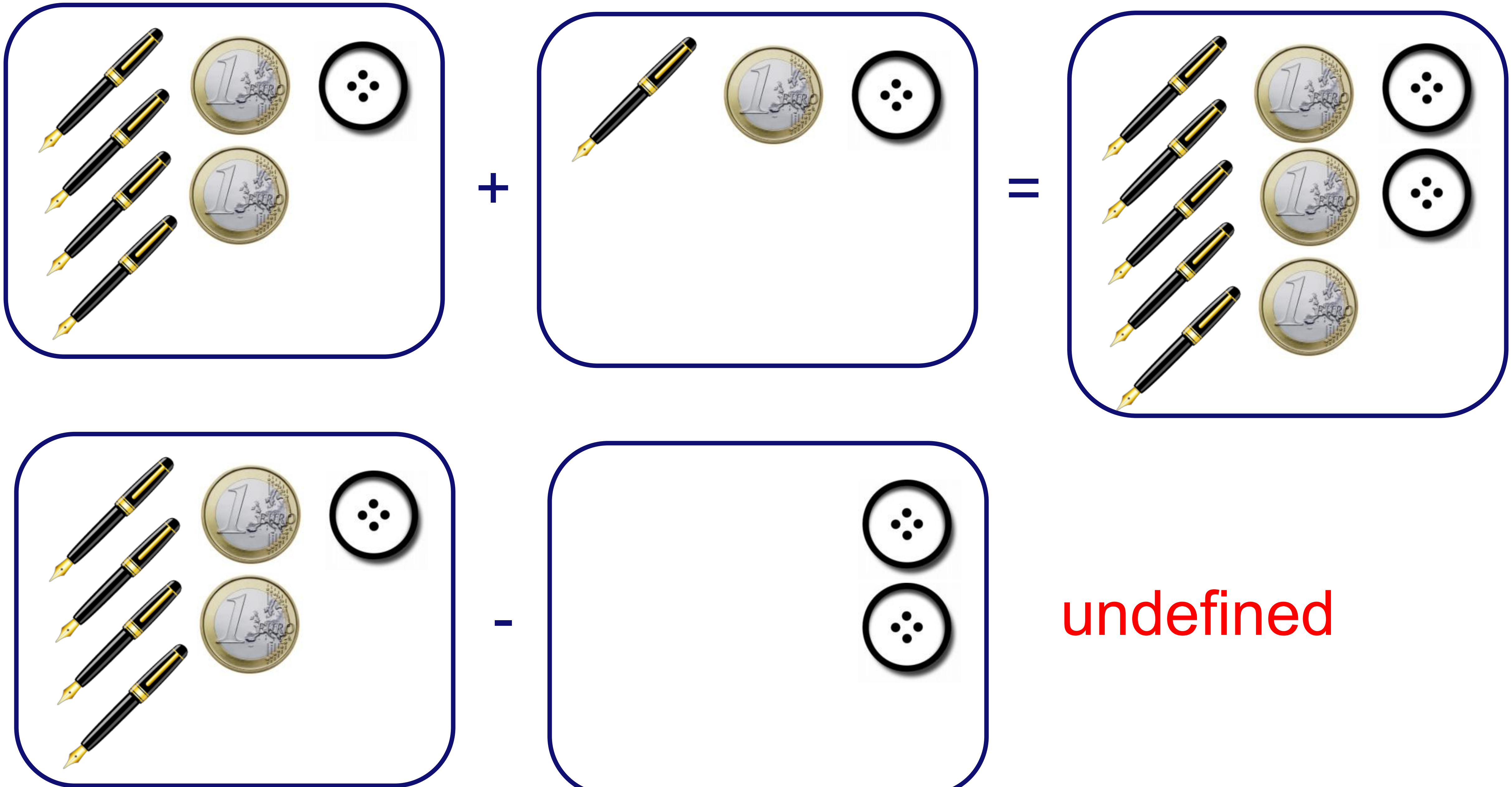
Multiset containment:
we write $M \subseteq M'$ if $M(x) \leq M'(x)$ for all $x \in S$

Multiset strict containment:
we write $M \subset M'$ if $M \subseteq M'$ and $M \neq M'$

Multiset union:
$M + M'$ is the multiset s.t. $(M + M')(x) = M(x) + M'(x)$ for all $x \in S$

Multiset difference (defined only if $M \supseteq M'$):
$M - M'$ is the multiset s.t. $(M - M')(x) = M(x) - M'(x)$ for all $x \in S$

Operations on Multisets



Notation: multisets

Multiset $M = \{ k_1 x_1, k_2 x_2, ..., k_n x_n\}$ as formal sum:

$$k_1 x_1 + k_2 x_2 + ... + k_n x_n$$

$$\sum_{i=1}^{n} k_i x_i$$

Question time

$$3a + 2b \overset{?}{\subseteq} 2a + 3b + c$$

$$3a + 2b \overset{?}{\supseteq} 2a + 3b + c$$

$$a + 2b \overset{?}{\subset} 2a + 3b$$

$$(a + 2b) + (2a + c) = ?$$

$$(2a + 3b) - (2a + b) = ?$$

$$(2a + 2b) - (a + c) = ?$$

Marking

A **marking** $M : P \to \mathbb{N}$ denotes the number of tokens in each place

The marking of a Petri net represents its state

$M(a) = 0$ denotes the absence of tokens in place a

Petri nets

A **Petri net** is a tuple (P, T, F, M_0) where

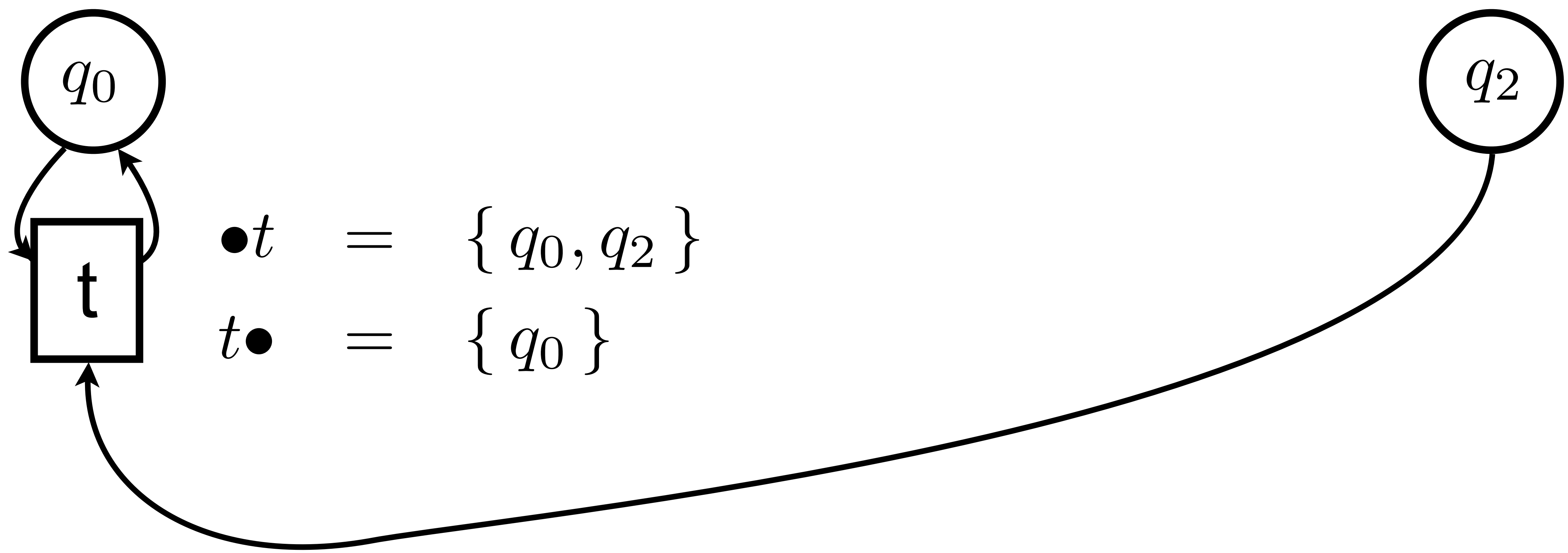
- P is a finite set of **places**;
- T is a finite set of **transitions**;
- $F \subseteq (P \times T) \cup (T \times P)$ is a **flow relation**;
- $M_0 : P \to \mathbb{N}$ is the **initial marking**.
  (i.e. $M_0 \in \mu(P)$)

Pre-set and post-set

A place p is an input place for transition t iff
$(p, t) \in F$
We let $\bullet t$ denote the set of input places of t.
(pre-set of t)

A place p is an output place for transition t iff
$(t, p) \in F$
We let $t\bullet$ denote the set of output places of t.
(post-set of t)

Example: pre and post



$$\begin{aligned} \bullet t &= \{\, q_0, q_2 \,\} \\ t\bullet &= \{\, q_0 \,\} \end{aligned}$$

Pre-set and post-set

Analogously, we let

$\bullet p$ denote the set of transitions that share p as output place

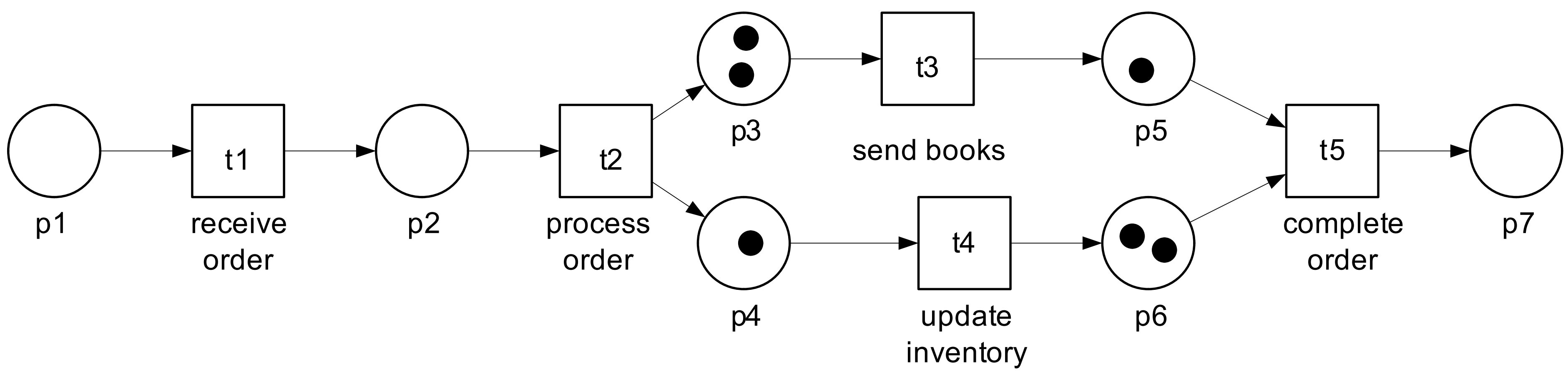
$p\bullet$ denote the set of transitions that share p as input place

Formally:

$$\bullet x = \{\ y \ \mid\ (y, x) \in F\ \}$$

$$x \bullet = \{\ y \ \mid\ (x, y) \in F\ \}$$

Exercises



$$P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$$
$$T = \{t_1, t_2, t_3, t_4, t_5\}$$
$$F = \{(p_1, t_1), (t_1, p_2), \ldots ?\}$$
$$M_0 = 2p_3 + \ldots ?$$

$\bullet t_1 = ?$ $\quad t_1 \bullet = ?$

$\bullet t_2 = ?$ $\quad t_2 \bullet = ?$

$\bullet t_3 = ?$ $\quad t_3 \bullet = ?$

$\bullet t_4 = ?$ $\quad t_4 \bullet = ?$

$\bullet t_5 = ?$ $\quad t_5 \bullet = ?$

$\bullet p_1 = ?$ $\quad p_1 \bullet = ?$

$\bullet p_2 = ?$ $\quad p_2 \bullet = ?$

$\bullet p_3 = ?$ $\quad p_3 \bullet = ?$

$\bullet p_4 = ?$ $\quad p_4 \bullet = ?$

$\bullet p_5 = ?$ $\quad p_5 \bullet = ?$

$\bullet p_6 = ?$ $\quad p_6 \bullet = ?$

$\bullet p_7 = ?$ $\quad p_7 \bullet = ?$

Petri nets:
enabling and firing

Enabling M[t>

A transition t is **enabled** at marking M iff $\bullet t \subseteq M$
and we write $M \xrightarrow{t}$ (also $M\,[t\rangle$)

A transition is enabled if each of its input places contains at least one token

Firing M[t>M'

A transition t that is enabled at M can **fire**.
The **firing** of t at M changes the state to

$$M' = M - \bullet t + t\bullet$$

and we write $M \xrightarrow{t} M'$ (also $M\,[t\rangle\,M'$)

When a transition fires
it consumes a token from each input place
it produces a token into each output place

Some remarks

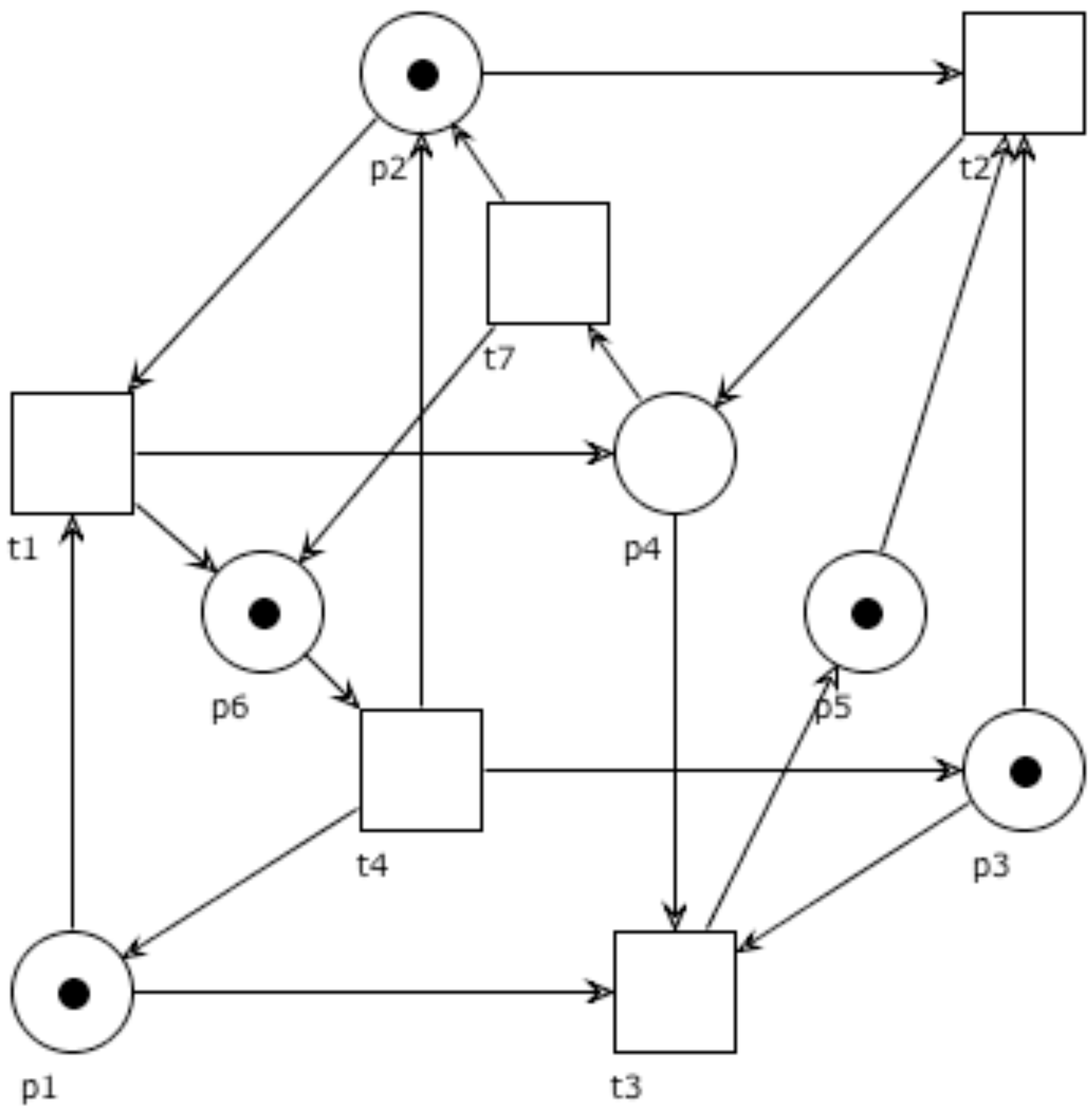
Firing is an atomic action

Our semantics is interleaving:
multiple transitions may be enabled,
but only one fires at a time

The network is static, but
the overall number of tokens may vary over time
(if transitions are fired for which the number of input places is not equal to the number of output places)

Question time

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$

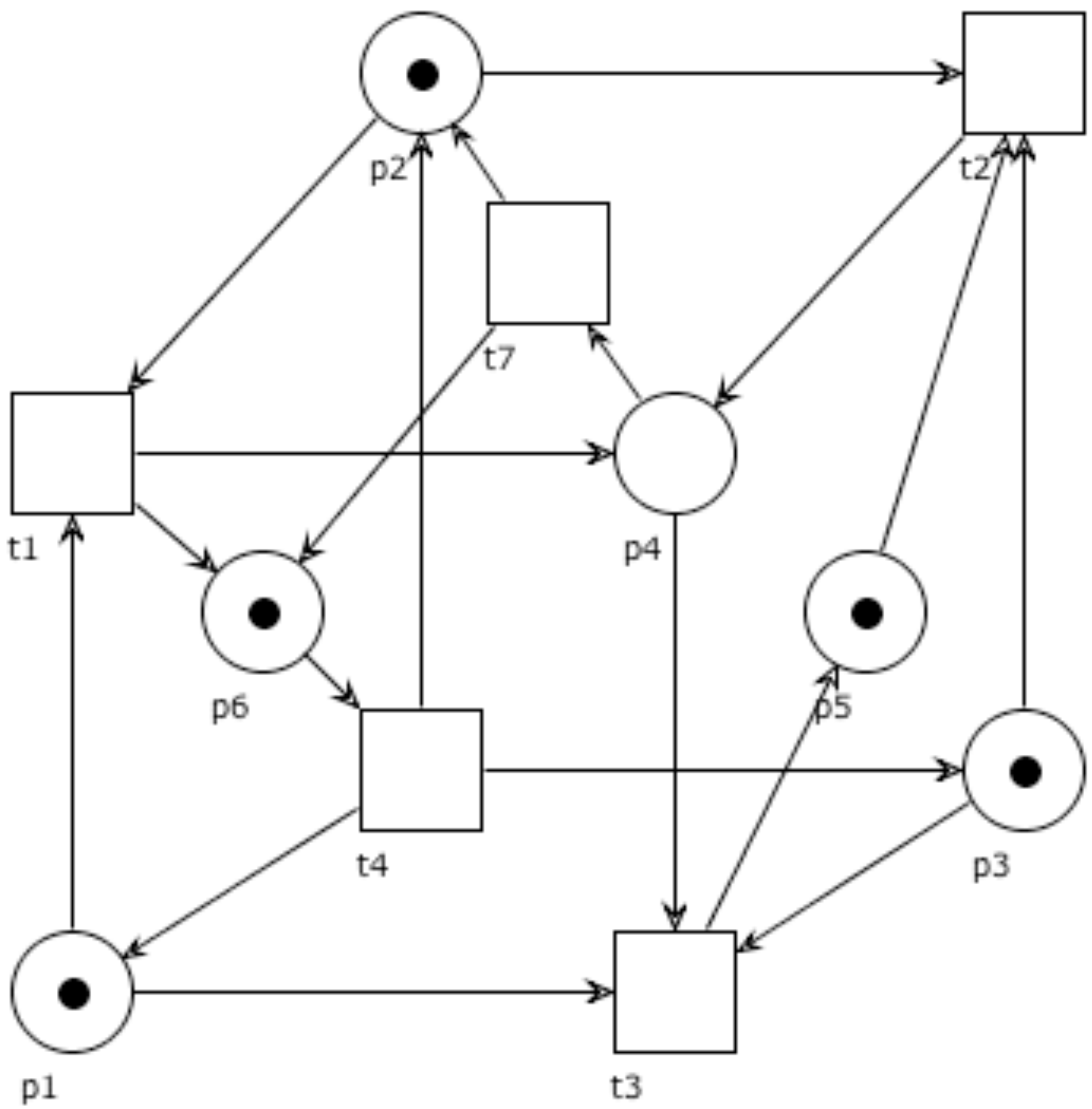


Which of the following holds true?

- $M_0 \xrightarrow{t_1}$
- $M_0 \xrightarrow{t_2}$
- $M_0 \xrightarrow{t_3}$
- $M_0 \xrightarrow{t_7}$

Question time

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$



Which of the following holds true?

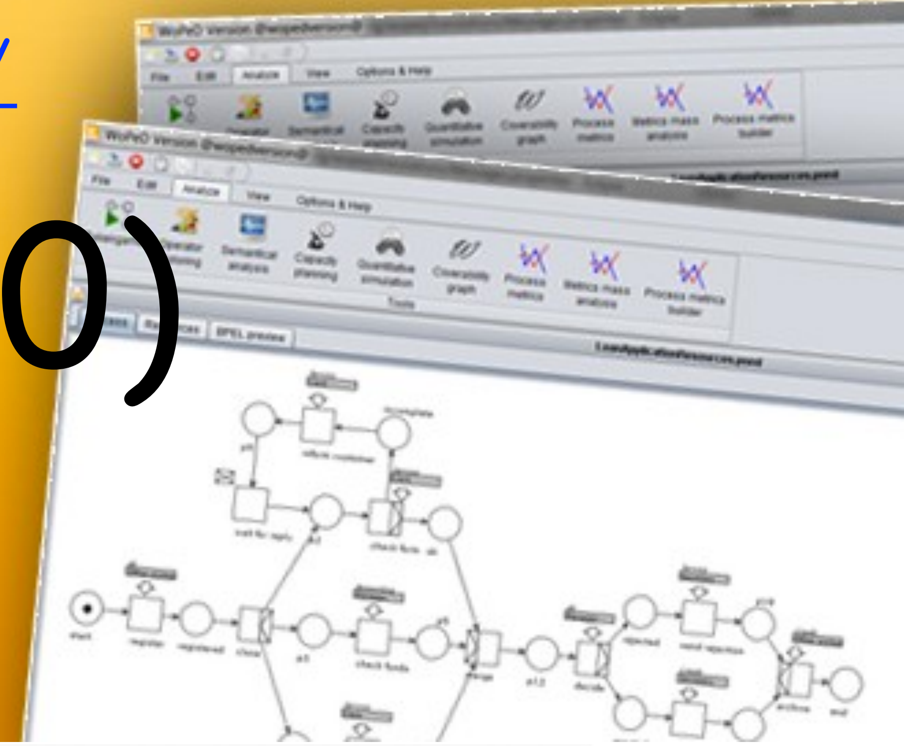
- $M_0 \xrightarrow{t_1} p_3 + p_4 + p_5 + p_6$
- $M_0 \xrightarrow{t_2} p_1 + p_4 + p_6$
- $M_0 \xrightarrow{t_4} 2p_1 + 2p_2 + 2p_3 + p_5$

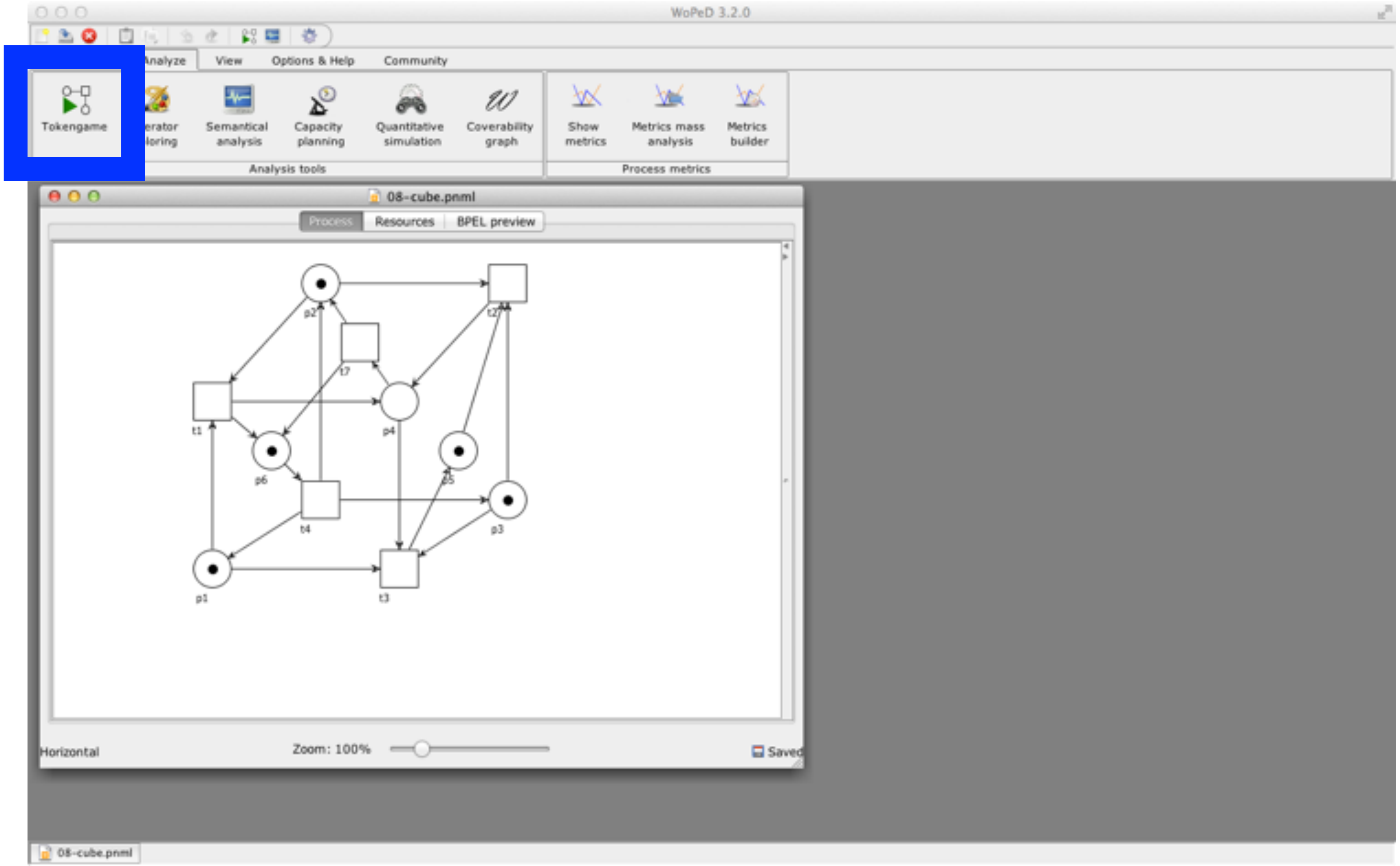
http://woped.dhbw-karlsruhe.de/woped/

WoPeD (3.2.0)

Workflow Petri Net Designer

Download WoPeD at sourceforge!





Notation

We write $M \rightarrow$ if $M \xrightarrow{t}$ for some transition t

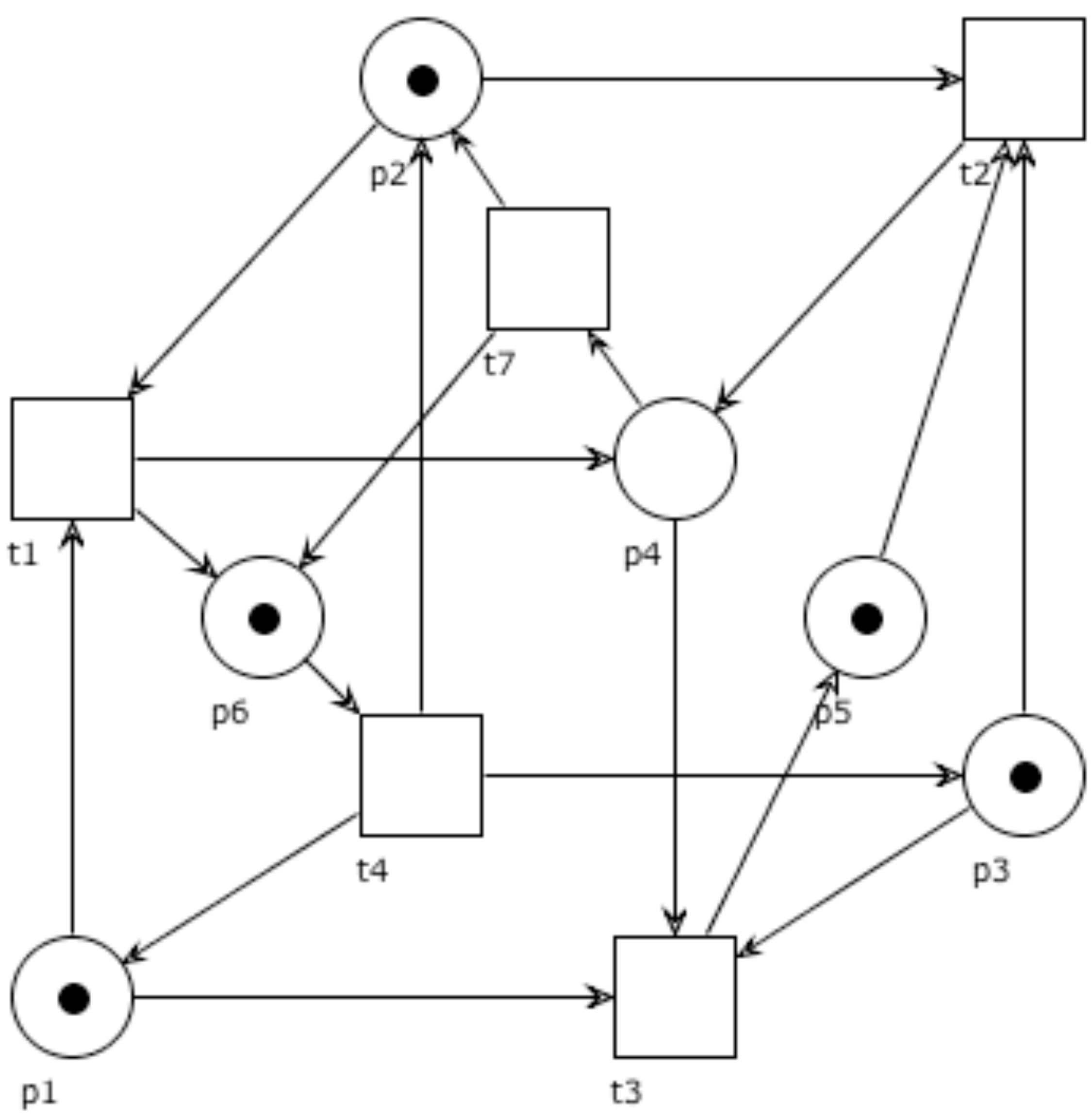
We write $M \rightarrow M'$ if $M \xrightarrow{t} M'$ for some transition t

We write $M \stackrel{t}{\not\rightarrow}$ if transition t is not enabled at M

We write $M \not\rightarrow$ if no transition is enabled at M

Example

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$



We can write that

- $M_0 \longrightarrow$
- $M_0 \longrightarrow p_1 + p_4 + p_6$
- $M_0 \not\xrightarrow{t_7}$
- $p_1 + p_5 \not\longrightarrow$

Firing sequence

Let $\sigma = t_1 t_2 ... t_{n-1} \in T^*$ be a sequence of transitions.

We write $M \xrightarrow{\sigma} M'$ (and $M \xrightarrow{\sigma}$) if:

there is a sequence of markings $M_1, ..., M_n$

with $M = M_1$ and $M' = M_n$

and $M_i \xrightarrow{t_i} M_{i+1}$ for $1 \leq i < n$

(i.e. $M = M_1 \xrightarrow{t_1} M_2 \xrightarrow{t_2} ... \xrightarrow{t_{n-1}} M_n = M'$)

Reachable markings [M>

We write $M \xrightarrow{*} M'$ if $M \xrightarrow{\sigma} M'$ for some $\sigma \in T^*$

A marking M' is **reachable from** M if $M \xrightarrow{*} M'$

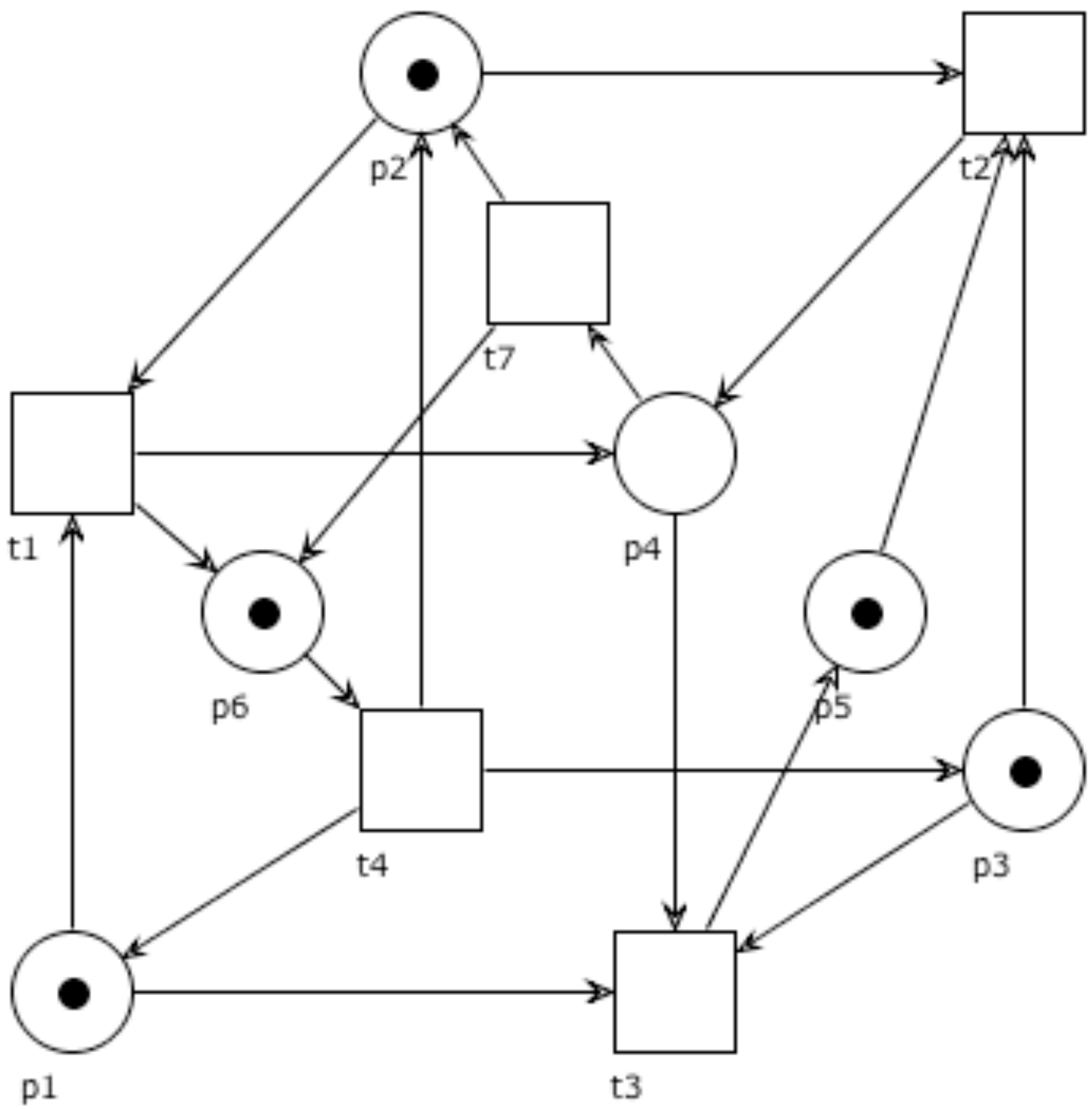
Note that $M \xrightarrow{\epsilon} M$ for ϵ the empty sequence

The set of markings reachable from M is often denoted:

$$reach(M) \text{ or also } [M\rangle$$

Question time

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$

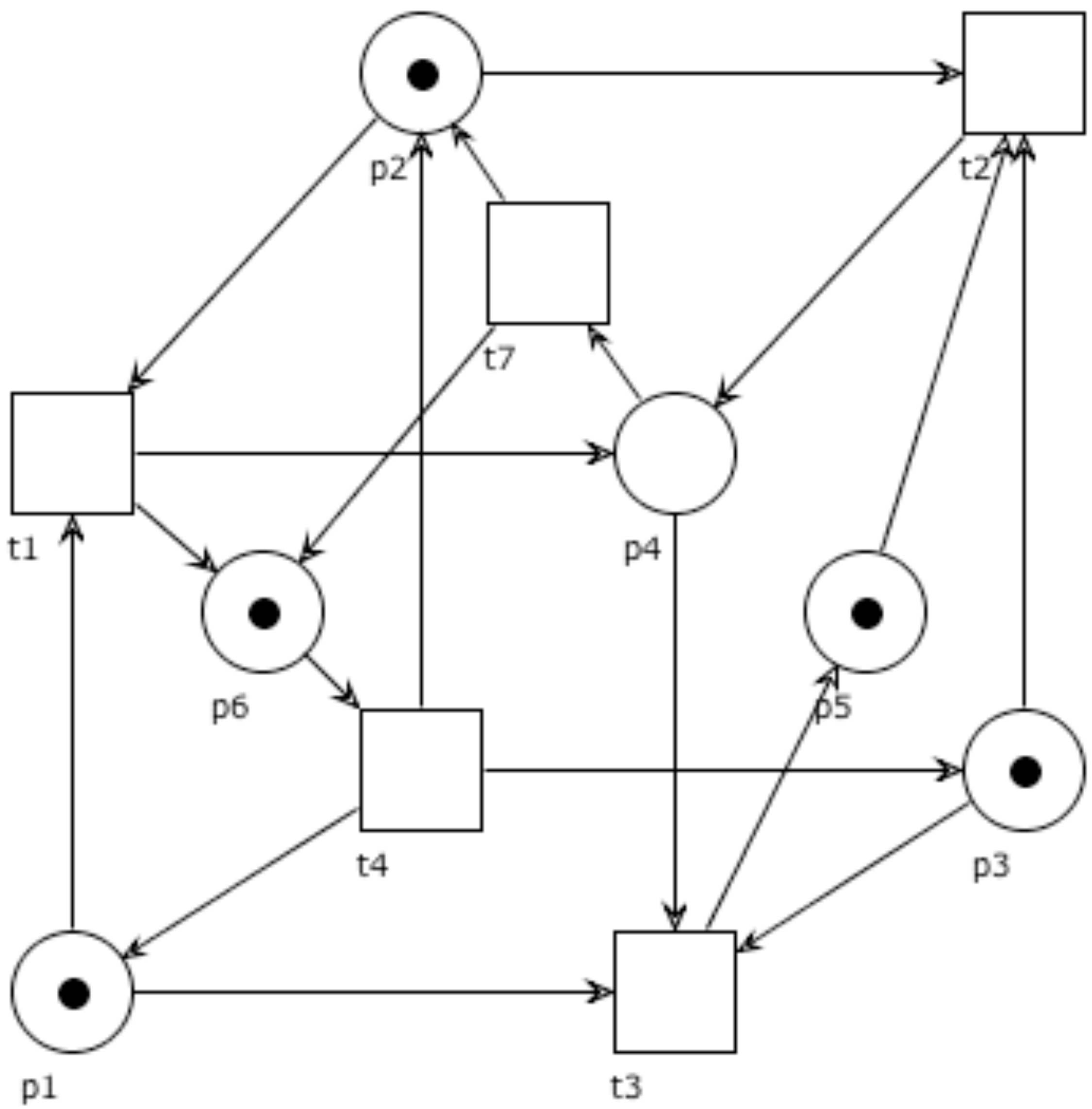


Which of the following holds true?

- $M_0 \xrightarrow{t_1 t_4 t_2 t_3}$
- $M_0 \xrightarrow{t_2 t_7 t_4}$
- $M_0 \xrightarrow{t_1 t_2 t_7}$
- $M_0 \xrightarrow{t_1 t_4 t_2 t_1}$

Example

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$



We have that

- $M_0 \xrightarrow{t_1 t_4 t_2 t_3} p_4 + p_5 + p_6$
- $M_0 \xrightarrow{t_2 t_7 t_4} 2p_1 + 2p_2 + p_3 + p_6$
- $M_0 \xrightarrow{t_1 t_4 t_3 t_2 t_7} p_2 + p_5 + 2p_6$

Infinite sequence

Let $\sigma = t_1 t_2 ... \in T^\omega$ be an infinite sequence of transitions.

We write $M \xrightarrow{\sigma}$ if:

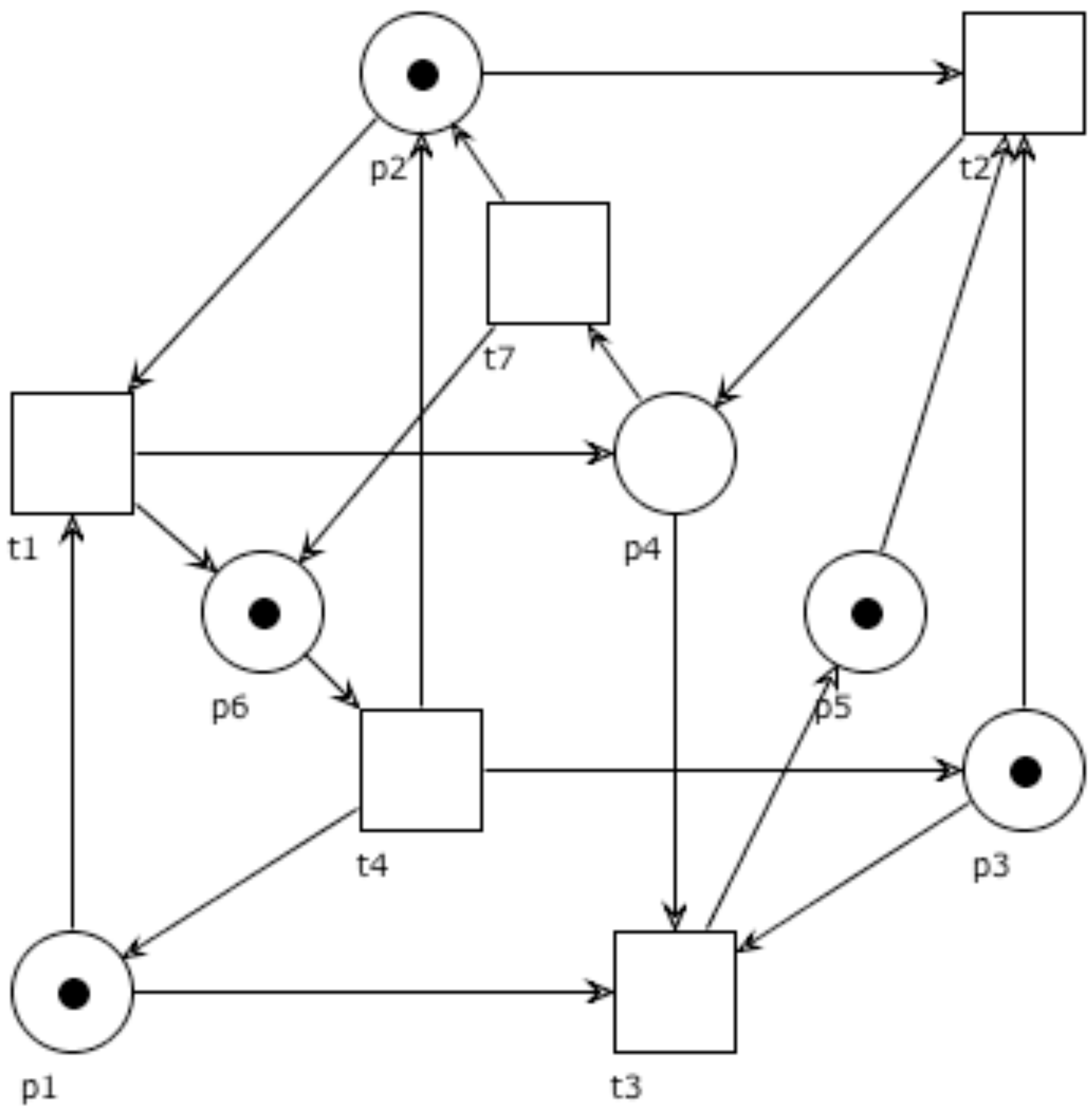
there is an infinite sequence of markings $M_1, M_2, ...$

with $M = M_1$ and $M_i \xrightarrow{t_i} M_{i+1}$ for $1 \leq i$

(i.e. $M = M_1 \xrightarrow{t_1} M_2 \xrightarrow{t_2} ...$)

Example

$$M_0 = p_1 + p_2 + p_3 + p_5 + p_6$$



We have that

- $M_0 \xrightarrow{t_1 t_4 t_1 t_4 t_1 t_4 \cdots}$
- $M_0 \xrightarrow{t_1 t_4 t_7 t_1 t_4 t_7 t_1 t_4 t_7 \cdots}$

Enabled sequence

We say that an occurrence sequence σ is **enabled** if $M \xrightarrow{\sigma}$

(σ can be finite or infinite)

Note that an infinite sequence can be represented as a map $\sigma : \mathbb{N} \rightarrow T$, where $\sigma(i) = t_i$

More on sequences: concatenation & prefix

Concatenation: finite + finite = finite

for $\sigma_1 = a_1 ... a_n$ and $\sigma_2 = b_1 ... b_m$, we let $\sigma_1 \sigma_2 = a_1 ... a_n b_1 ... b_m$

for $\sigma_1 = a_1 ... a_n$ and $\sigma_2 = b_1 b_2 ...$, we let $\sigma_1 \sigma_2 = a_1 ... a_n b_1 b_2 ...$

finite + infinite = infinite

σ is a **prefix** of σ' if $\sigma = \sigma'$ or $\sigma\sigma'' = \sigma'$ for some $\sigma'' \neq \epsilon$

σ is a **proper prefix** of σ' if $\sigma\sigma'' = \sigma'$ for some $\sigma'' \neq \epsilon$

Enabledness

Proposition: $M \xrightarrow{\sigma}$ iff $M \xrightarrow{\sigma'}$ for every prefix σ' of σ

($\Rightarrow$) immediate from definition

($\Leftarrow$) trivial if σ is finite (σ itself is a prefix of σ)

When σ is infinite: taken any $i \in \mathbb{N}$ we need to prove that $t_i = \sigma(i)$ is enabled after the firing of the prefix $\sigma' = t_1 t_2 ... t_{i-1}$ of σ.

But this is obvious, because

$$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} ... \xrightarrow{t_{i-1}} M_{i-1} \xrightarrow{t_i} M_i$$

is also a finite prefix of σ and therefore $M_{i-1} \xrightarrow{t_i}$

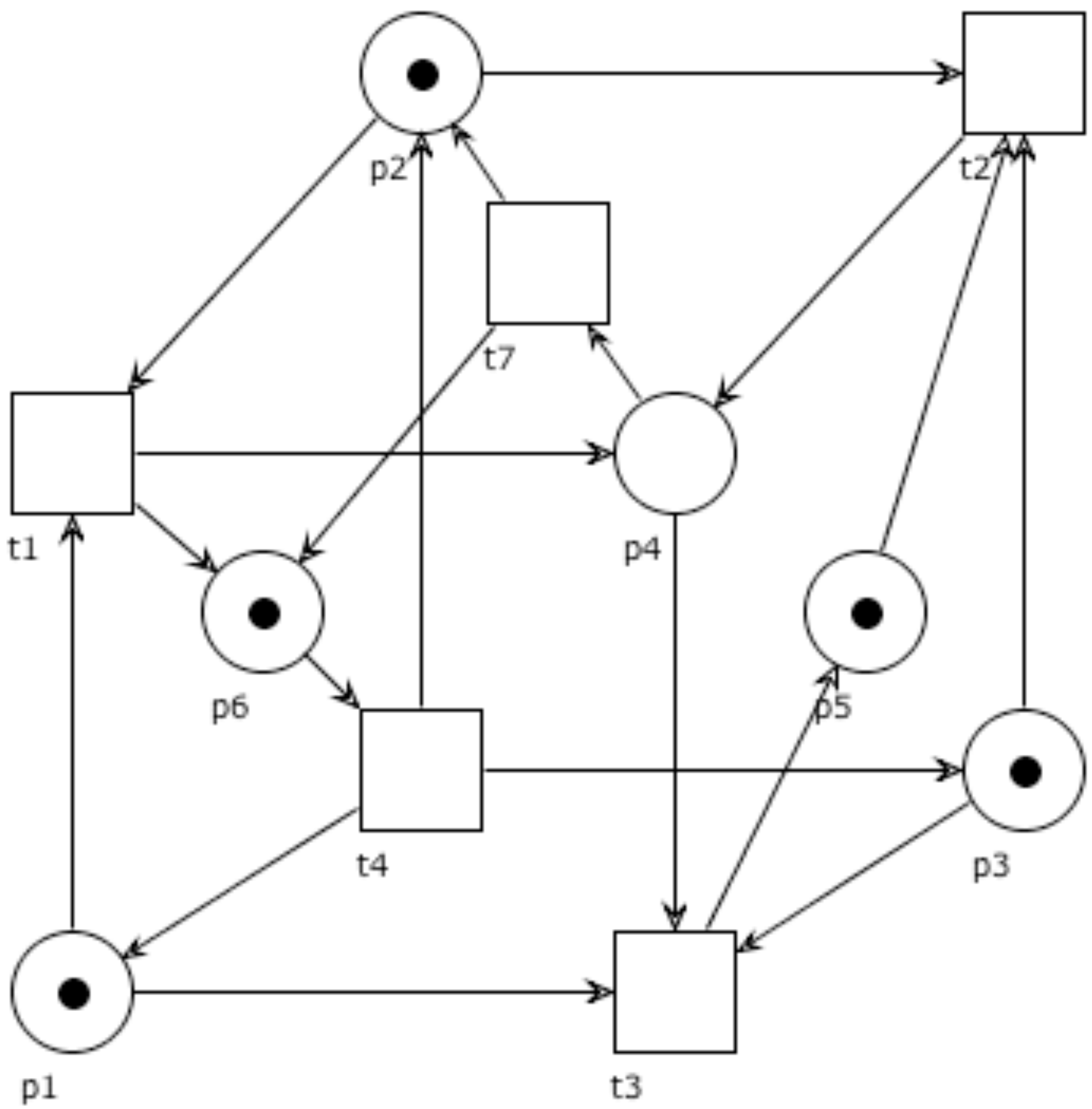
More on sequences: projection

Restriction: (also extraction / projection)
given $T' \subseteq T$ we inductively define $\sigma_{|T'}$ as:

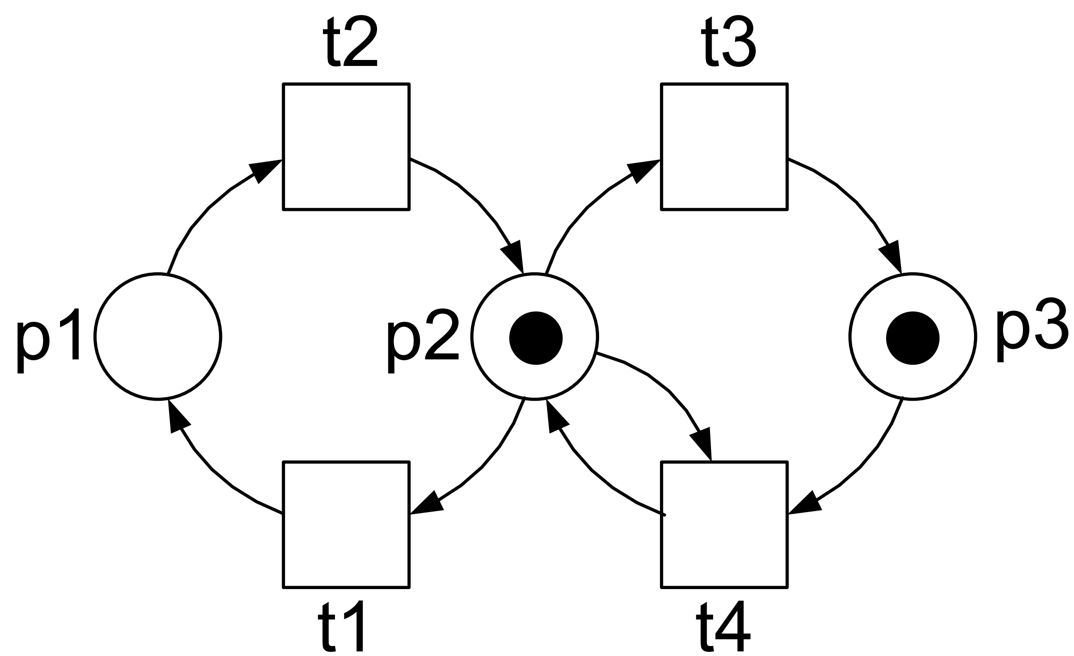
$$\epsilon_{|T'} = \epsilon \qquad (t\sigma)_{|T'} = \begin{cases} t(\sigma_{|T'}) & \text{if } t \in T' \\ \sigma_{|T'} & \text{if } t \notin T' \end{cases}$$

Example

$$
\begin{aligned}
(t_1 t_4 t_7 t_1 t_4 t_7)_{|\{t_1,t_4\}} &= t_1 (t_4 t_7 t_1 t_4 t_7)_{|\{t_1,t_4\}} \\
&= t_1 t_4 (t_7 t_1 t_4 t_7)_{|\{t_1,t_4\}} \\
&= t_1 t_4 (t_1 t_4 t_7)_{|\{t_1,t_4\}} \\
&= t_1 t_4 t_1 (t_4 t_7)_{|\{t_1,t_4\}} \\
&= t_1 t_4 t_1 t_4 (t_7)_{|\{t_1,t_4\}} \\
&= t_1 t_4 t_1 t_4 (t_7 \epsilon)_{|\{t_1,t_4\}} \\
&= t_1 t_4 t_1 t_4 (\epsilon)_{|\{t_1,t_4\}} \\
&= t_1 t_4 t_1 t_4 \epsilon \\
&= t_1 t_4 t_1 t_4
\end{aligned}
$$



Exercises

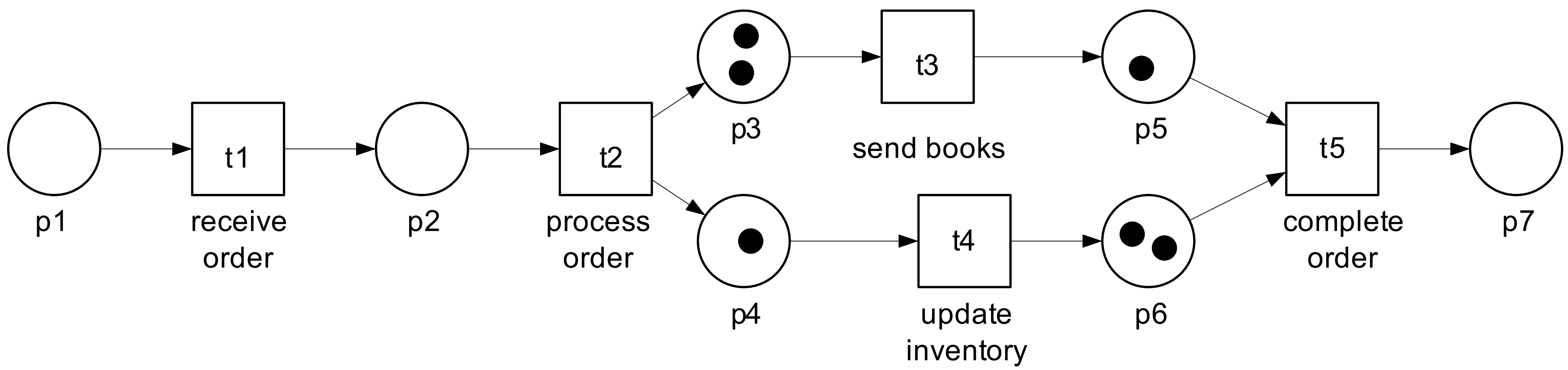


Determine the pre- and post-set of each element

Which are the currently enabled transitions?
For each of them, which state would the firing lead to?

What are the reachable states?

Exercises



Which are the currently enabled transitions?

For each of them, which state would the firing lead to?

What are the reachable states?

Petri nets:
occurrence graph

Occurrence graph (aka Reachability graph)

The reachability graph is a graph that represents all possible occurrence sequences of a net

Nodes of the graphs = reachable markings
Arcs of the graphs = firings

Formally, $OG(N) = ([M_0\rangle, A)$ where $A \subseteq [M_0\rangle \times T \times [M_0\rangle$ s.t.

$$(M, t, M') \in A \quad \text{iff} \quad M \xrightarrow{t} M'$$

How to compute OG(N)

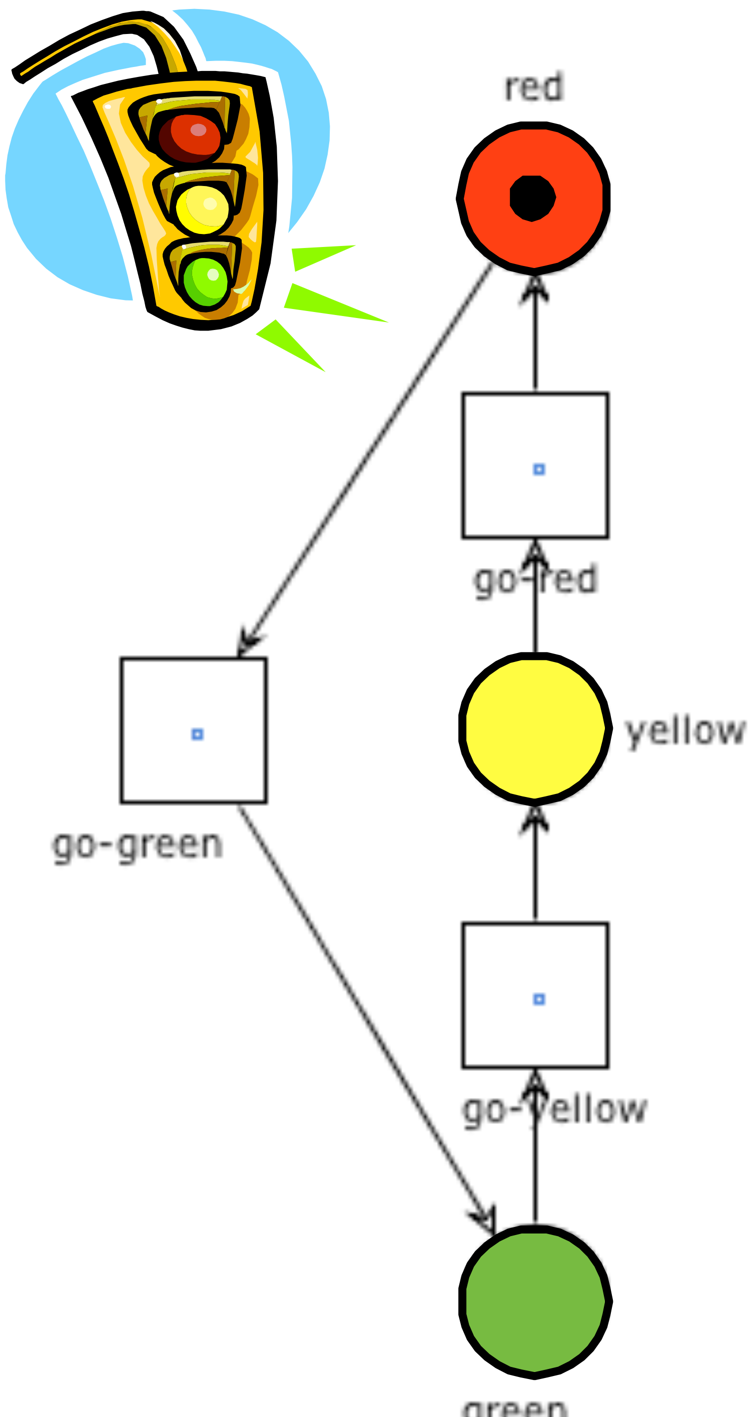
1. Initially R = { M_0 } and A = $\emptyset$
2. Take a marking M $\in$ R and a transition t $\in$ T such that
   1. M enables t and there is no arc labelled t leaving from M
3. Let M' = M - •t + t•
4. Add M' to R and (M,t,M') to A
5. Repeat steps 2,3,4 until no new arc can be added

How to compute OG(N)

The occurrence graph can be constructed as follows:

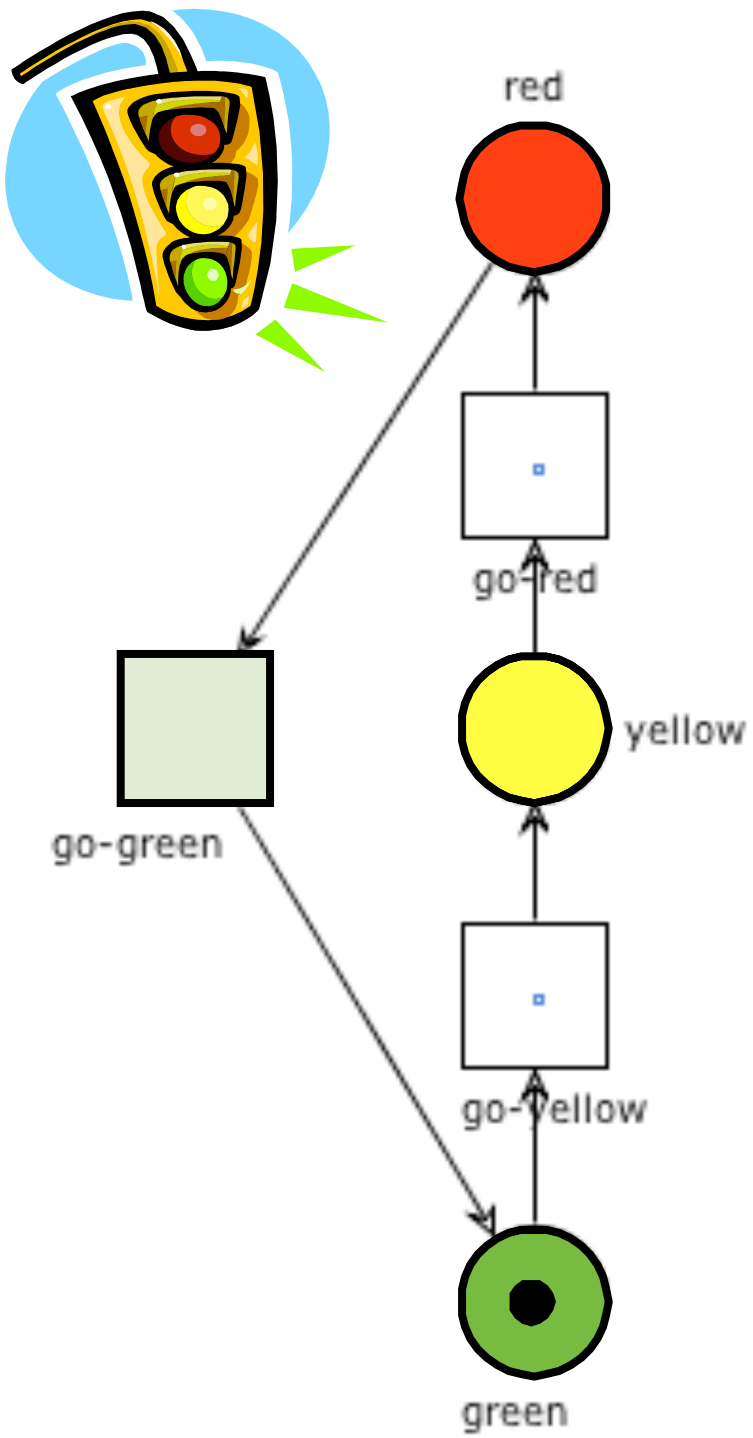
1. $Nodes = \{\},\ Arcs = \{\},\ Todo = \{M_0\}$
2. $M\ = next(Todo)$
3. $Nodes = Nodes \cup \{M\},\ Todo = Todo \setminus \{M\}$
4. $Firings = \{(M, t, M') \mid \exists t \in T, \exists M' \in \mu(P),\ M \xrightarrow{t} M'\}$
5. $New = \{M' \mid (M, t, M') \in Firings\} \setminus (Nodes \cup Todo)$
6. $Todo = Todo \cup New,\ Arcs = Arcs \cup Firings$
7. $isEmpty(Todo)\ ?\ \textbf{stop}\ :\ \textbf{goto}\, 2$

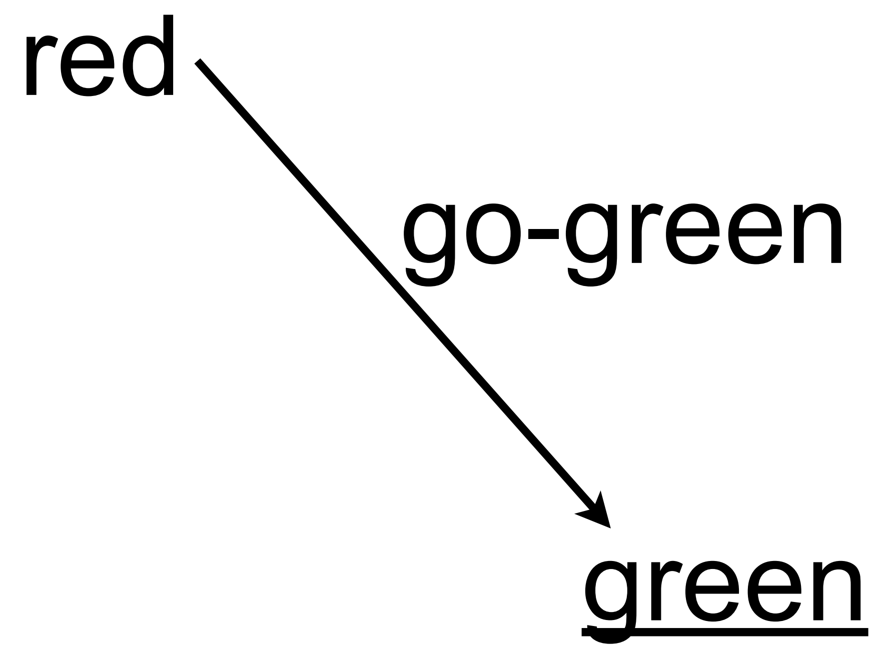
Example: traffic light



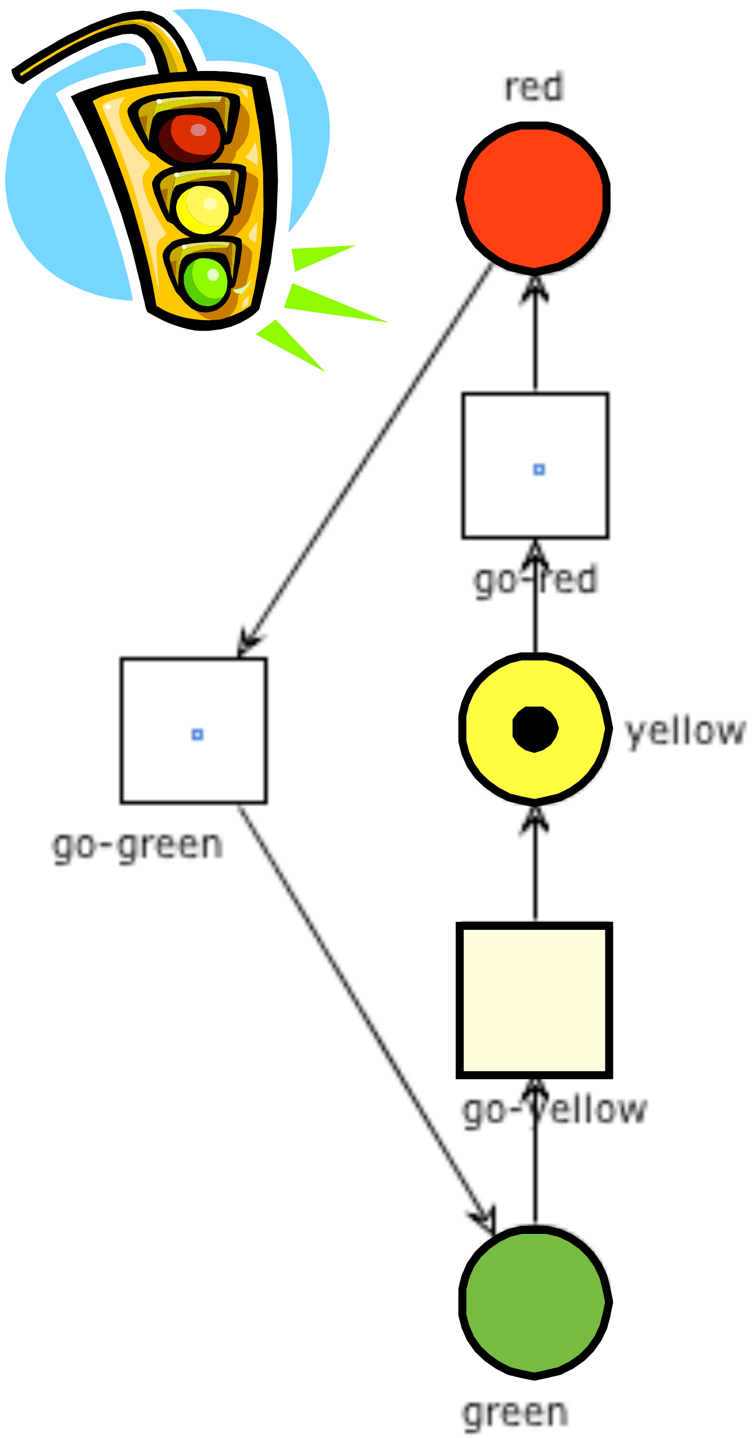
<u>red</u>

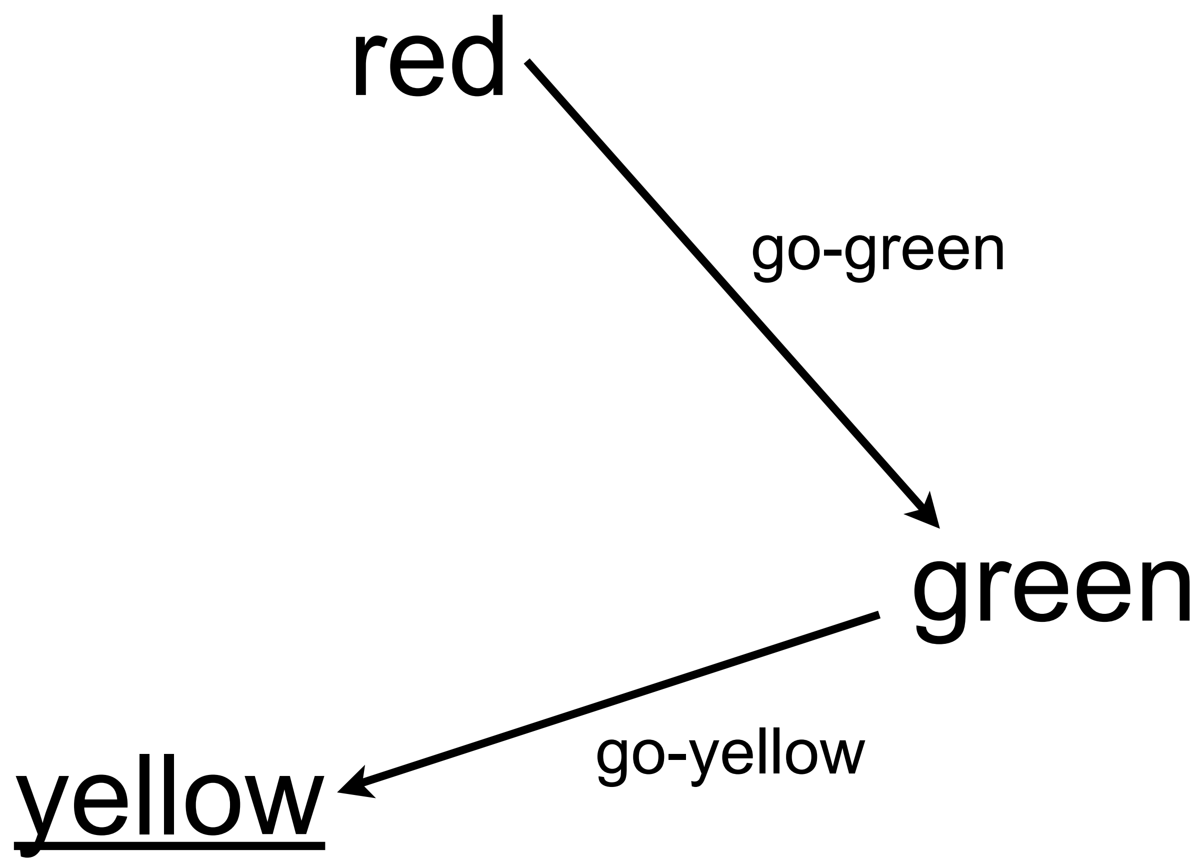
Example: traffic light



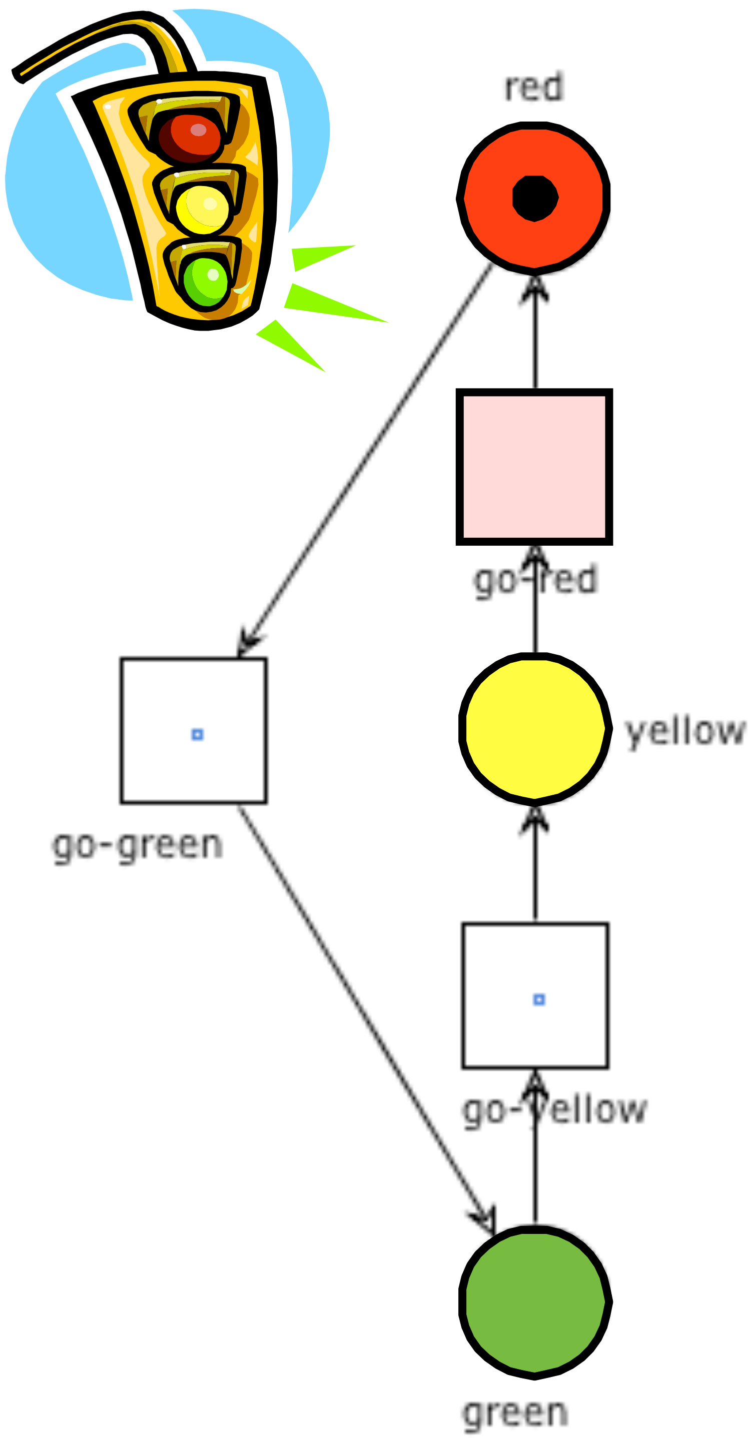


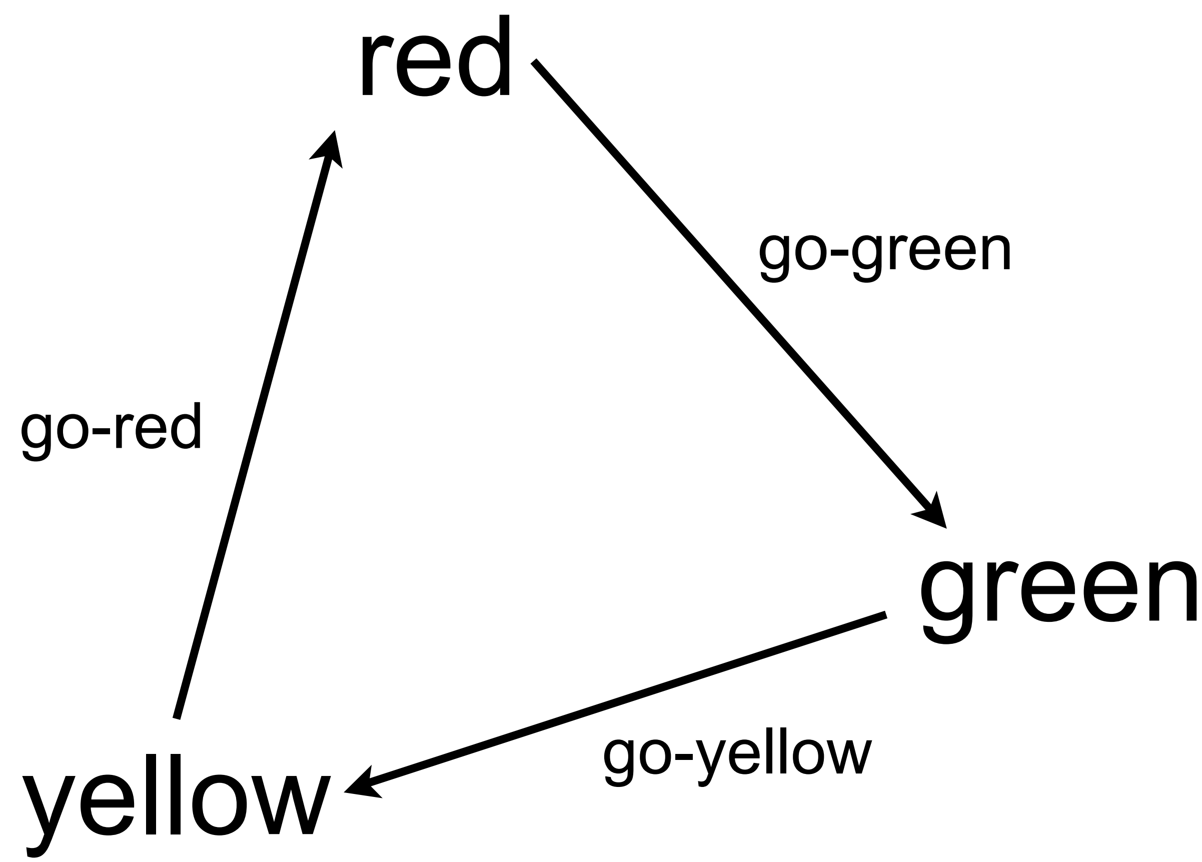
Example: traffic light



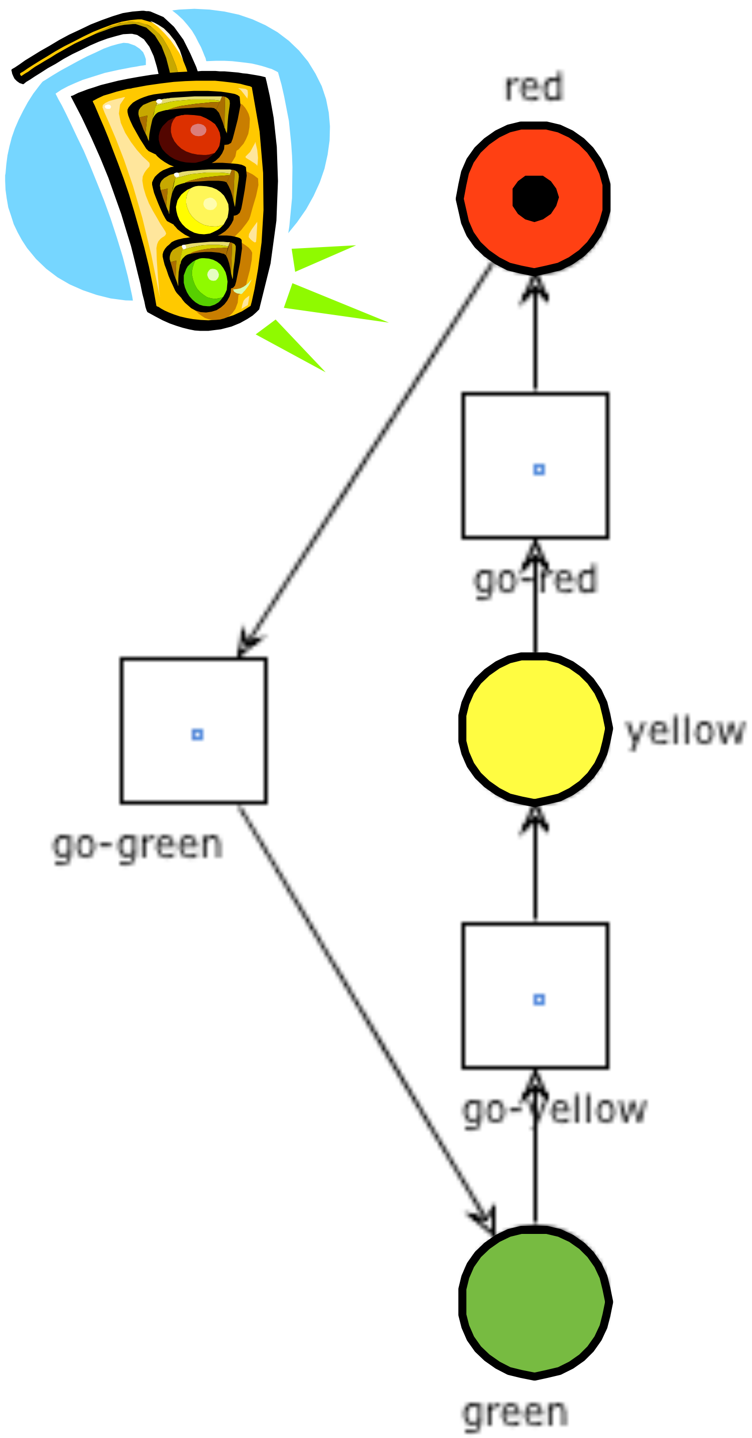


Example: traffic light

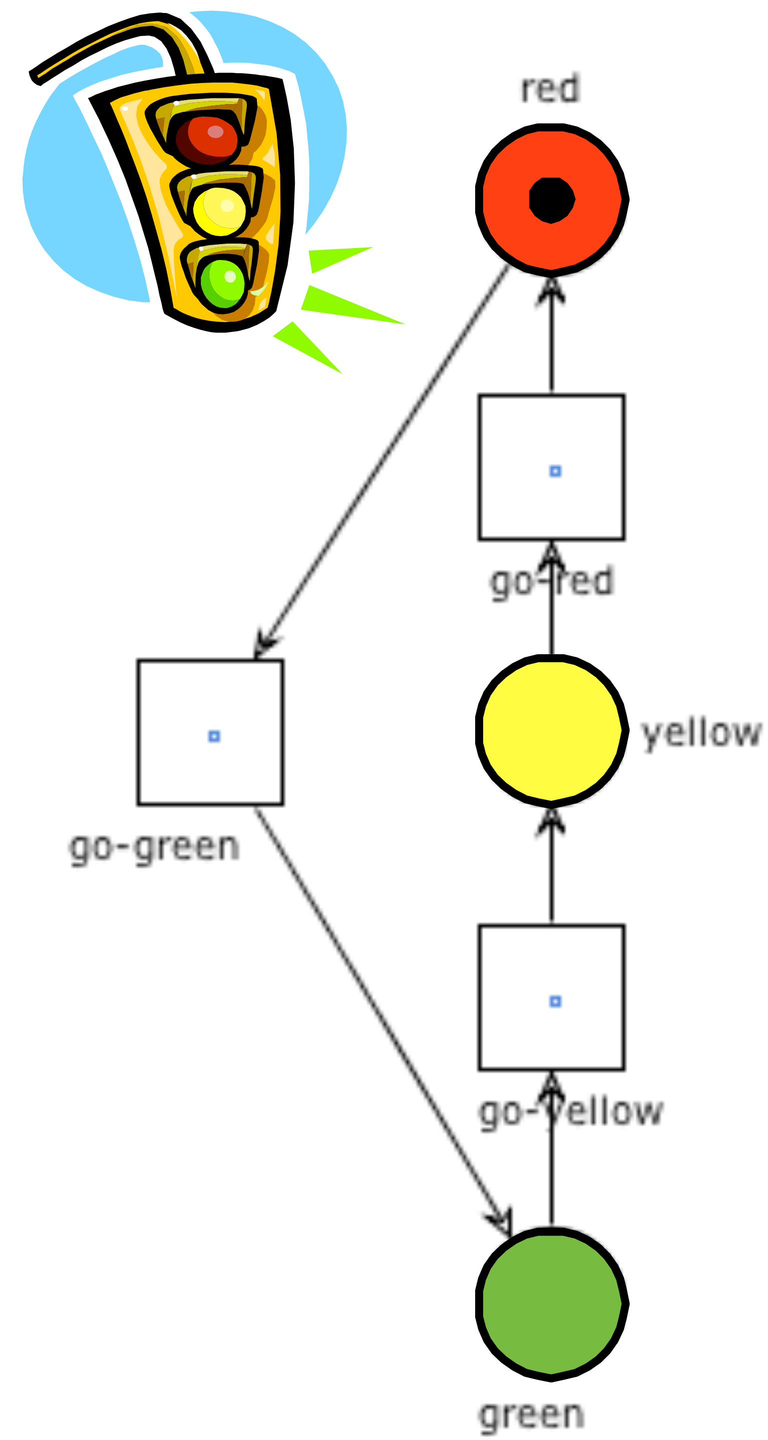




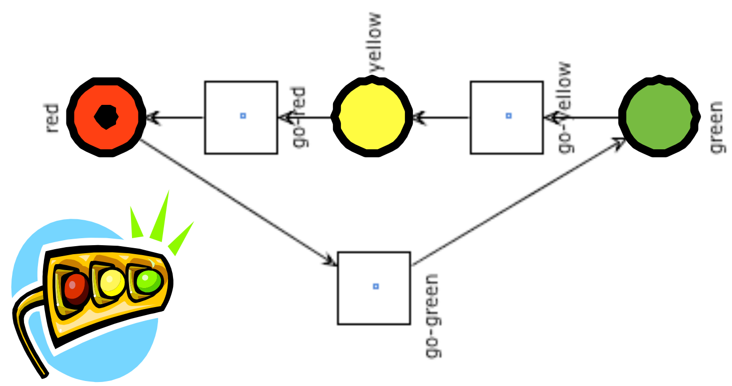
Example: two traffic lights

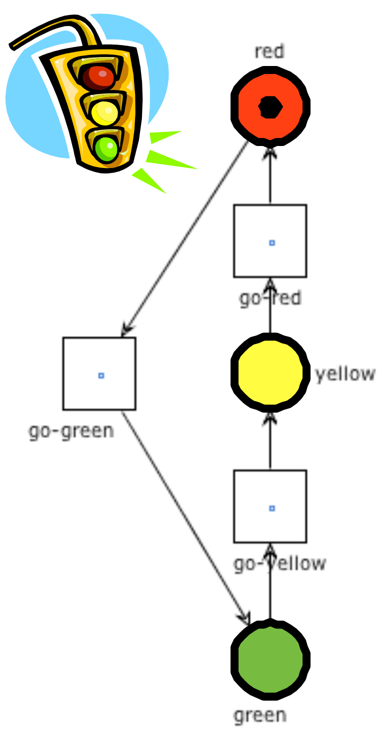


red + red’



Example: two traffic lights

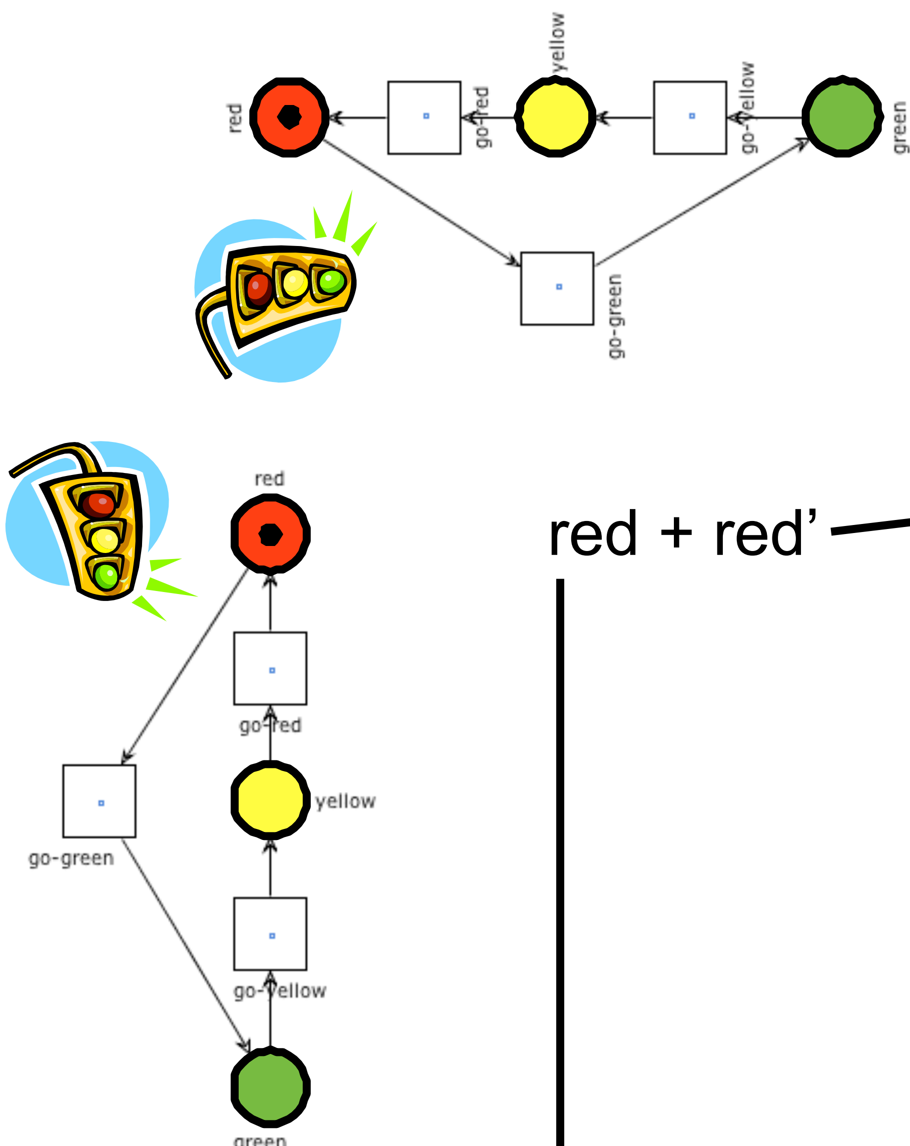


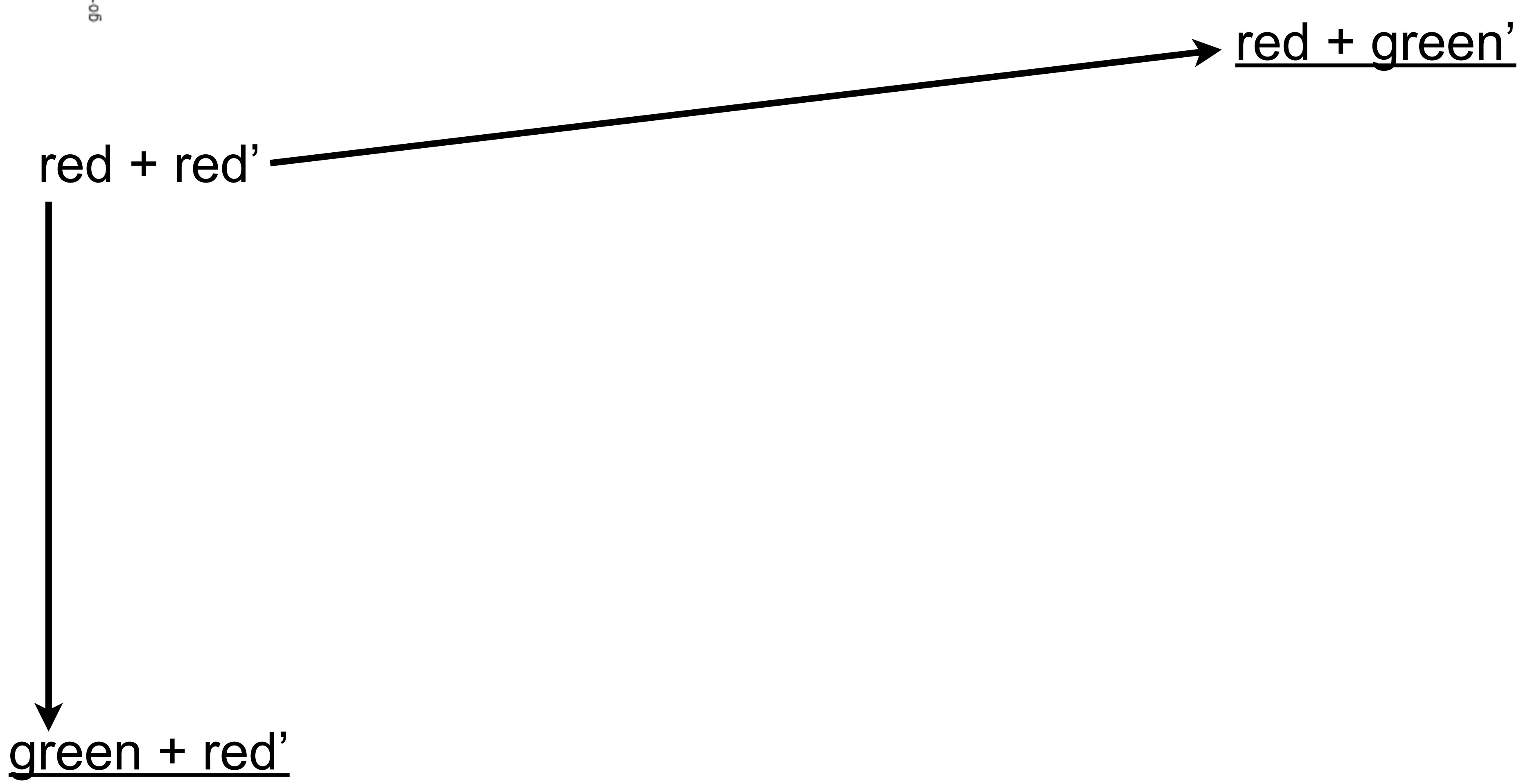


red + red’

(we omit arc labels
for readability issues)

Example: two traffic lights





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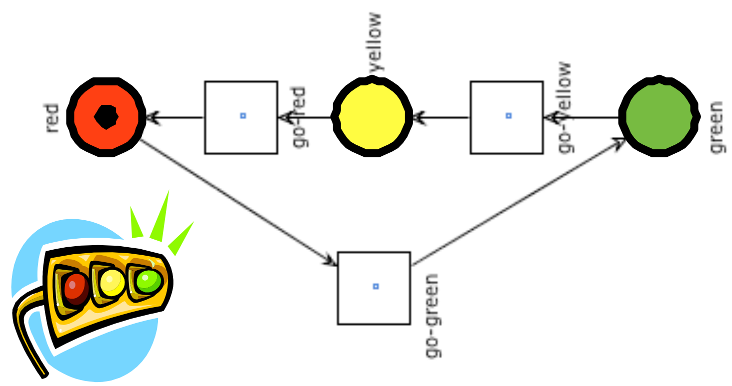

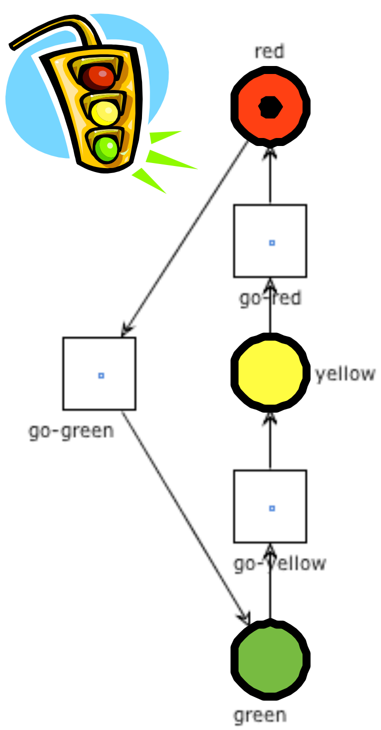

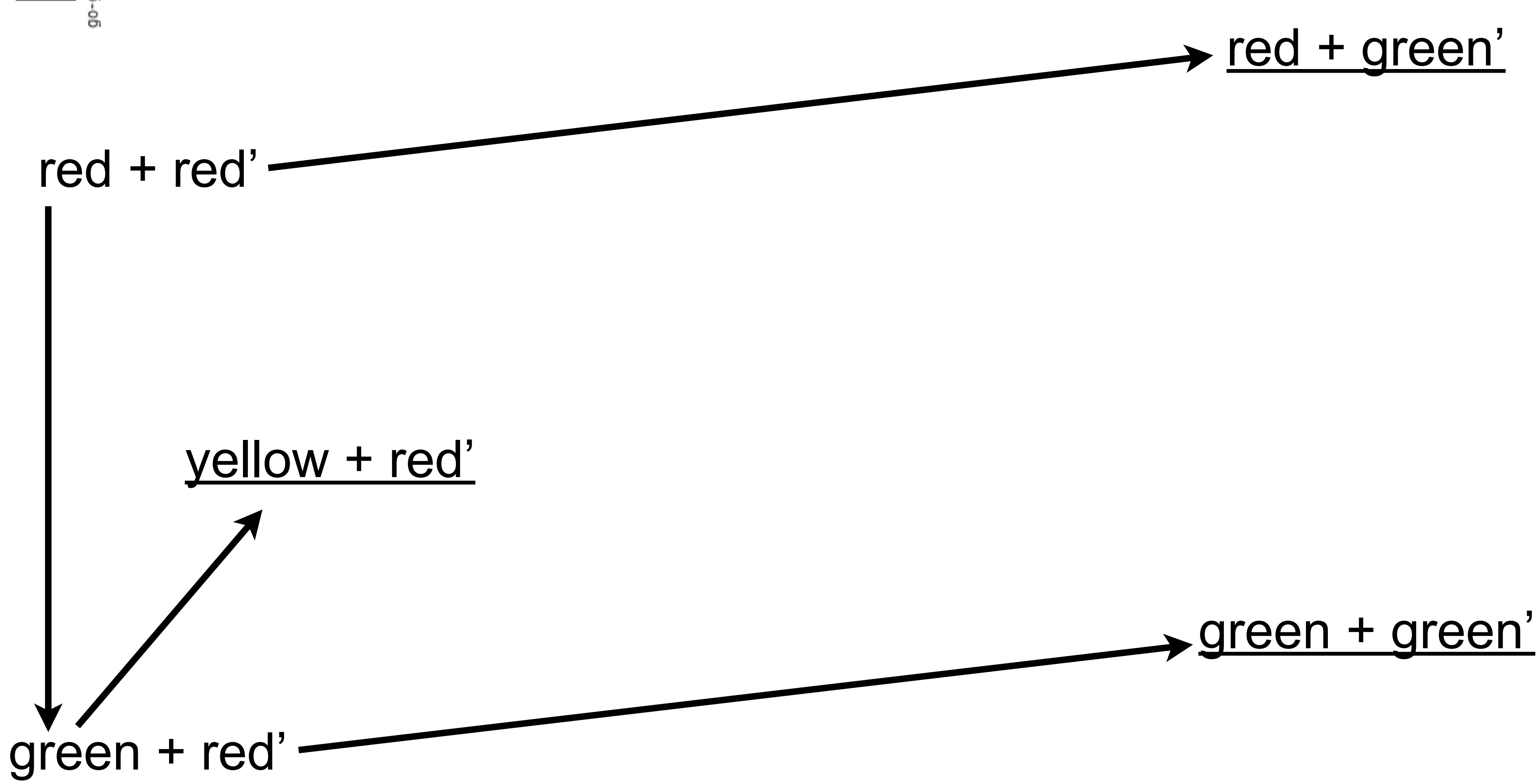

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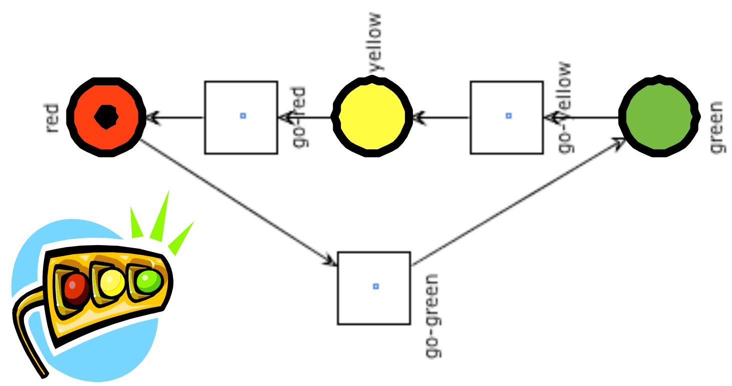

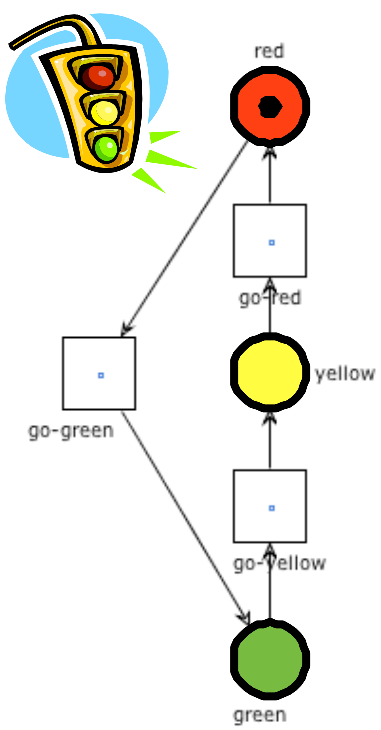

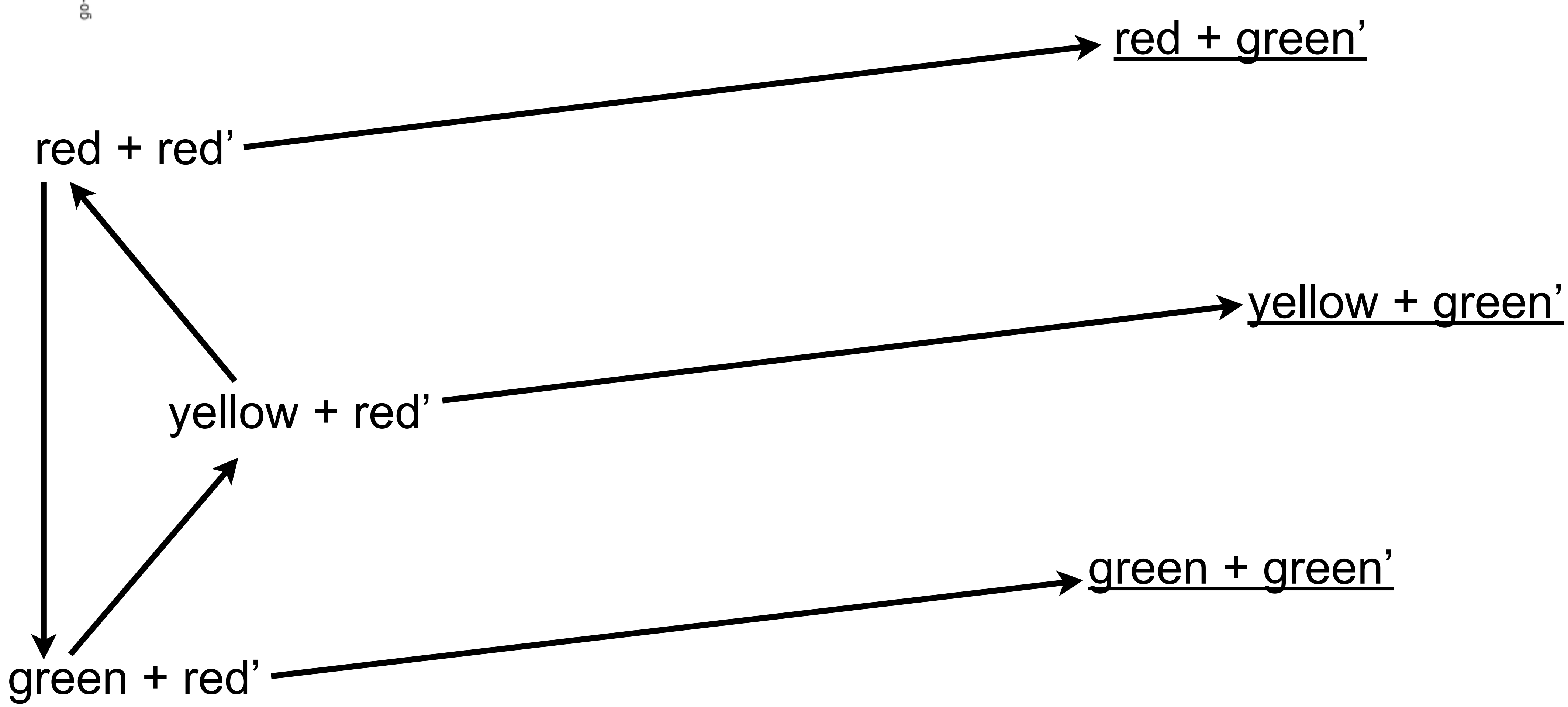

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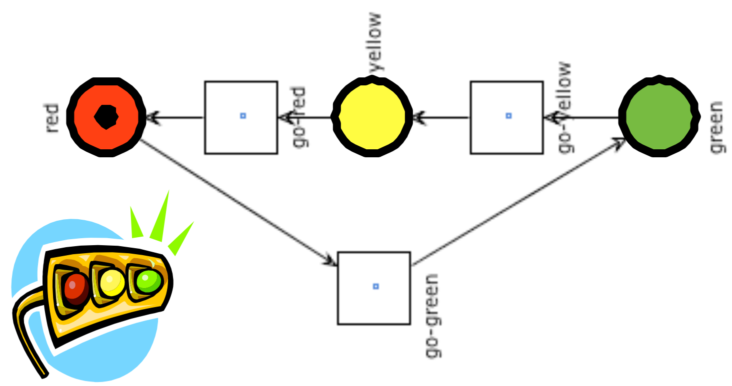

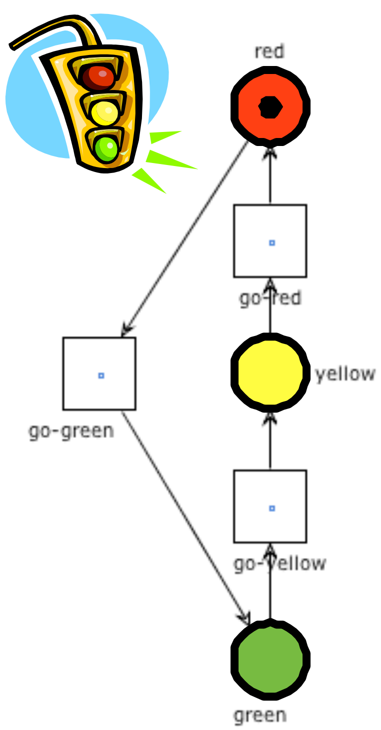

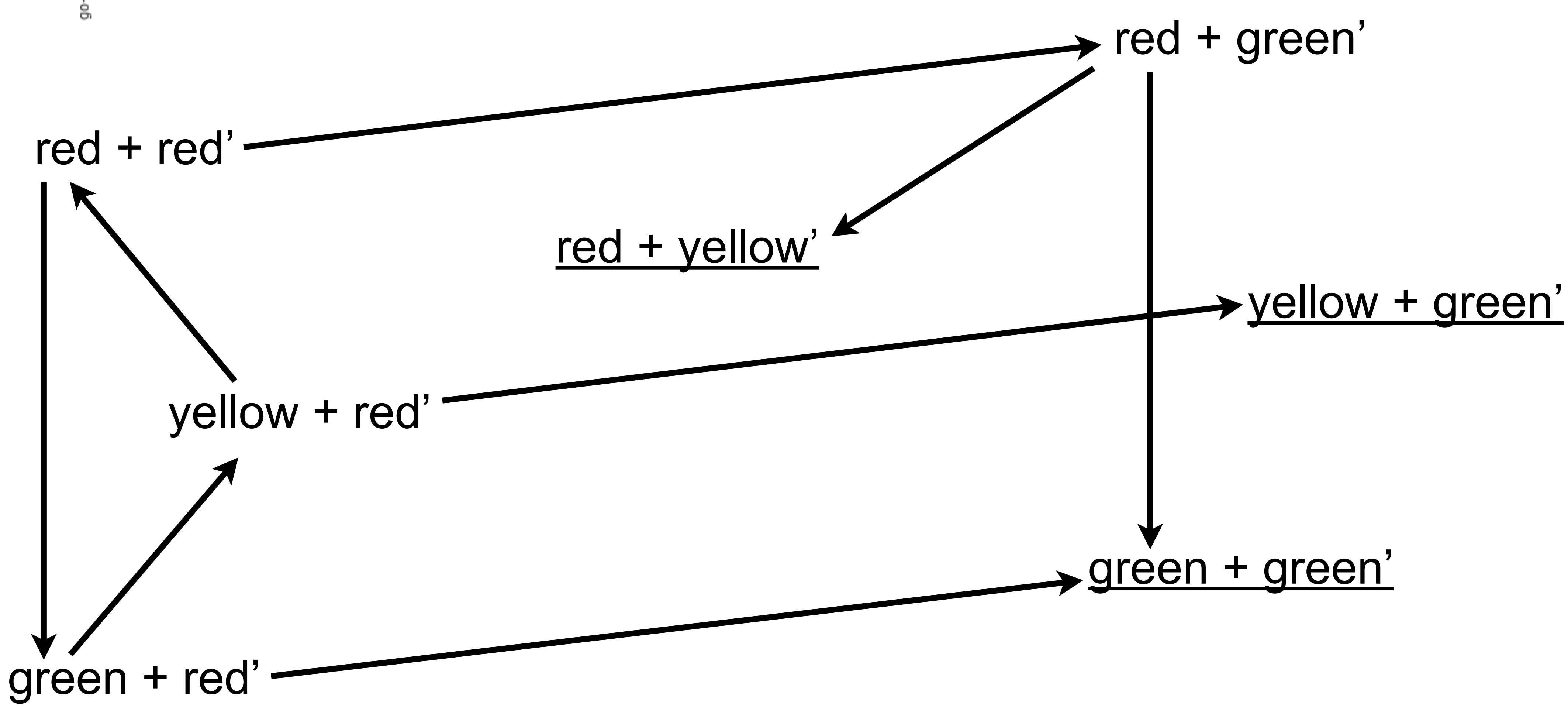

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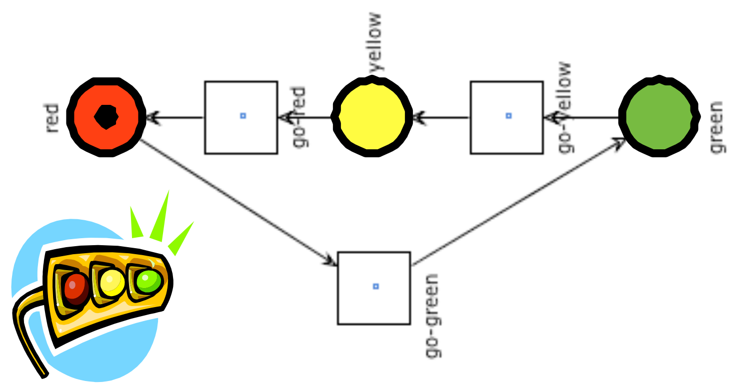

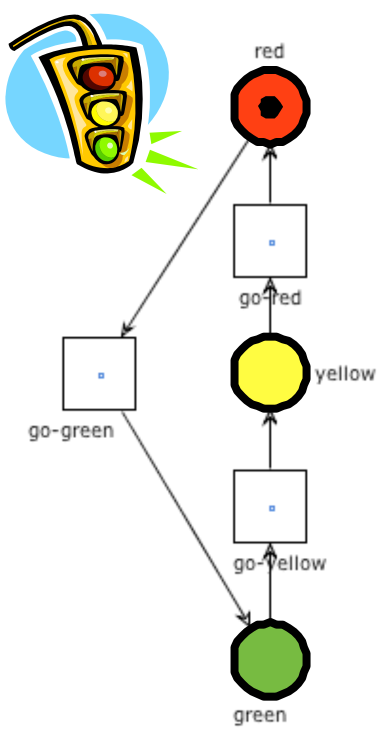

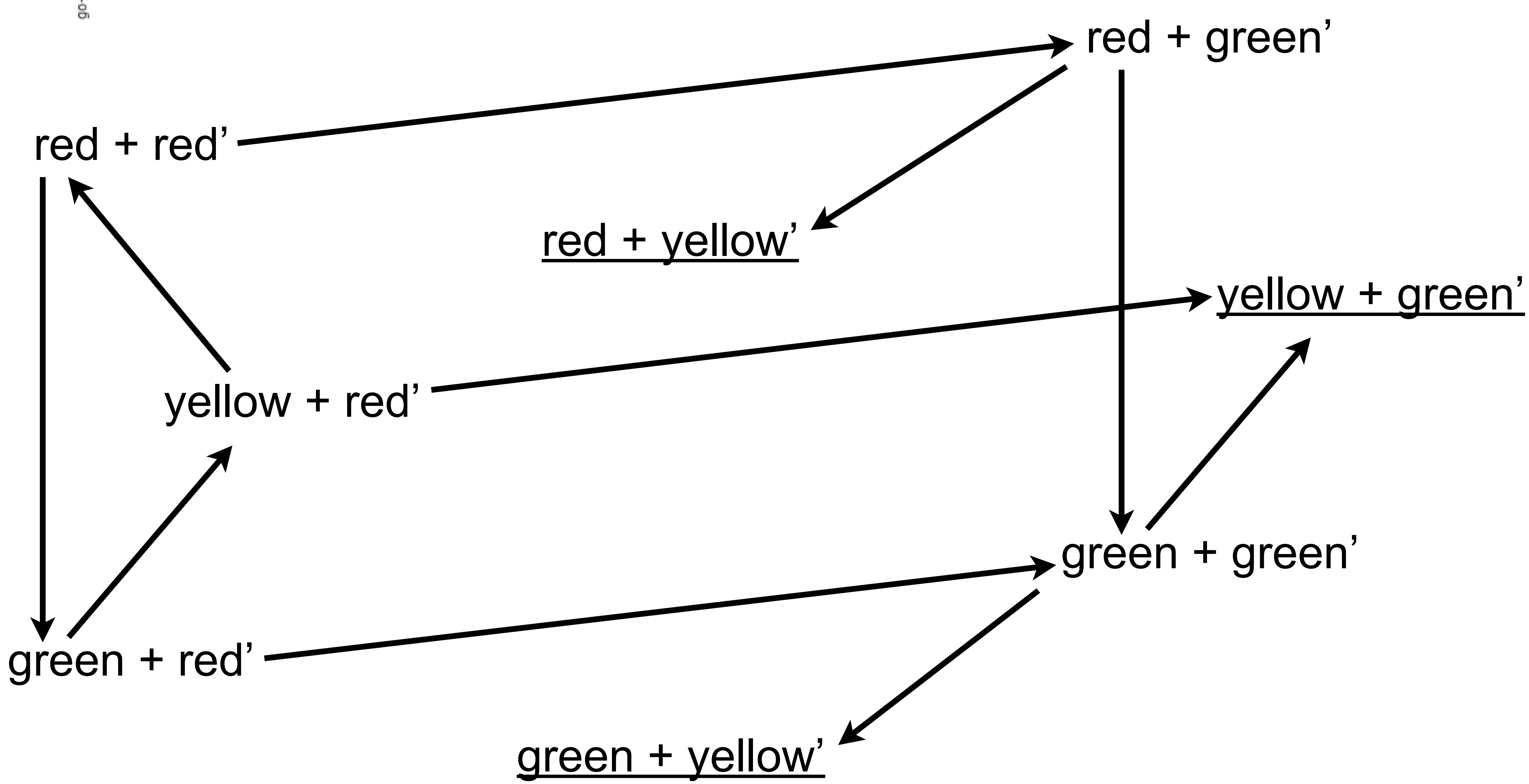

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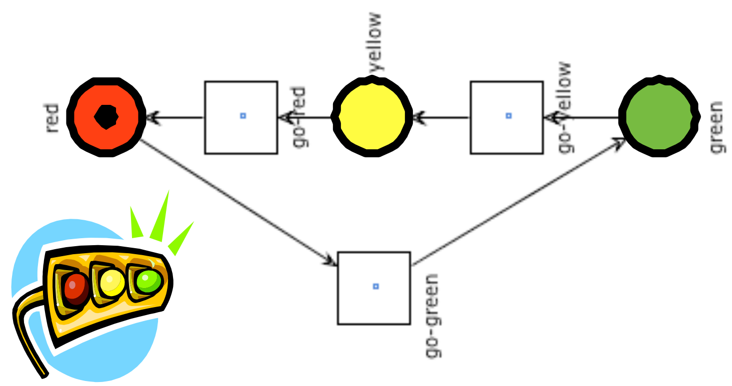

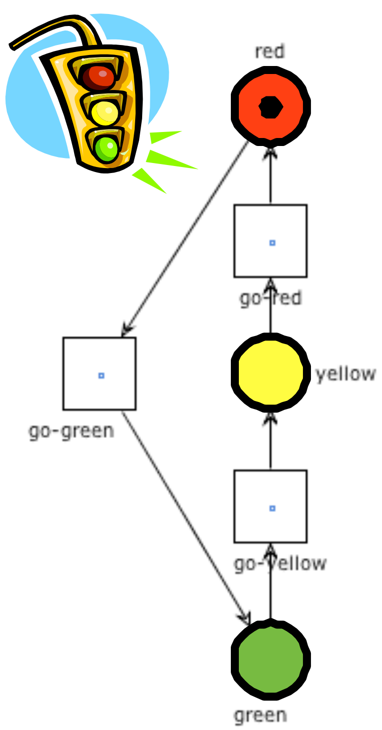

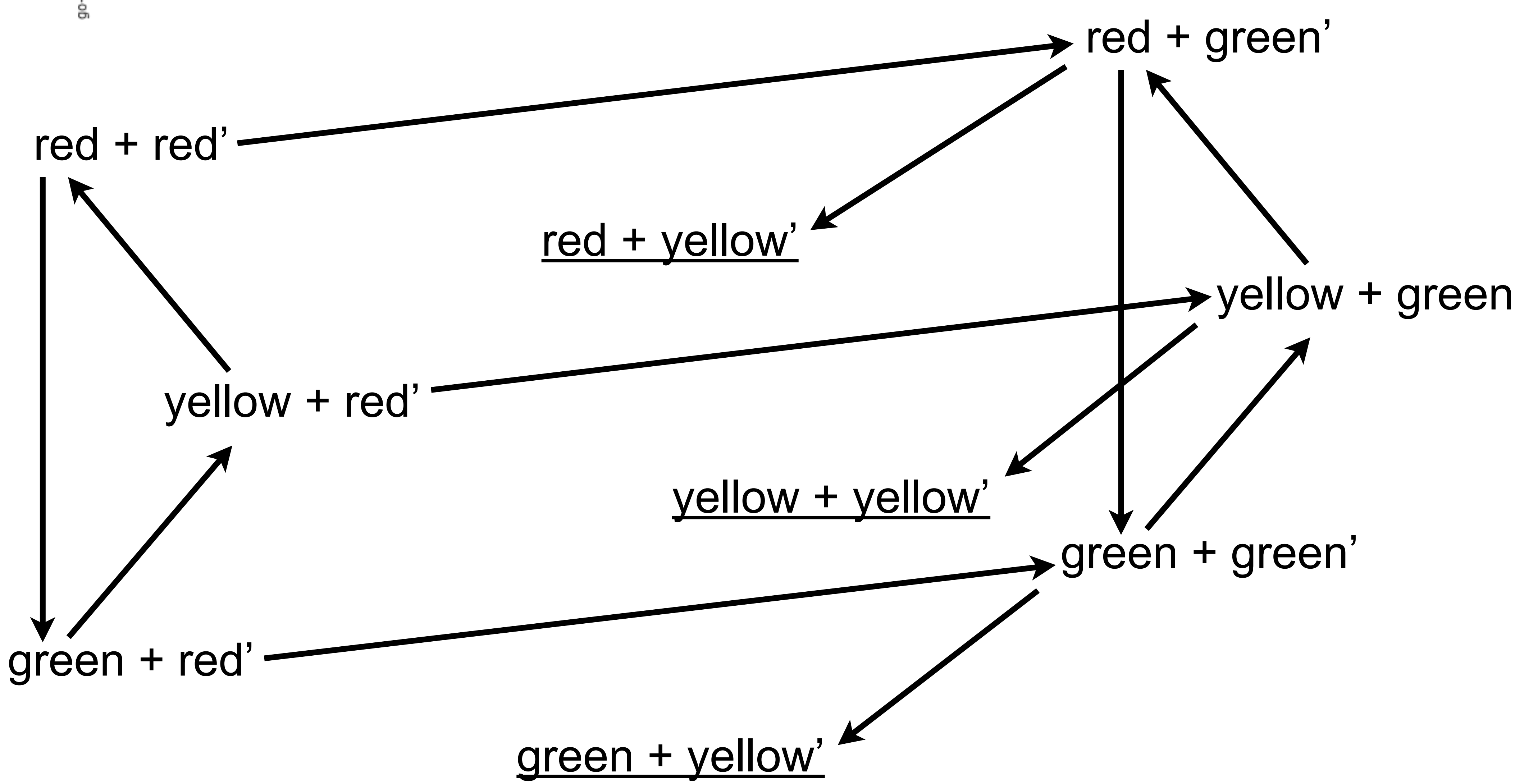

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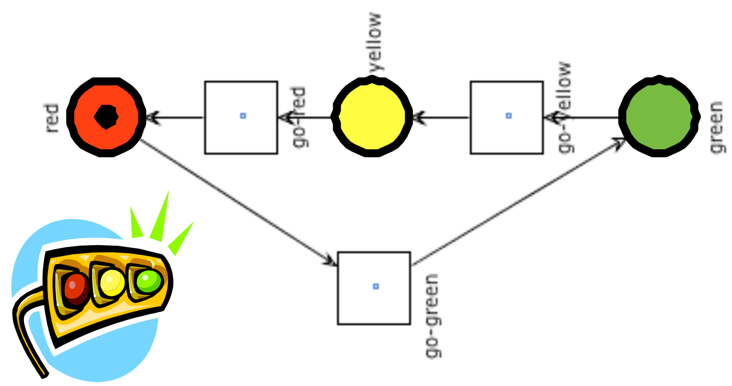

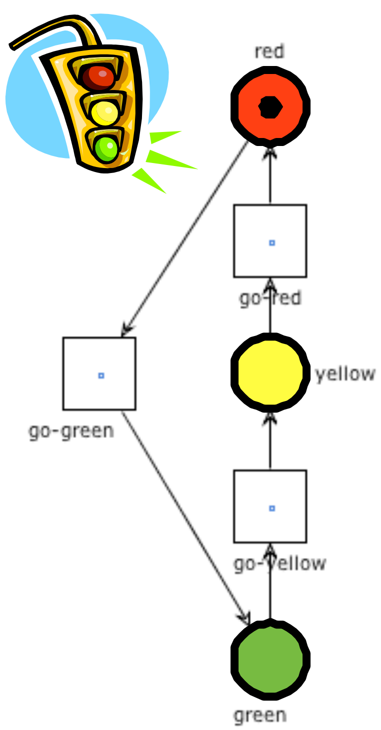

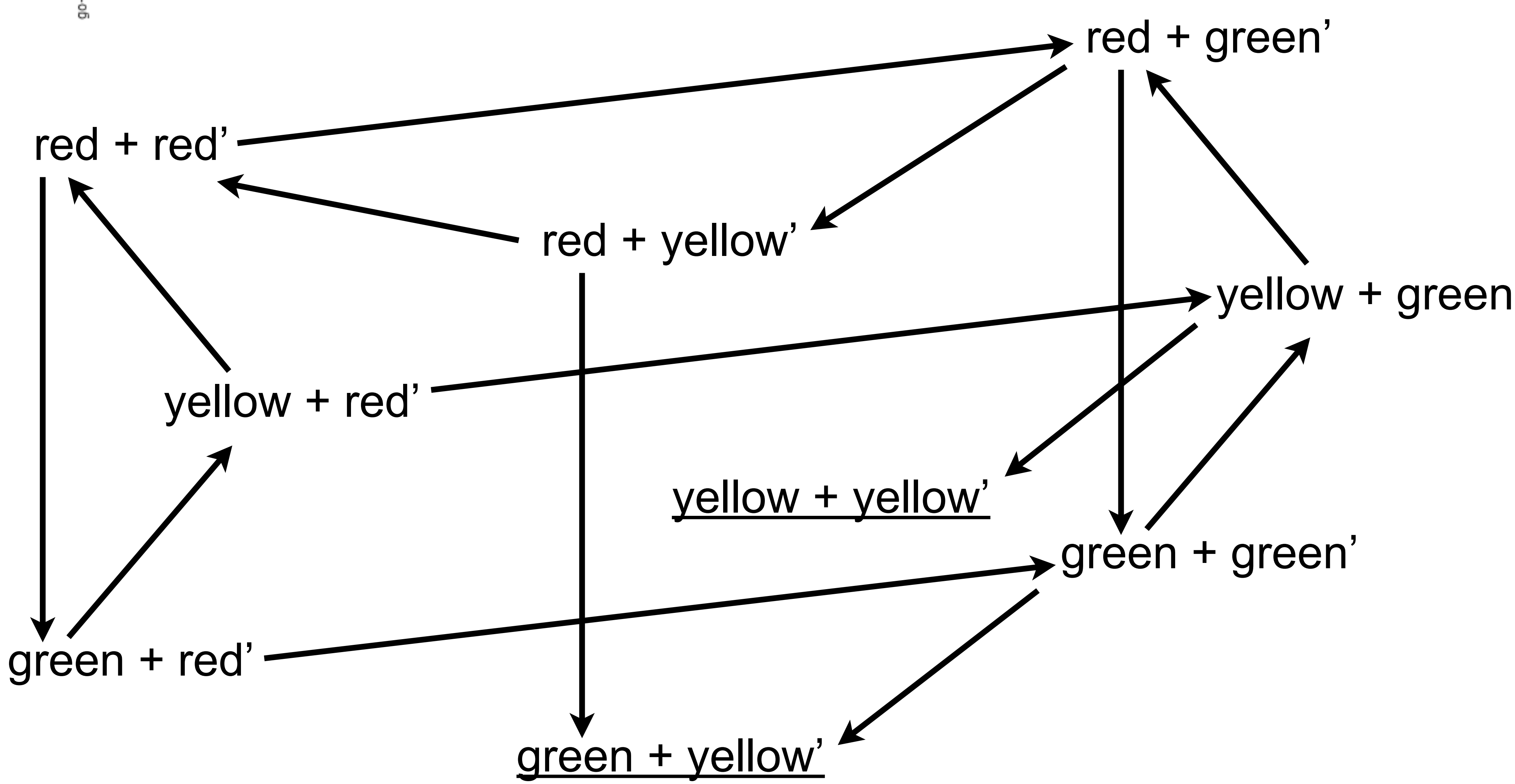

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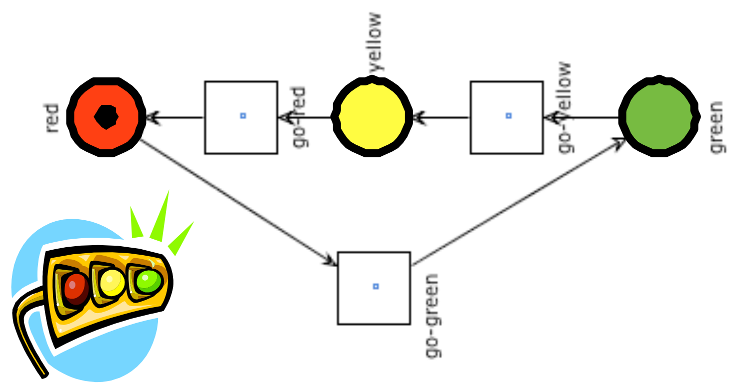

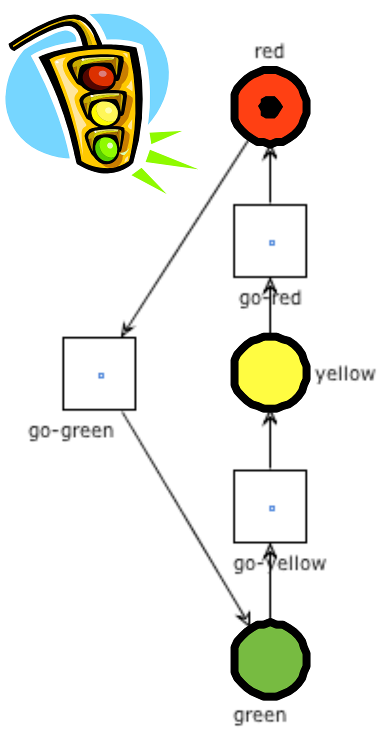

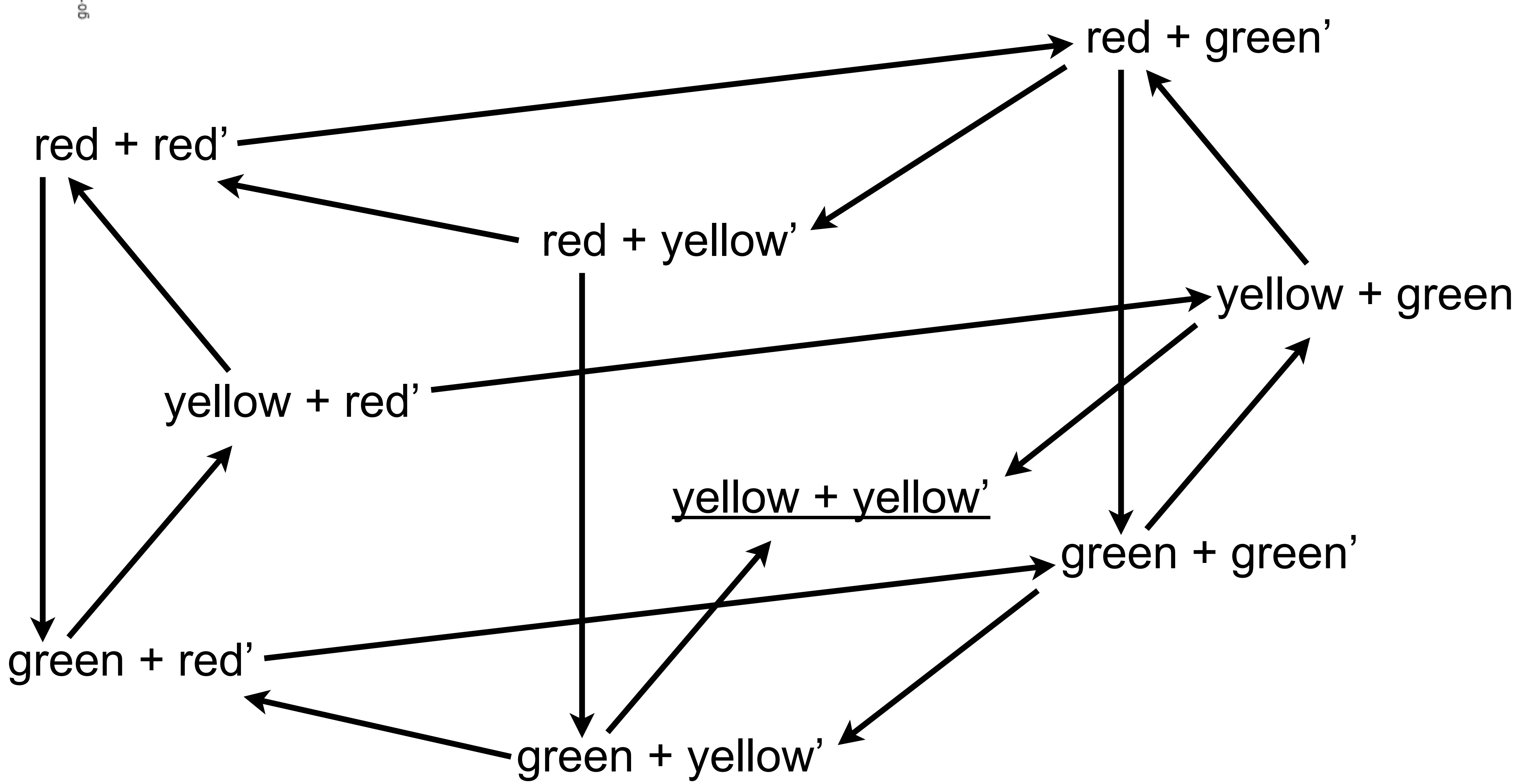

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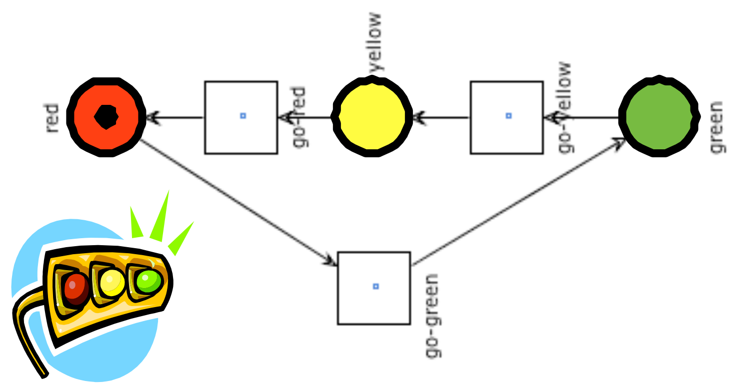

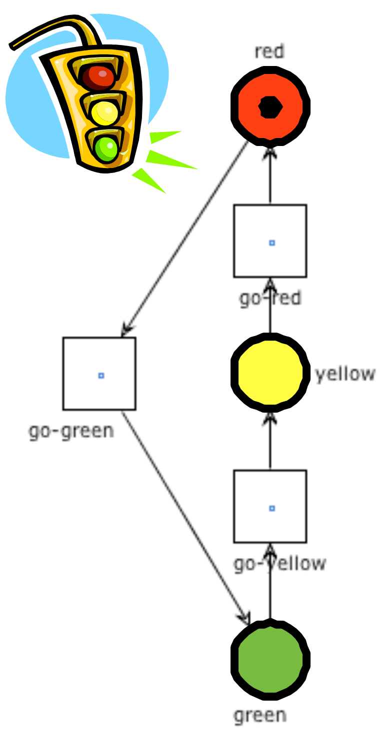

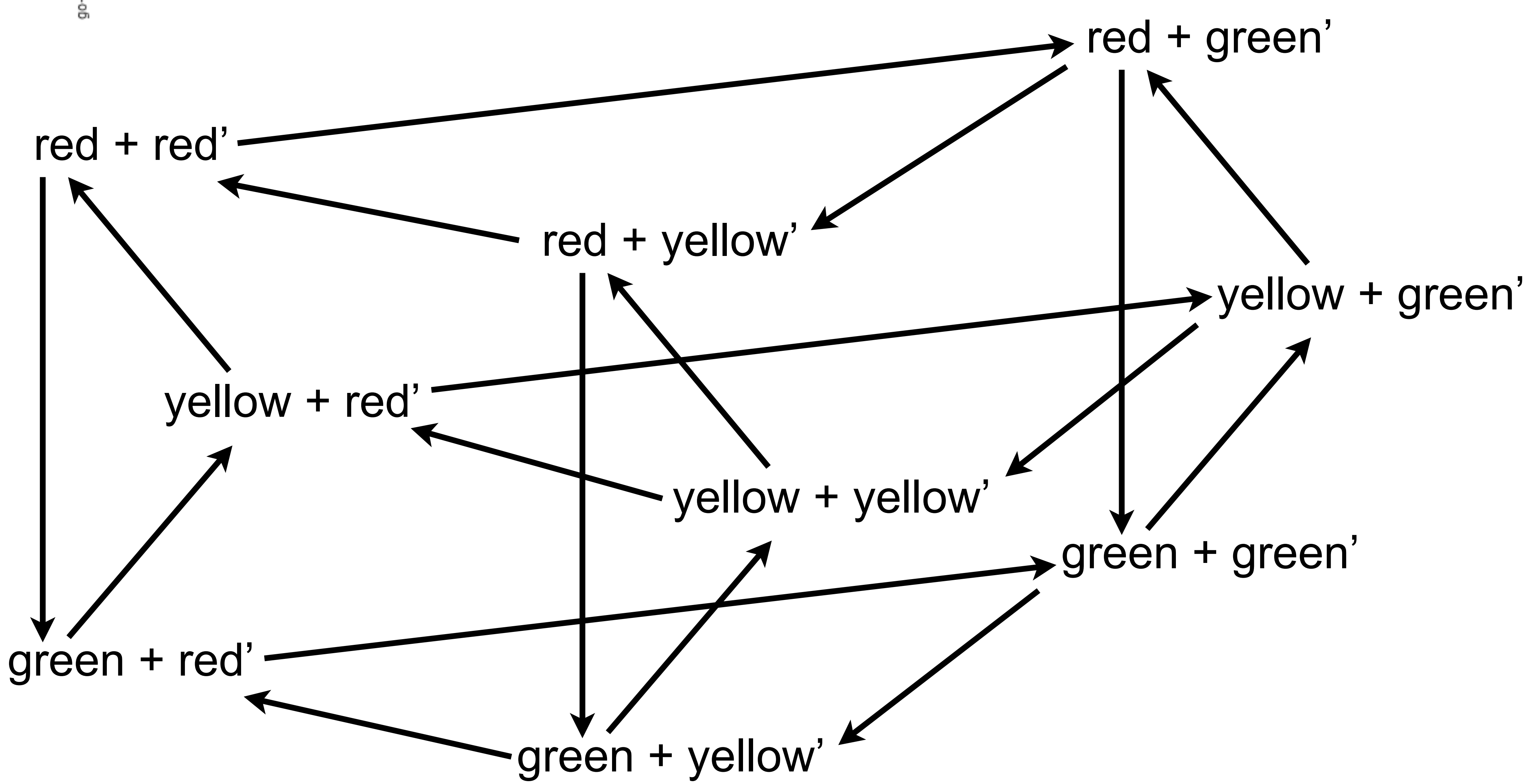

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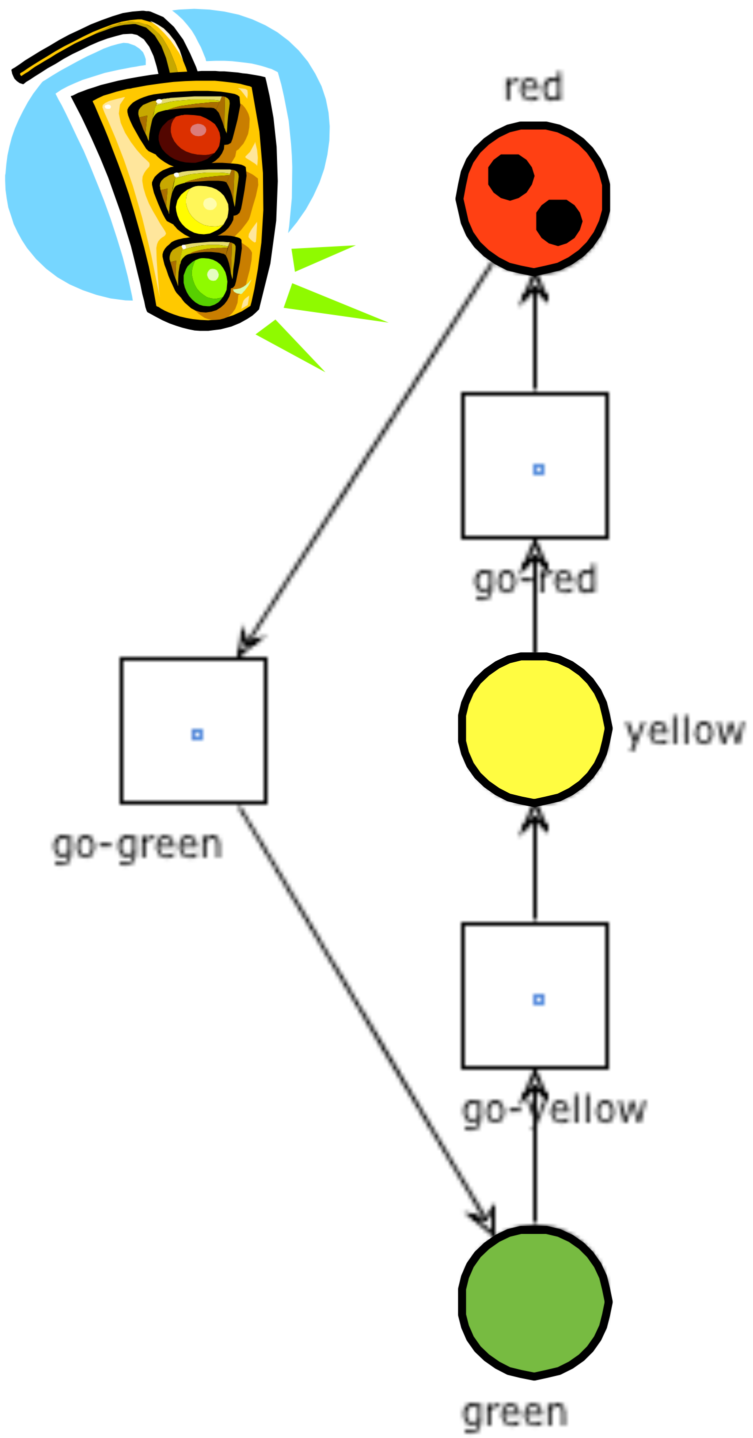
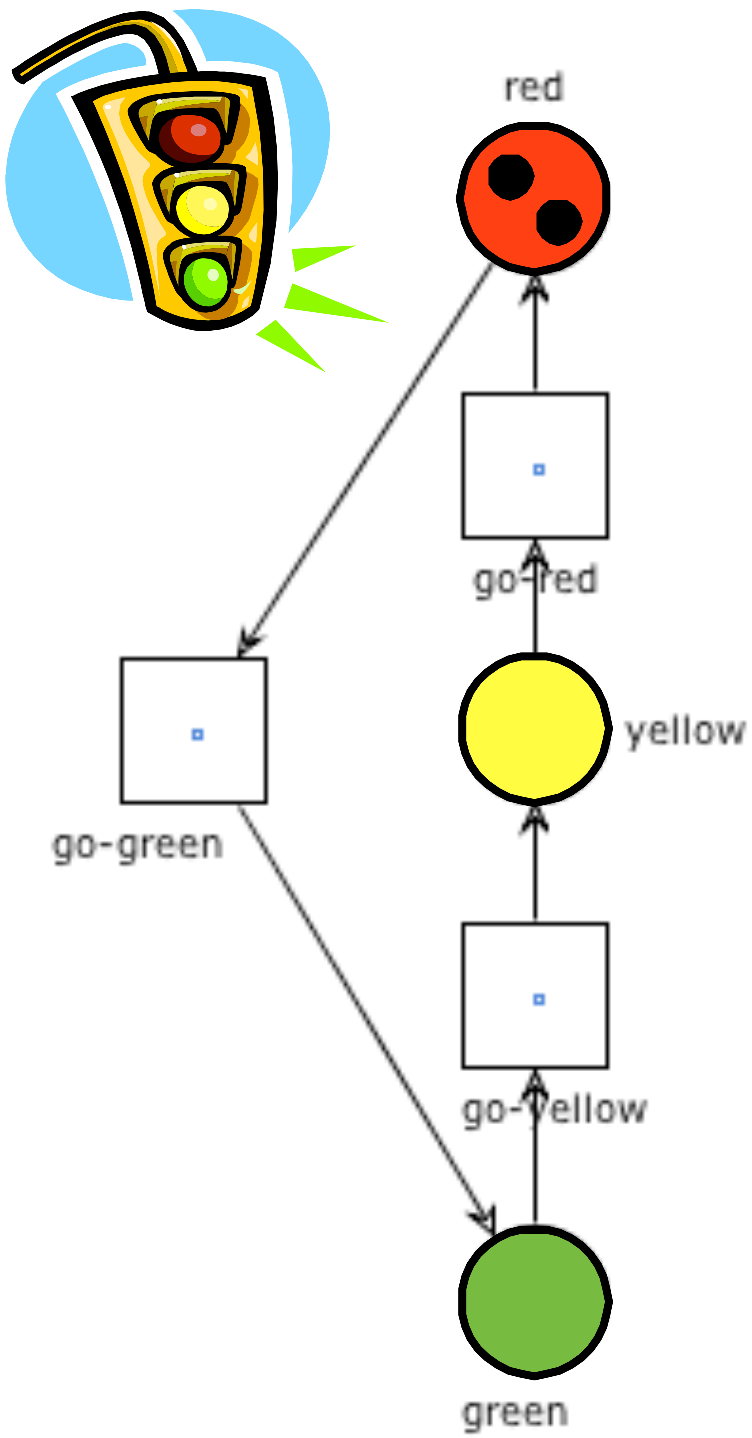


2 red

Example: two traffic lights

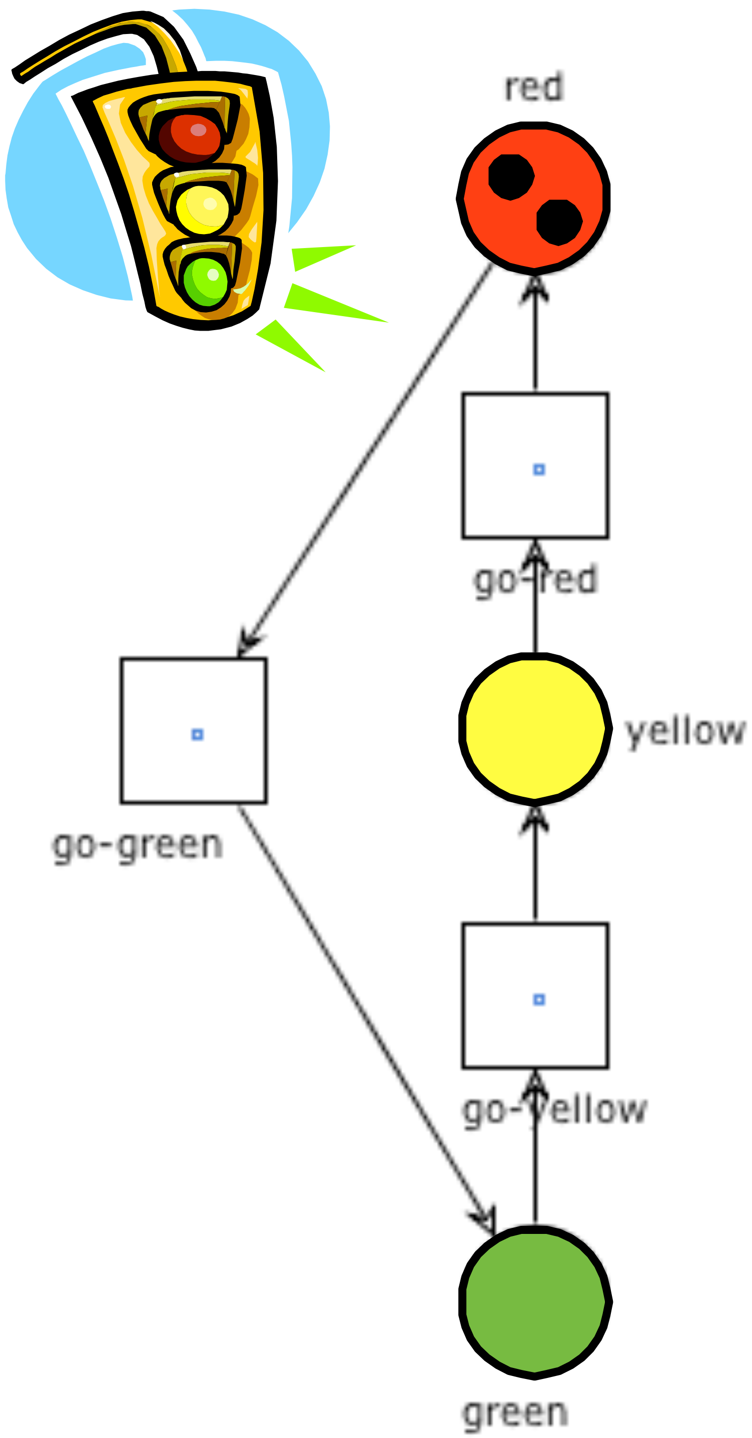


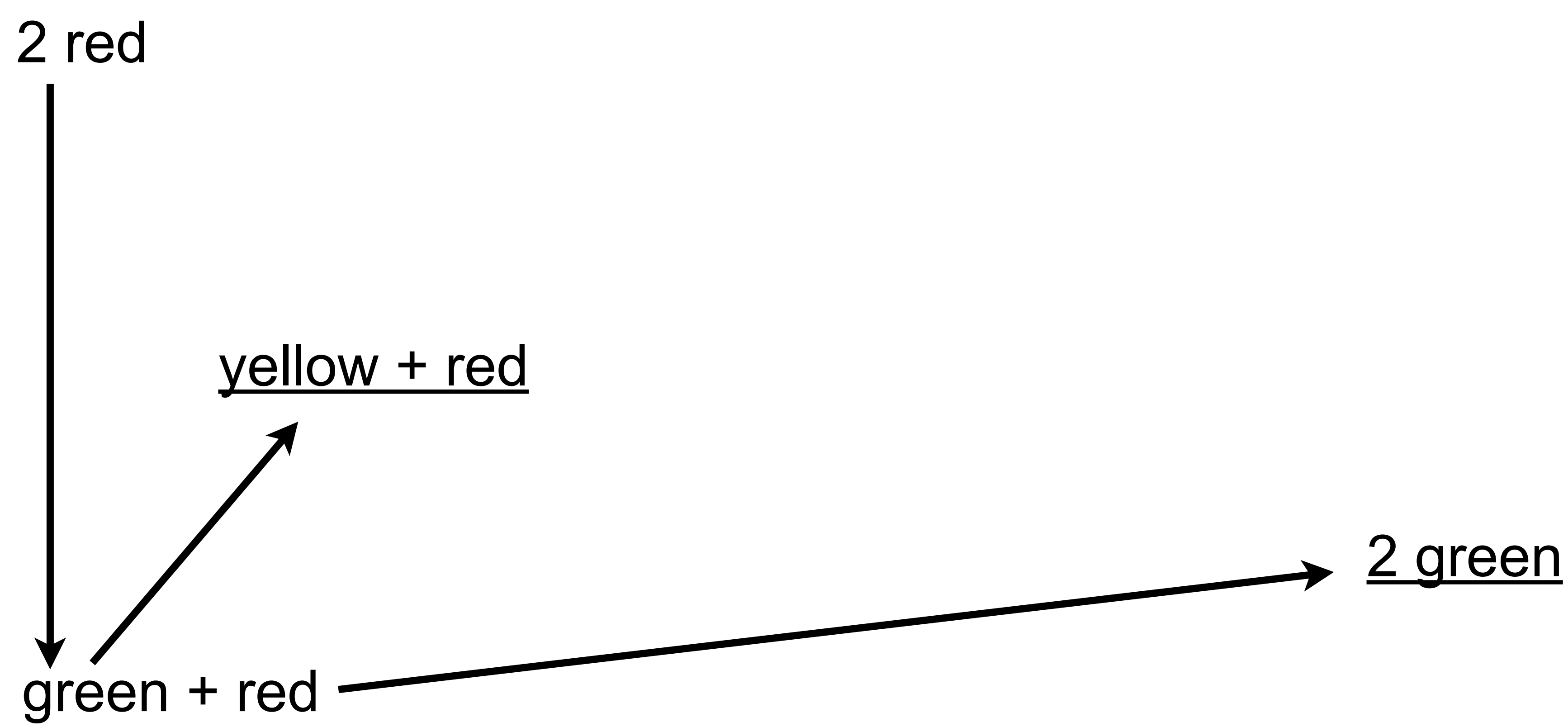
2 red

↓

<u>green + red</u>

Example: two traffic lights





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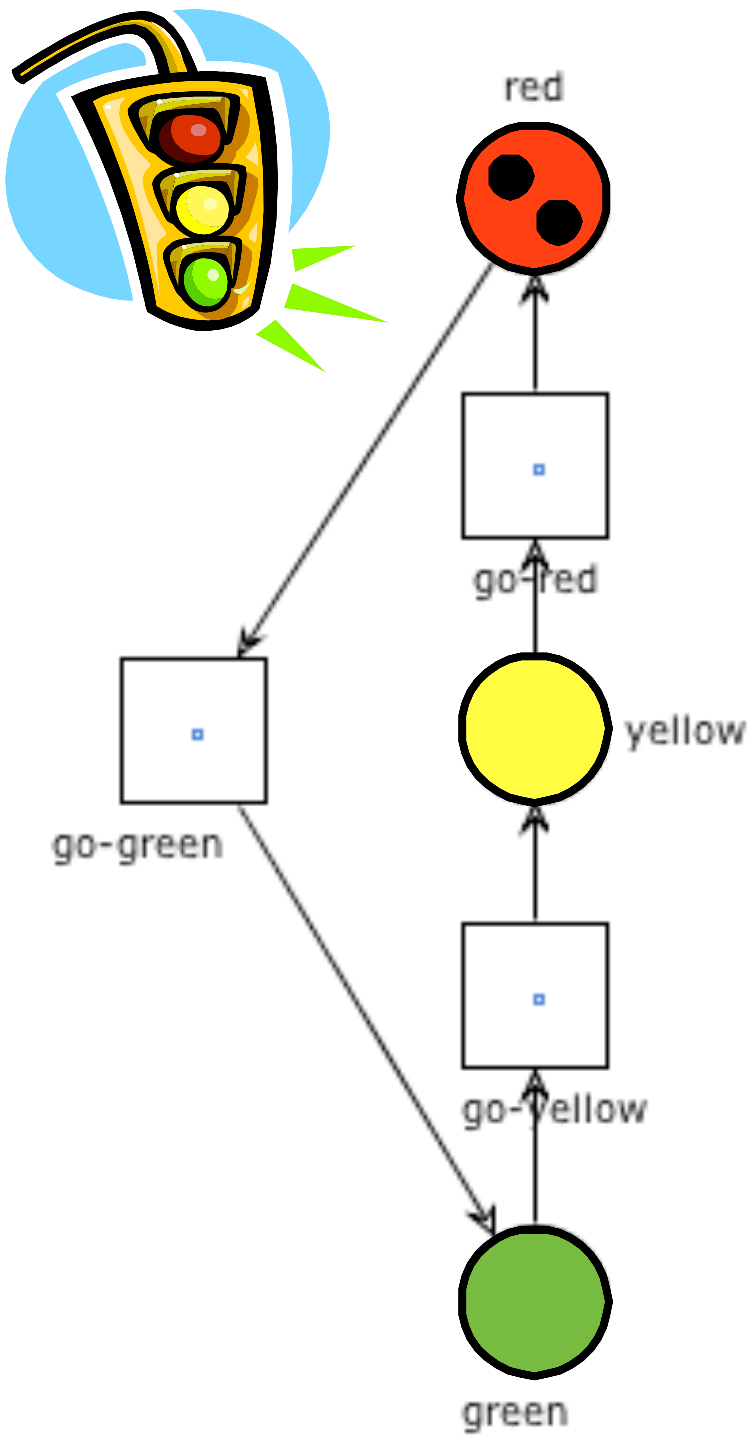

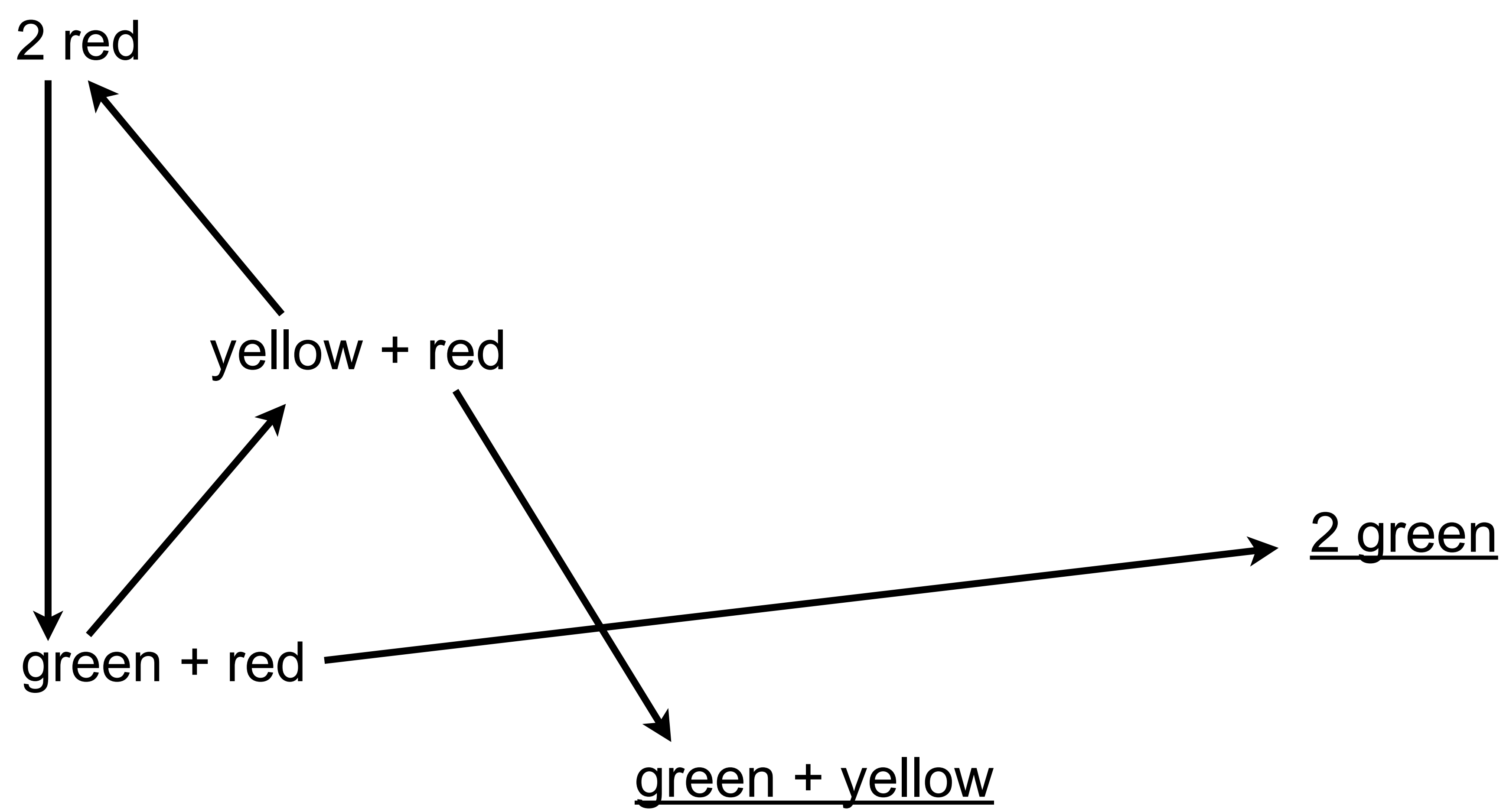

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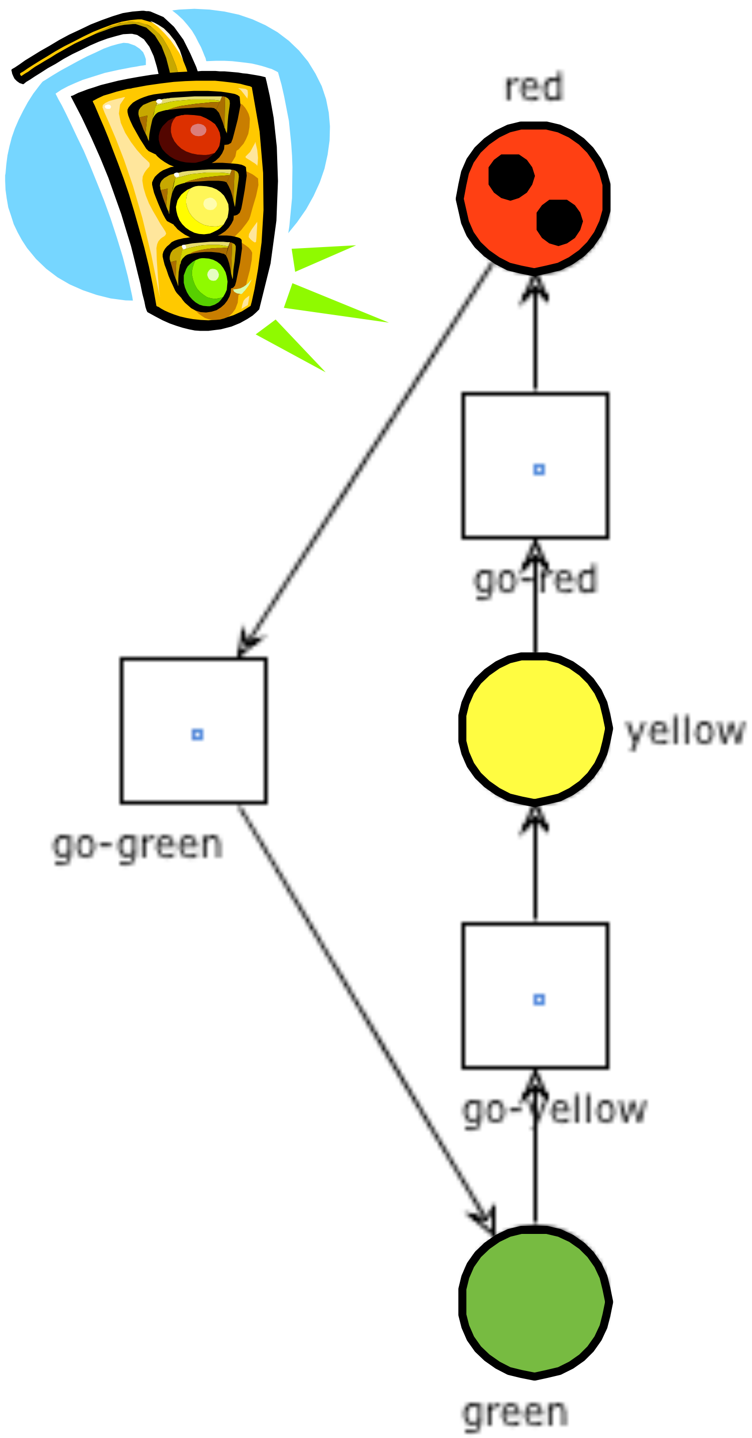

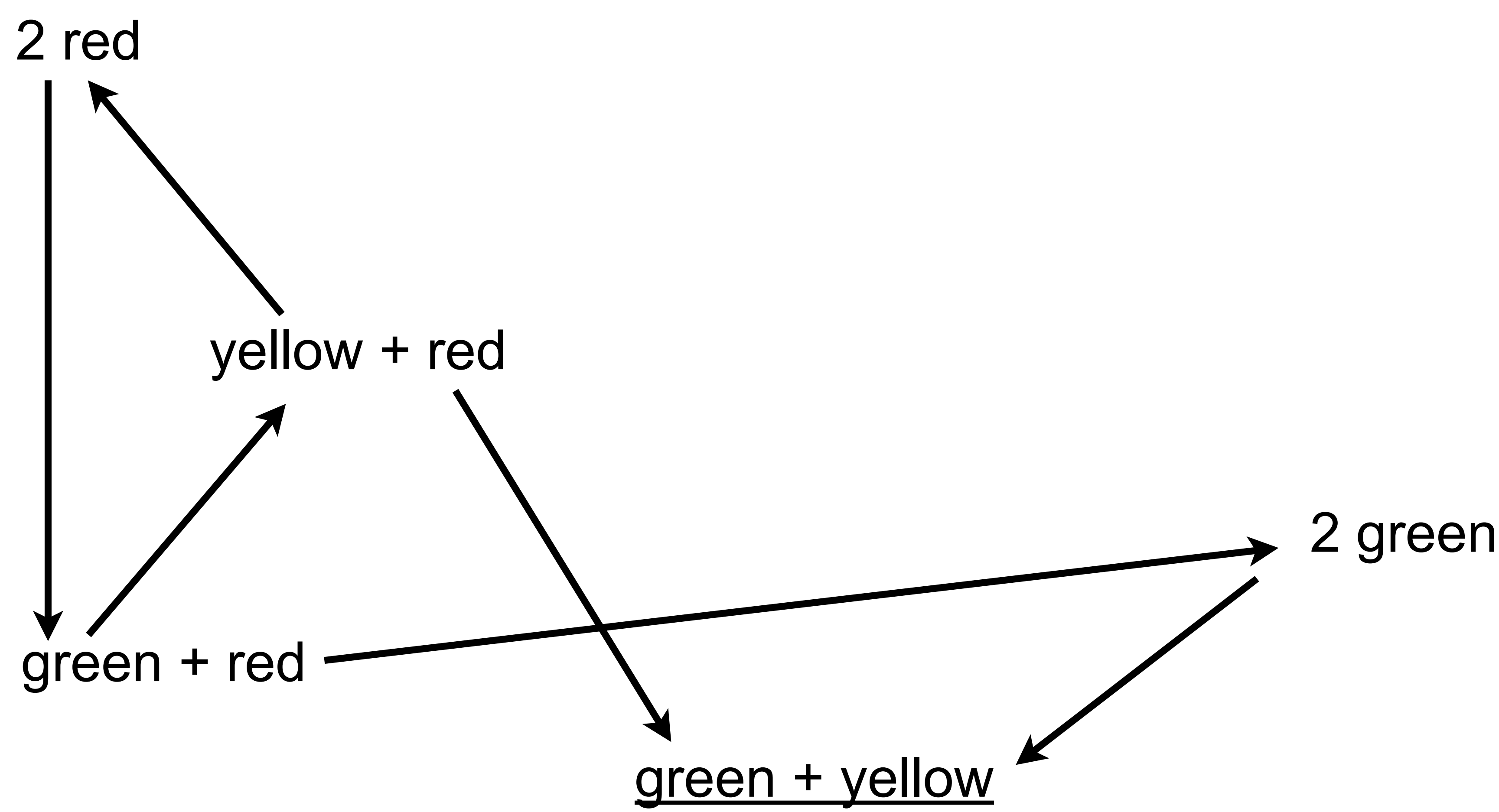

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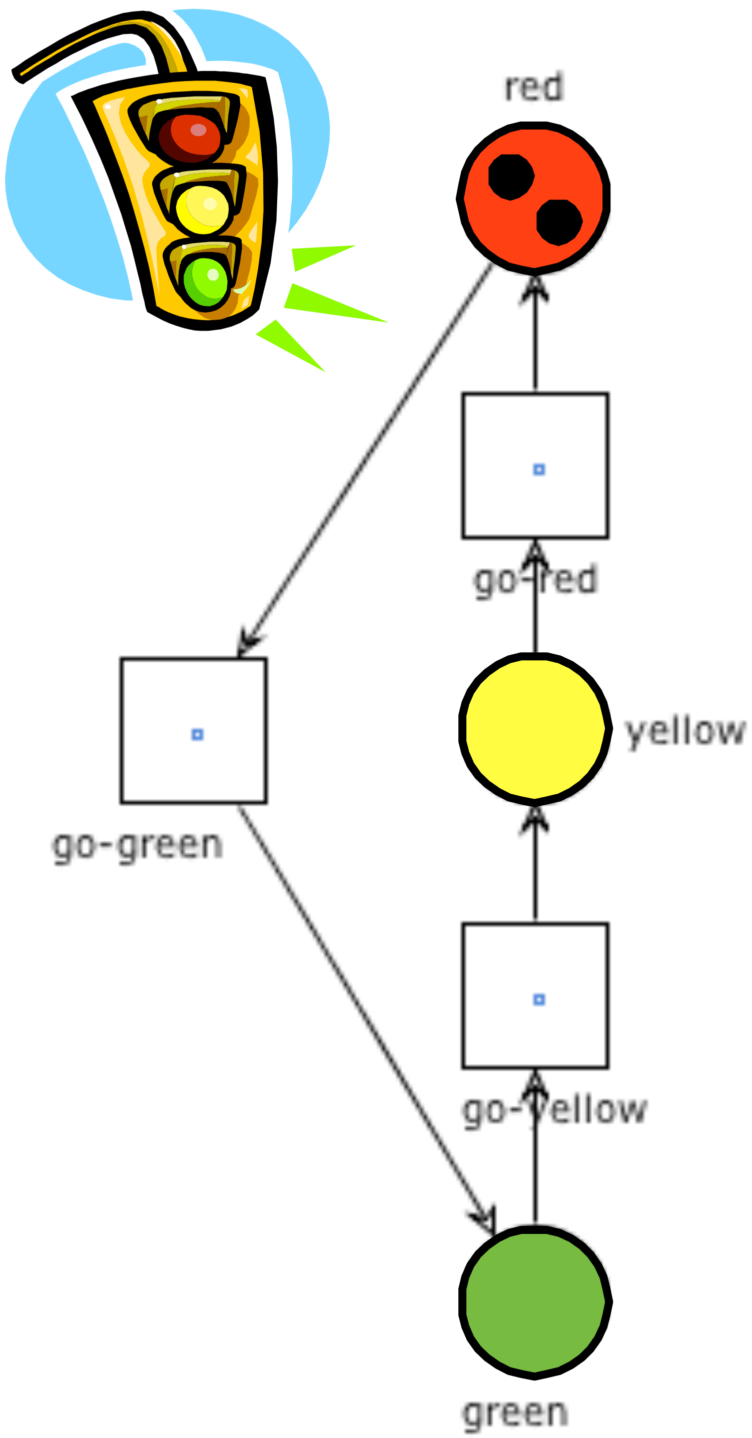

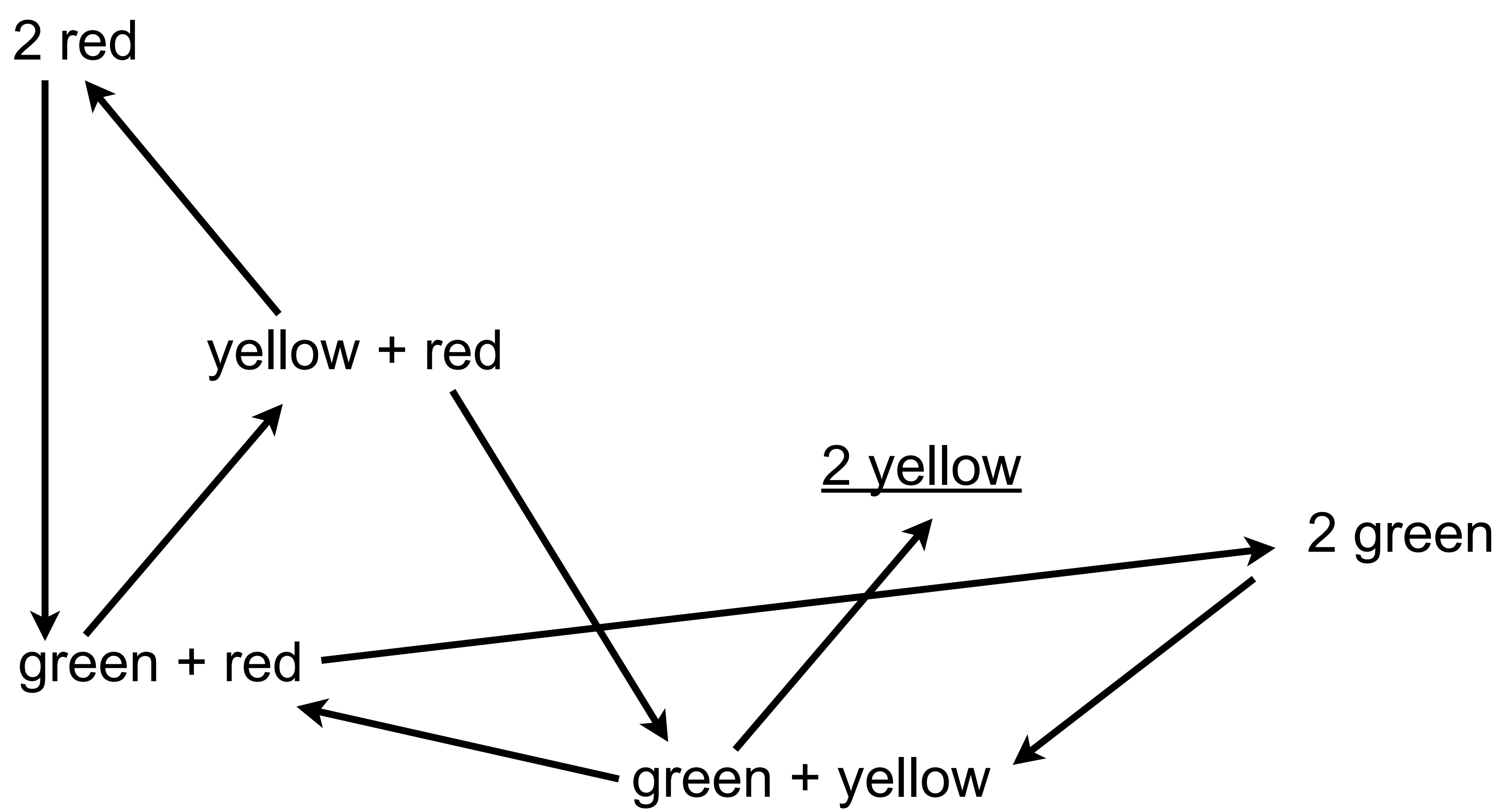

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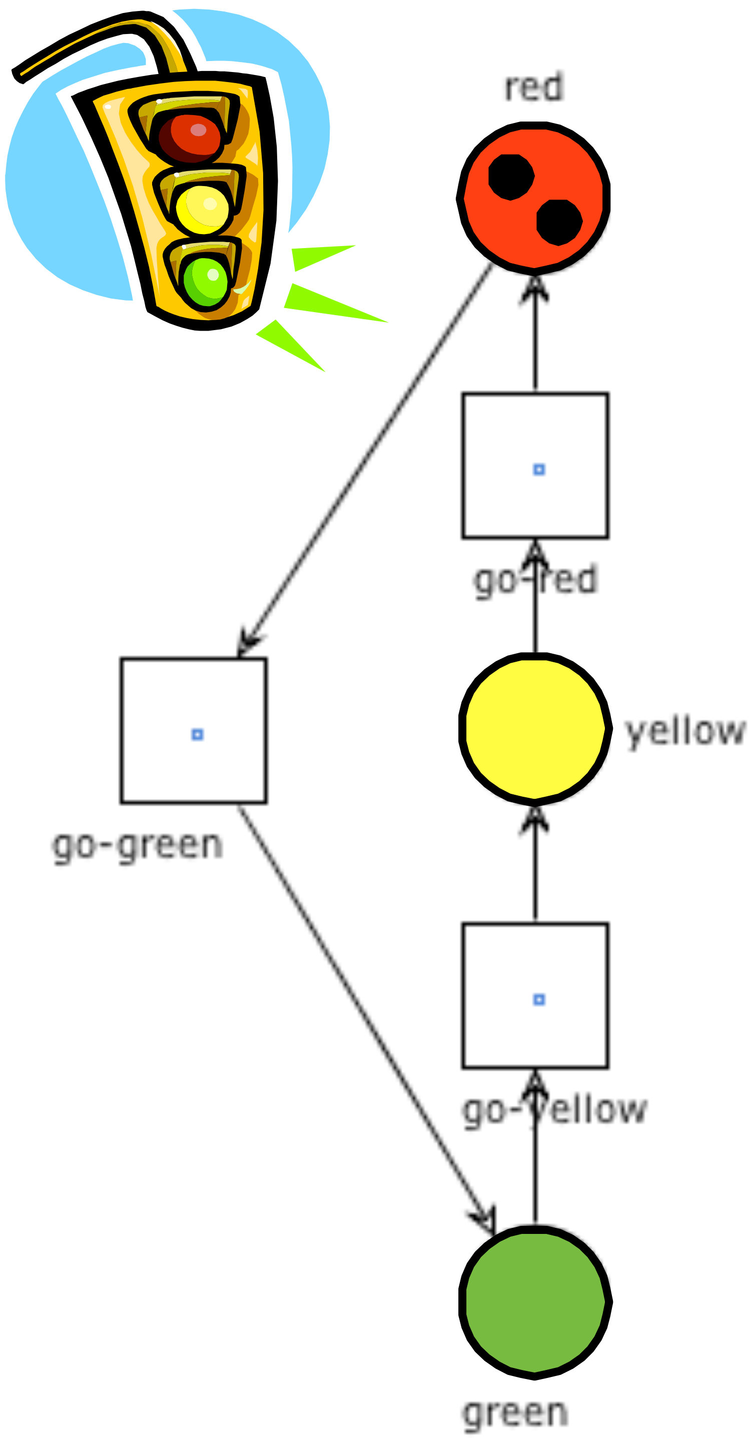

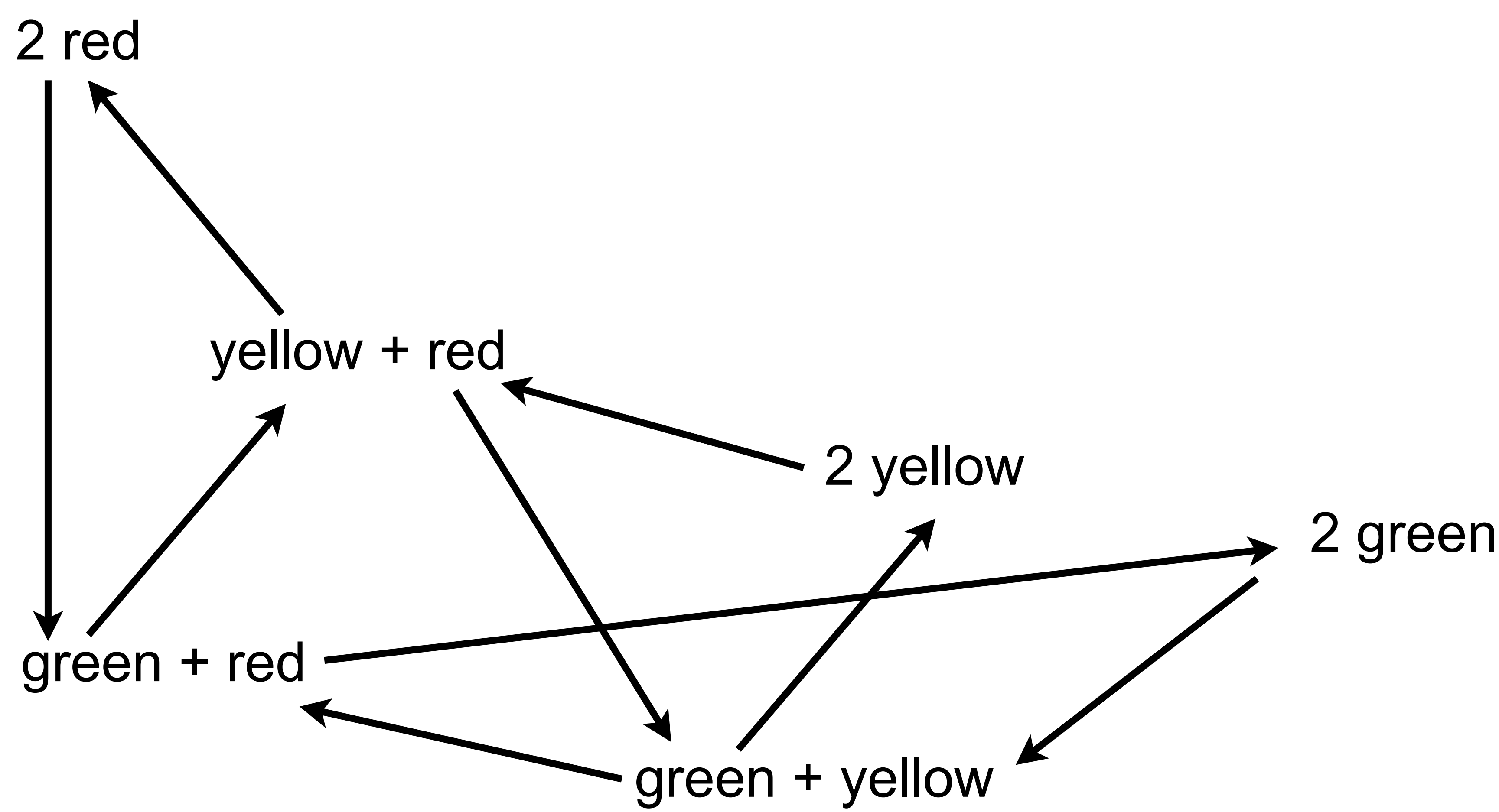

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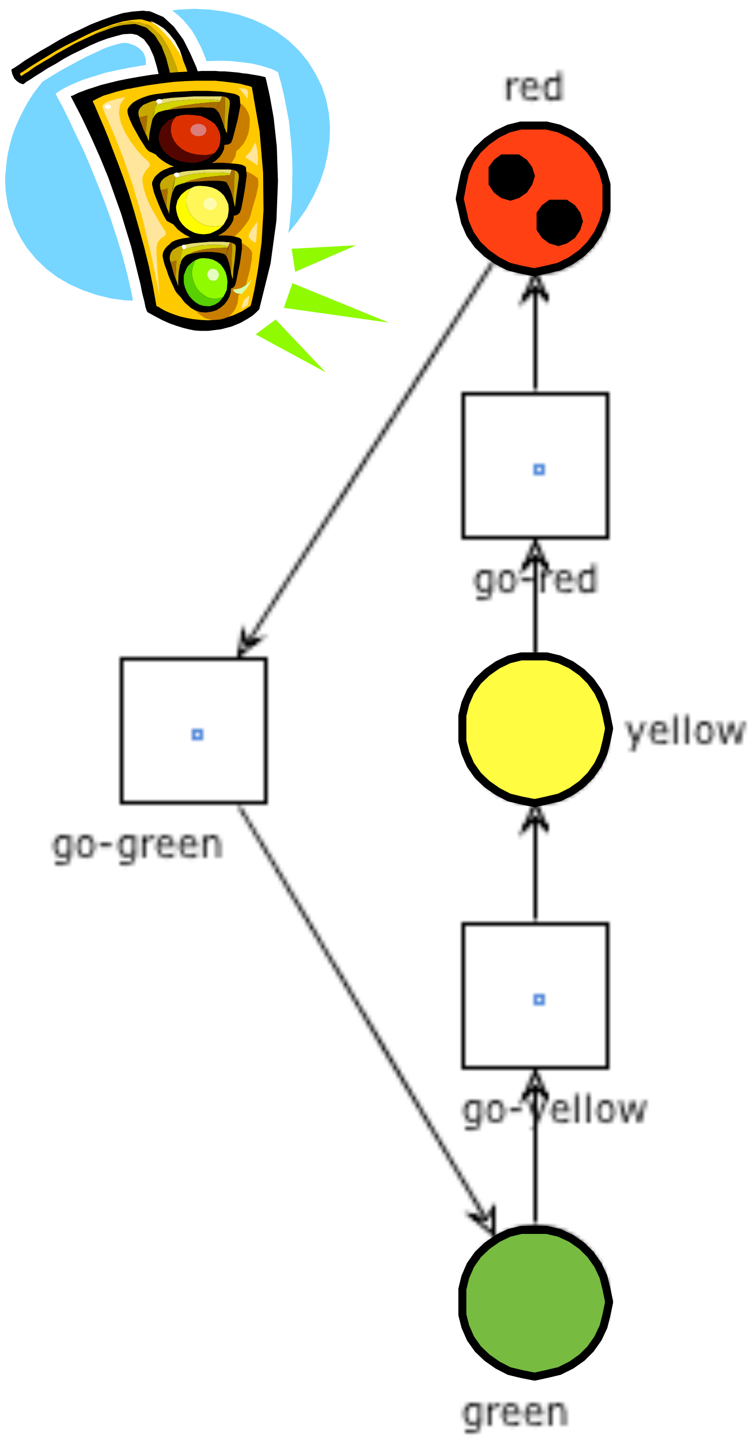
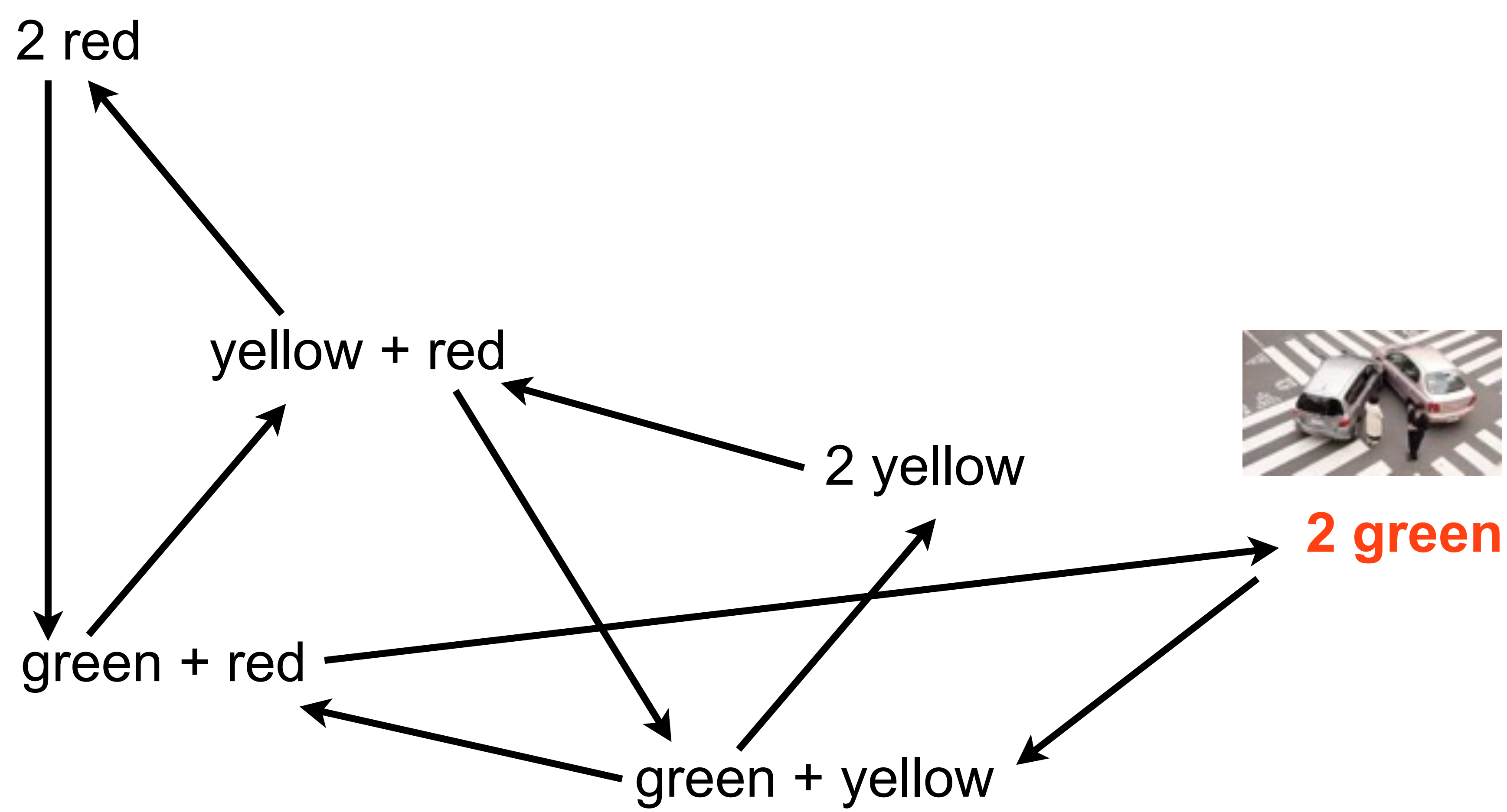
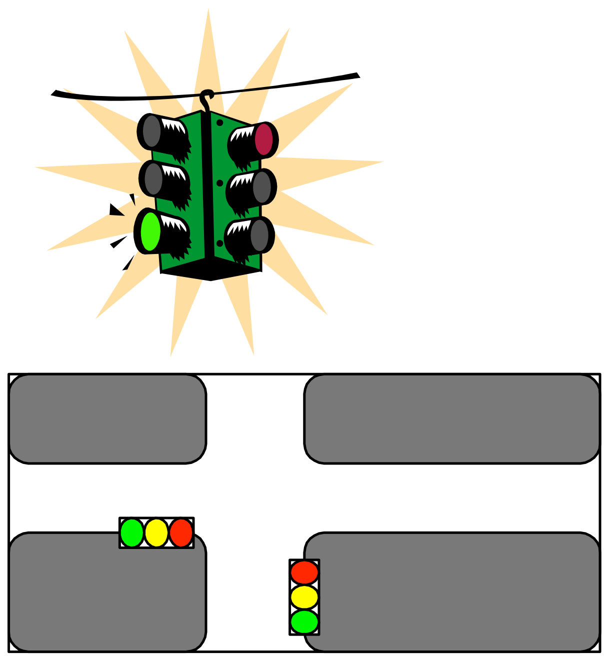
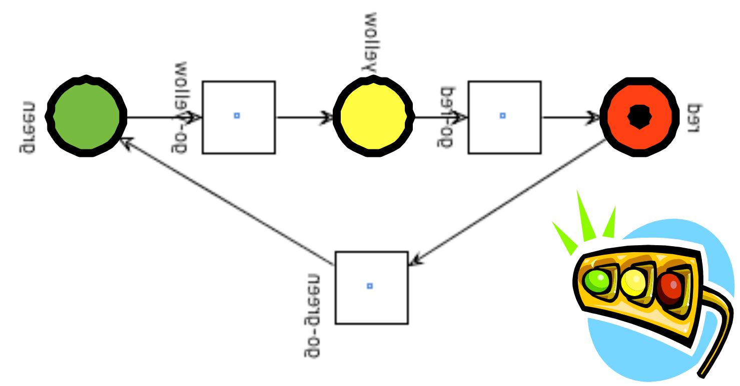


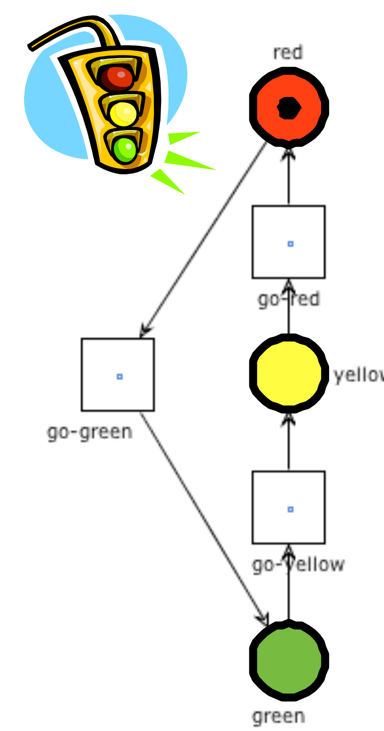


Question time



Complete the net in such a way that the two lights can never be green at the same time





Exercises

Draw the reachability graph of the last net

Modify the net so to guarantee that green alternate on the two traffic lights and then draw the reachability graph

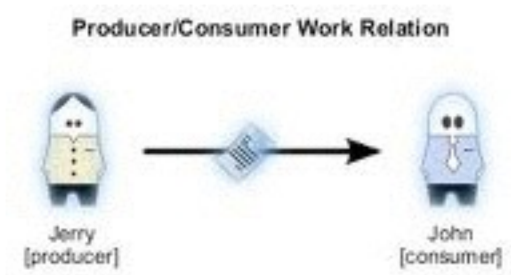
Play the “token games” on the above nets
On the web (Petri net applet):
http://wwwis.win.tue.nl/~wvdaalst/workflowcourse/pn_applet/pn_applet.htm
On your PC (Workflow Petri net Designer):
http://www.woped.org

Exercise: German traffic lights

German traffic lights have an extra phase: traffic lights turn not suddenly from red to green but give a red light together with a yellow light before turning to green.

Identify the possible states and model the transition system that lists all possible states and state transitions.

Provide a Petri net that is able to behave exactly like a German traffic light. There should be three places indicating the state of each light and make sure that the Petri net does not allow state transitions which should not be possible.

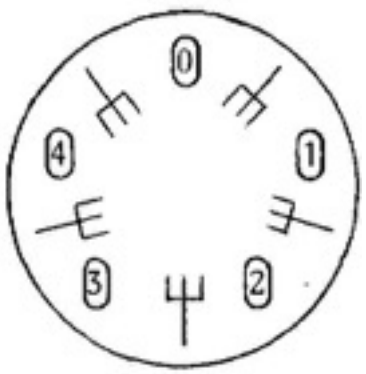


Exercise:

Producer and consumer

Model a process with one producer and one consumer:
Each one is either busy or free.
Each one alternates between these two states
After every production cycle the producer puts a product in a buffer and the consumer consumes one product from this buffer (when available) per cycle.

Draw the reachability graph
How to model 4 producers and 3 consumers connected through a single buffer?
How to limit the size of the buffer to 2 items?



Exercise: Dining philosophers

The problem is originally due to E.W. Dijkstra (and soon elaborated by T. Hoare) as an examination question on a synchronization problem where five computers competed for access to five shared tape drive peripherals.

It can be used to illustrate several important concepts in concurrency (mutual exclusion, deadlock, starvation)

Exercise: Dining philosophers

The life of a philosopher consists of an alternation of thinking and eating

Five philosophers are living in a house where the table laid for them, each philosopher having his own place at the table

Their only problem (besides those of philosophy) is that the dish served is a very difficult kind of spaghetti, that has to be eaten with two forks. There are two forks next to each plate, so that presents no difficulty: as a consequence, however, no two neighbours may be eating simultaneously.

Exercise: Dining philosophers

Design a net for representing the dining philosophers problem, then use WoPeD to compute the reachability graph

image taken from wikipedia
philosophers clockwise from top:
Plato, Konfuzius, Socrates,
Voltaire and Descartes

Exercise

Use a Petri net to model a circular railway system with four stations (st_1, st_2, st_3, st_4) and one train

At each station passengers may "hop on" or "hop off" (this is impossible when the train is moving)

The train has a capacity of 50 persons (if the train is full no passenger can hop on, if the train is empty no passenger can hop off)

What is the number of reachable states?

Petri nets: behavioural properties

Properties of Petri nets

We describe, in an informal way, some of the properties of Petri nets that can play an important role in the verification of business processes

Liveness
Deadlock-freedom
Boundedness
Cyclicity (also Reversibility)

Liveness

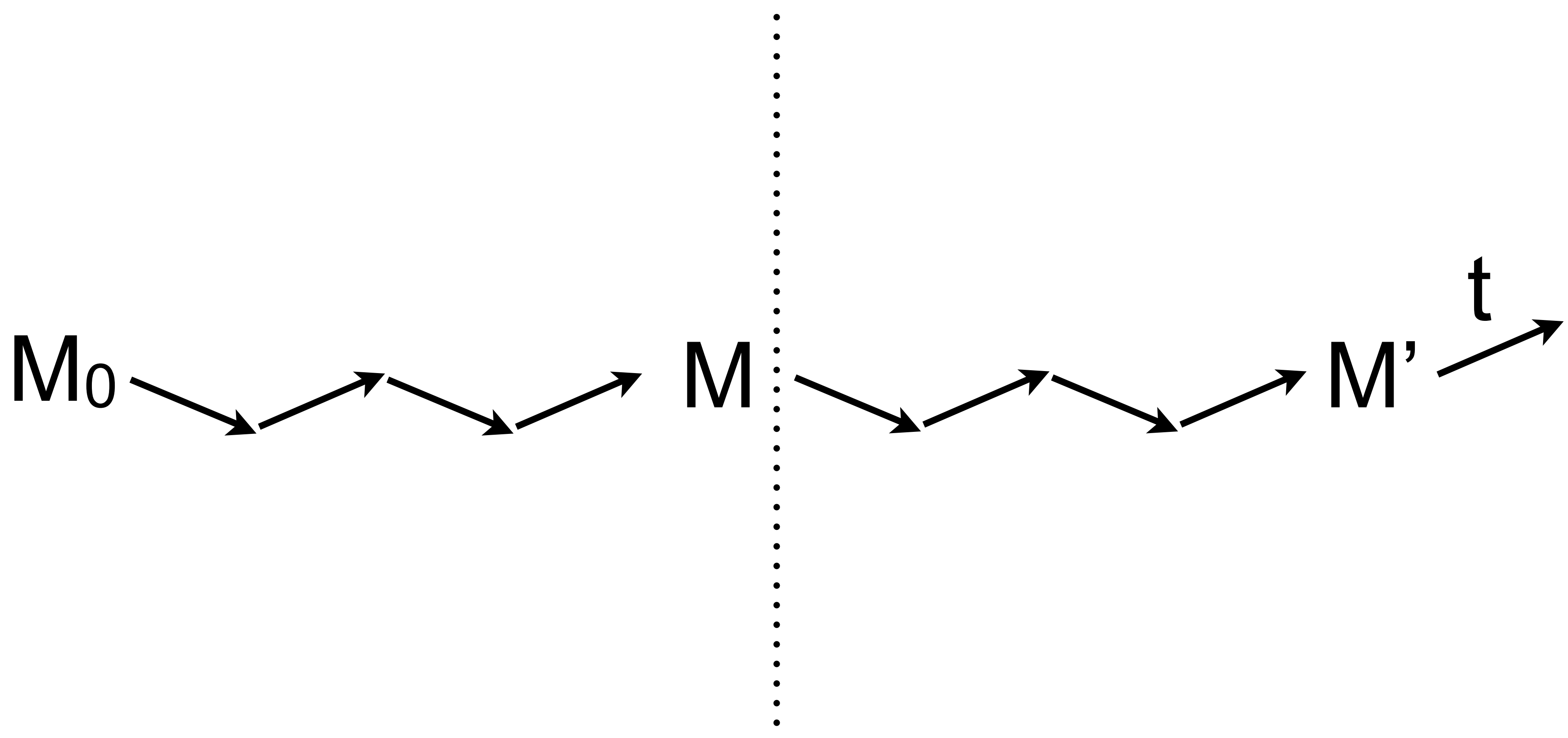
A transition t is **live**, if from any reachable marking M another marking M' can be reached where t is enabled

In other words, at any point in time of the computation, we cannot exclude that t will fire in the future

A Petri net is **live** if all of its transitions are live

Liveness illustrated

For any reachable marking M



Can we find a way to enable t?

Liveness, formally

$$(P, T, F, M_0)$$

$$\forall t \in T, \quad \forall M \in [\, M_0 \,\rangle, \quad \exists M' \in [\, M \,\rangle, \quad M' \xrightarrow{t}$$

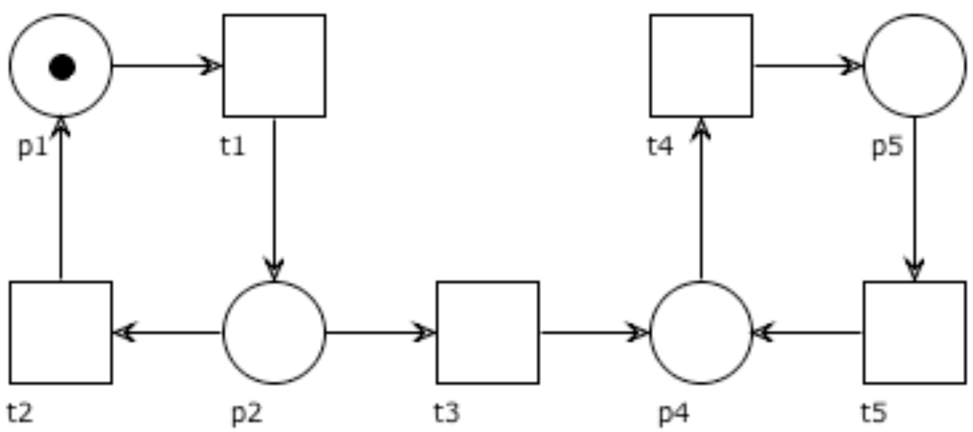
Liveness: pay attention!

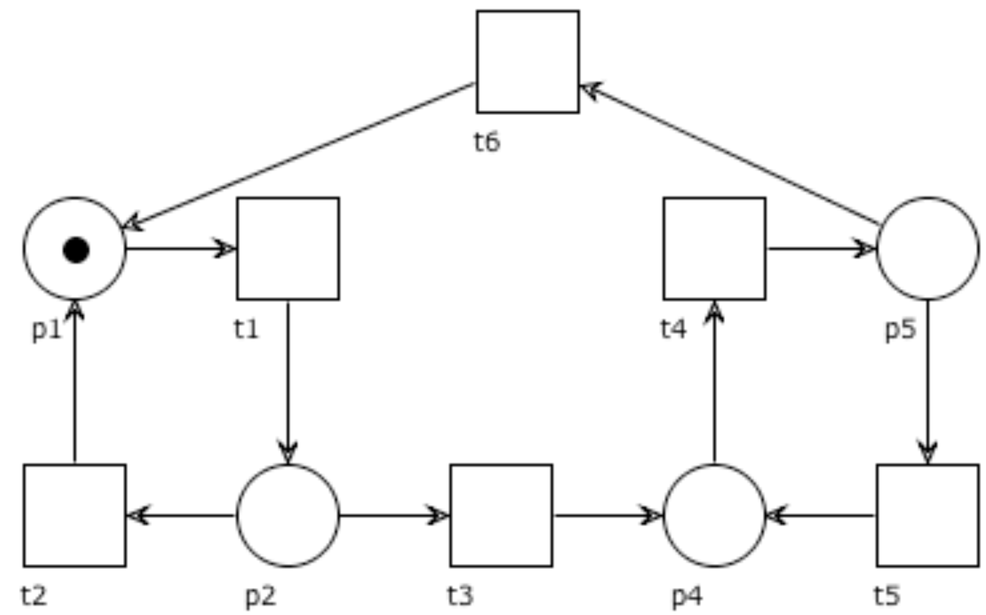
Liveness should not be confused with the following property:

"starting from the initial marking M_0 it is possible to reach a marking M that enables t"

(this property just ensures that t is not "dead" in M_0)

Liveness: example





Which transitions are live?
Which are not?
Is the net live?

Marked place

Given a marking M

We say that a place p is marked (in M)
if $M(p) > 0$
(i.e., there is a token in p in the marking M)

We say that p is unmarked
if $M(p) = 0$
(i.e., there is no token in p in the marking M)

Live place, intuitively

A place p is live

if every time it becomes unmarked

there is still the possibility to be marked in the future

(or if it is always marked)

Live place

Definition: Let (P, T, F, M_0) be a net system.

A place $p \in P$ is **live** if $\forall M \in [\, M_0 \,\rangle.\, \exists M' \in [\, M \,\rangle.\, M'(p) > 0$

Place liveness

Definition:
A net system (P, T, F, M_0) is **place-live** if every place $p \in P$ is live

Liveness implies place-liveness

Proposition: Live systems are also place-live

Take any p and any $t \in \bullet p \cup p \bullet$

Let $M \in [\, M_0 \,\rangle$

By liveness: there is $M', M'' \in [\, M \,\rangle$ s.t. $M' \xrightarrow{t} M''$

Then $M'(p) > 0$ or $M''(p) > 0$

Dead nodes, intuitively

Given a marking M

A transition t is dead at M
if t will never be enabled in the future
(i.e., t is not enabled in any marking reachable from M)

A place p is dead at M
if p will never be marked in the future
(i.e., there is no token in p in any marking reachable from M)

Dead nodes

Definition: Let (P, T, F) be a net

A transition $t \in T$ is **dead** at M if $\forall M' \in [\, M \,\rangle.\, M' \not\xrightarrow{t}$

A place $p \in P$ is **dead** at M if $\forall M' \in [\, M \,\rangle.\, M'(p) = 0$

Some obvious facts

If a system is not live, it must have a transition dead at some reachable marking

If a system is not place-live, it must have a place dead at some reachable marking

If a place / transition is dead at M, then it remains dead at any marking reachable from M
(the set of dead nodes can only increase during a run)

Every transition in the pre- or post-set of a dead place is also dead

Exercises

Prove each of the following properties
or give some counterexamples

If a system is not place-live, then it is not live

If a system is not live, then it is not place-live

If a system is place-live, then it is deadlock-free

If a system is deadlock-free, then it is place-live

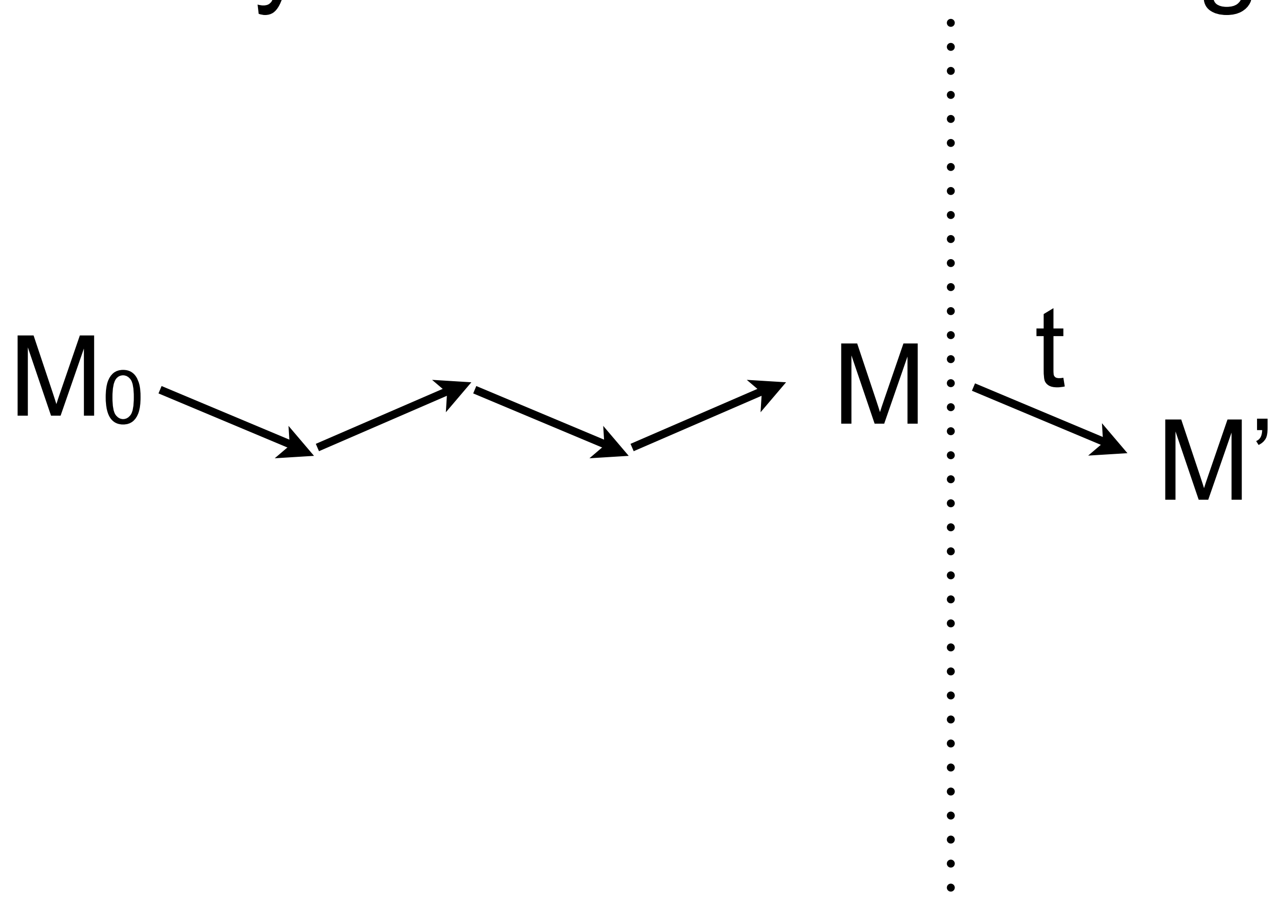
Deadlock-freedom

A Petri net is **deadlock free**, if every reachable marking enables some transition

In other words, we are guaranteed that at any point in time of the computation, some transition can be fired

Deadlock-freedom illustrated

For any reachable marking M



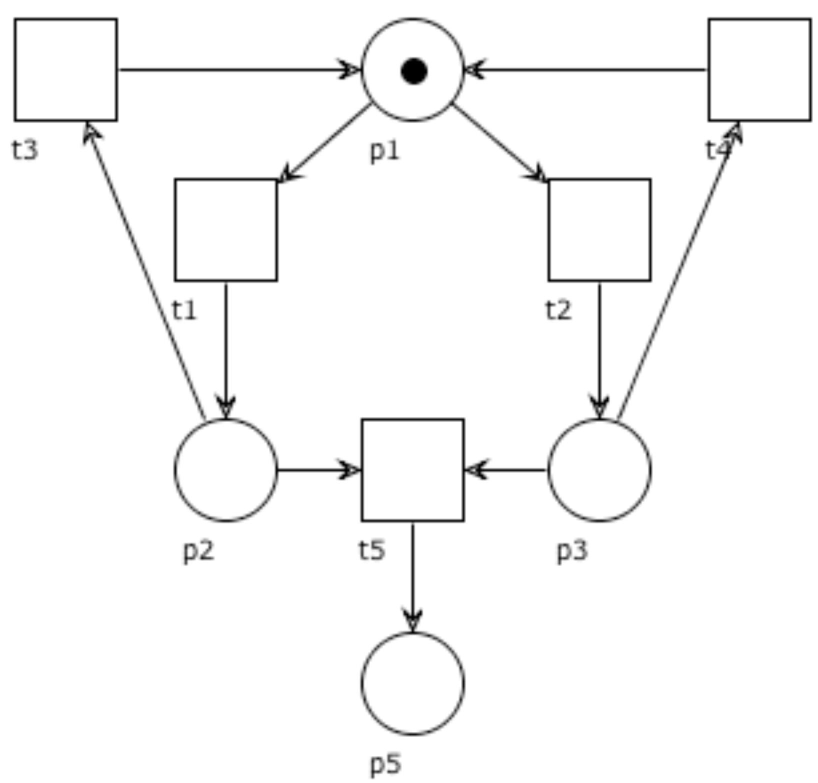
Can we fire some transition?

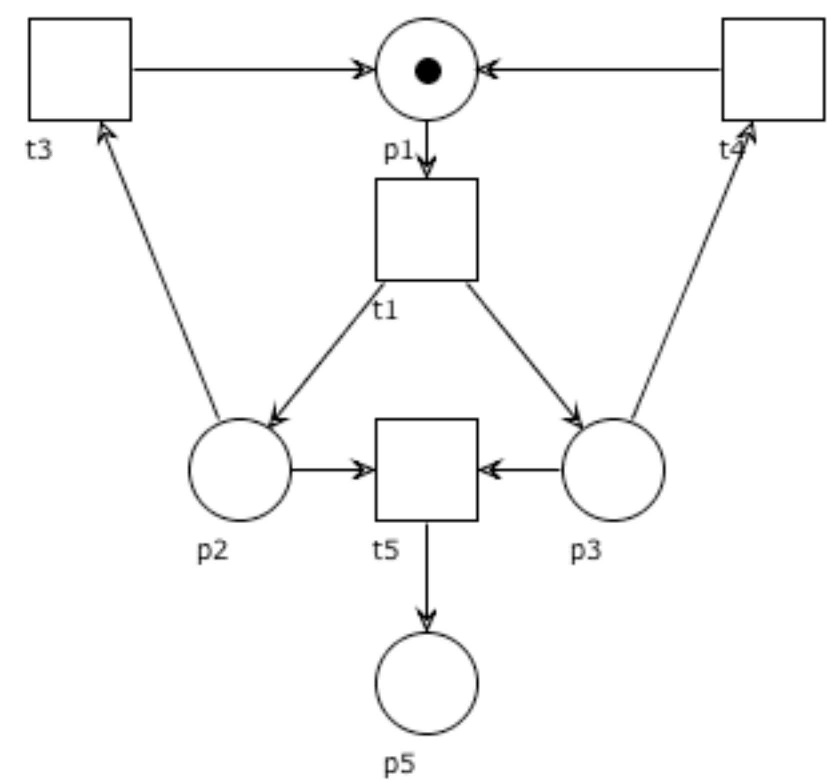
Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [\, M_0 \,\rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

Deadlock-freedom: example





Is the net deadlock-free?

Question time

Does liveness imply deadlock-freedom?
(Can you exhibit a live Petri net that is not deadlock-free?)

Does deadlock-freedom imply liveness?
(Can you exhibit a deadlock-free net that is not live?)

Liveness implies deadlock freedom

Lemma If (P, T, F, M_0) is live, then it is deadlock-free

By contradiction, let $M \in [\, M_0 \,\rangle$, with $M \not\rightarrow$

Let $t \in T$ (T cannot be empty).
By liveness, $\exists M' \in [\, M \,\rangle$ with $M' \xrightarrow{t}$.

Since M is dead, $[\, M \,\rangle = \{\, M \,\}$.
Therefore $M = M' \xrightarrow{t}$, which is absurd.

k-Boundedness

Let k be a natural number

A place p is **k-bounded** if no reachable marking has more than k tokens in place p

A net is **k-bounded** if all of its places are k-bounded

In other words, if a net is k-bounded, then k is a capacity constraint that can be imposed over places without any risk of causing “overflow”

Safe nets

A place p is **safe** if it is 1-bounded

A net is **safe** if all of its places are safe

In other words, if the net is safe, then we know that, in any reachable marking, each place contains one token at most

Boundedness

A place p is **bounded** if it is k-bounded for some natural number k

A net is **bounded** if all of its places are bounded

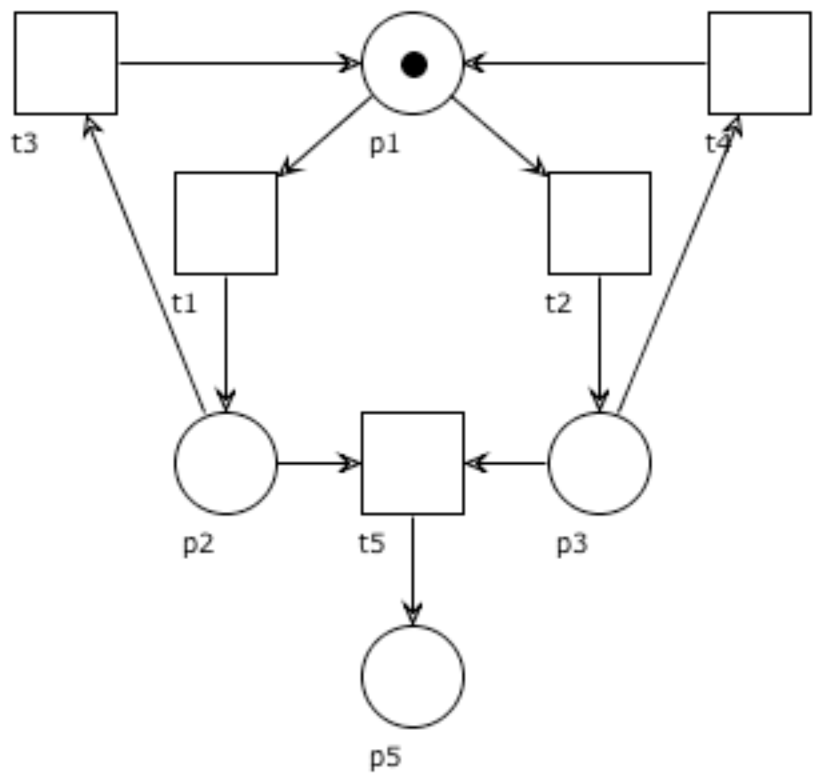
A net is **unbounded** if it is not bounded

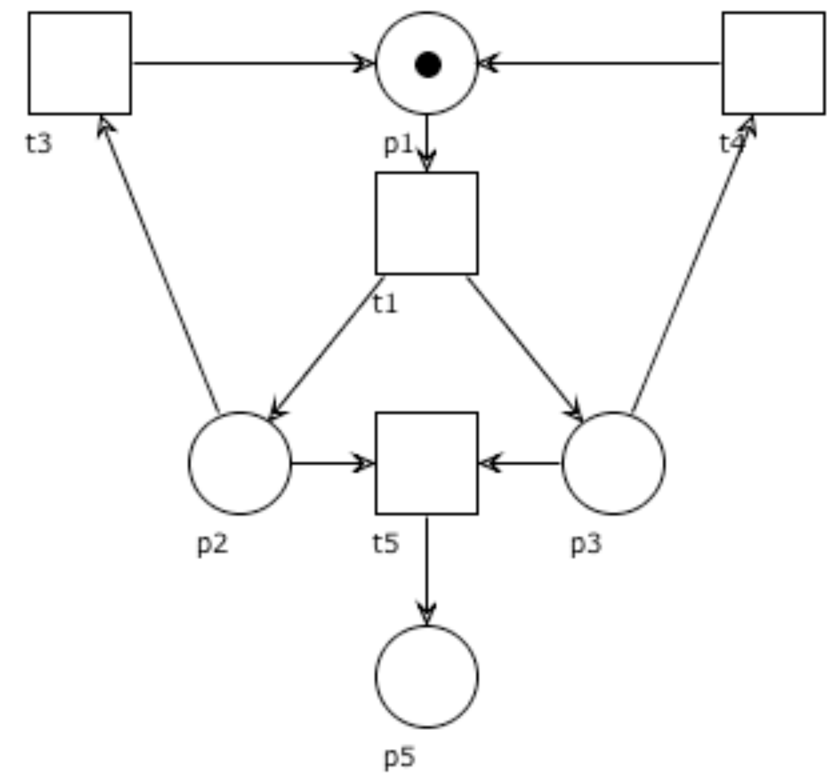
Boundedness, formally

$$(P, T, F, M_0)$$

$$\exists k \in \mathbb{N}, \quad \forall M \in [\, M_0 \,\rangle, \quad \forall p \in P, \quad M(p) \leq k$$

Boundedness: example





Which places are bounded?
Is the net bounded?
Which places are safe?
Is the net safe?

A puzzle about reachability

Theorem: If a system is... then its reachability graph is finite

Theorem: A system is... iff its reachability graph is finite

(fill the dots and the proofs)

A puzzle about boundedness

Theorem: If a system is k-bounded then any reachable marking contains a number of tokens less than or equal to …

Theorem: If a system is safe then any reachable marking contains a number of tokens less than or equal to …

(fill the dots and the proofs)

Cyclicity (aka Reversibility)

A marking M is a **home marking** if it can be reached from every reachable marking

A net is **cyclic** (or **reversible**) if its initial marking is a home marking

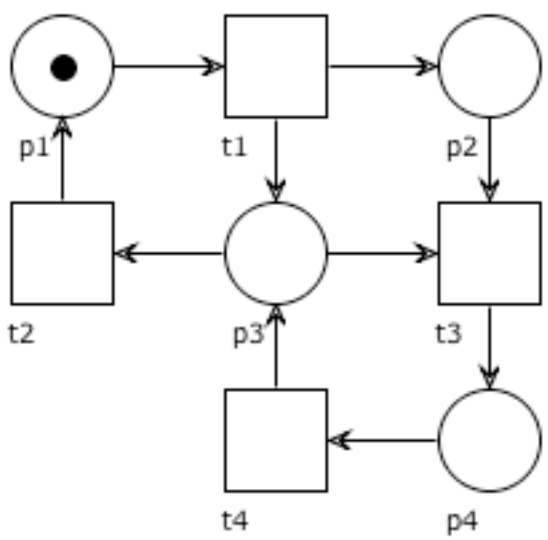
Orthogonal properties

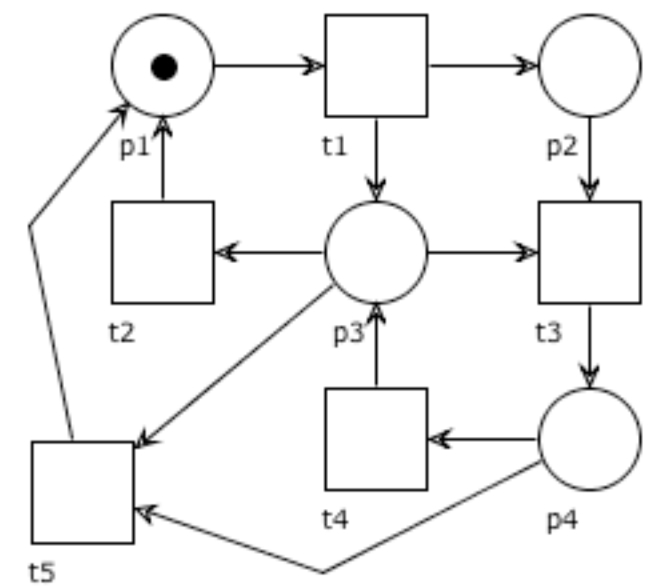
Liveness, boundedness and cyclicity are independent of each other

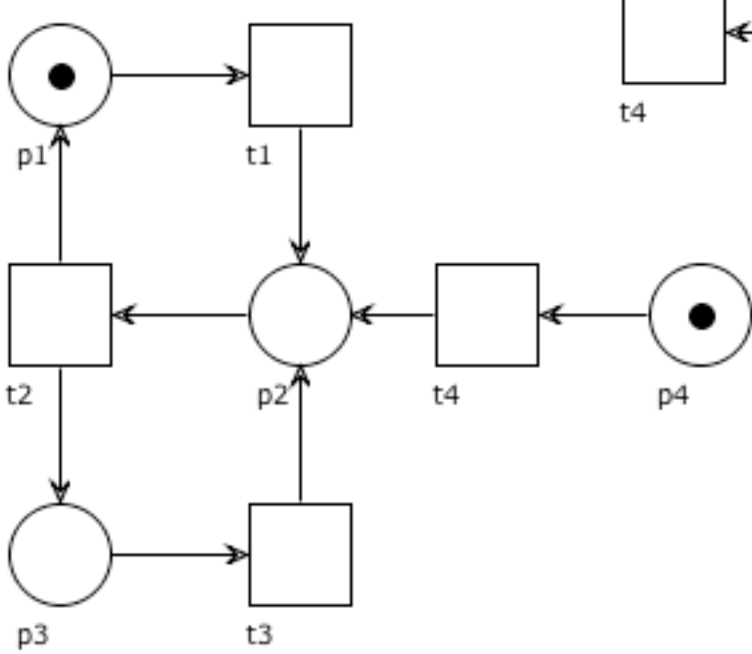
In other words, you can find nets that satisfy any arbitrary combination of the above three properties (and not the others)

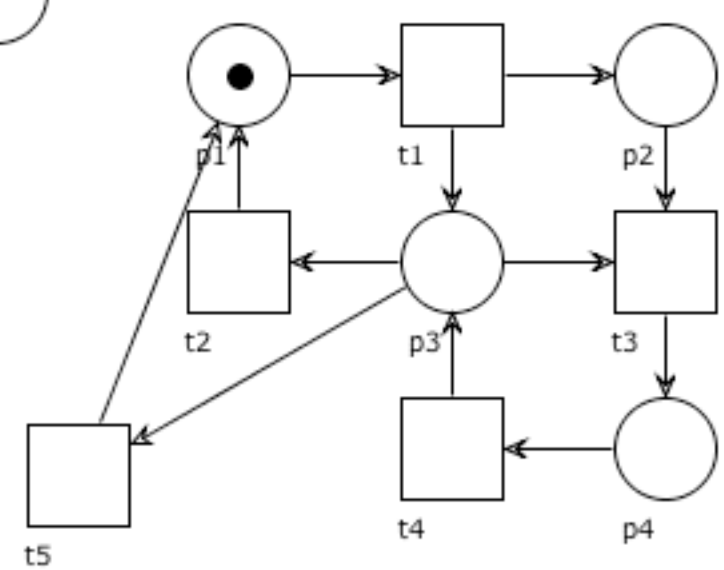
Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic



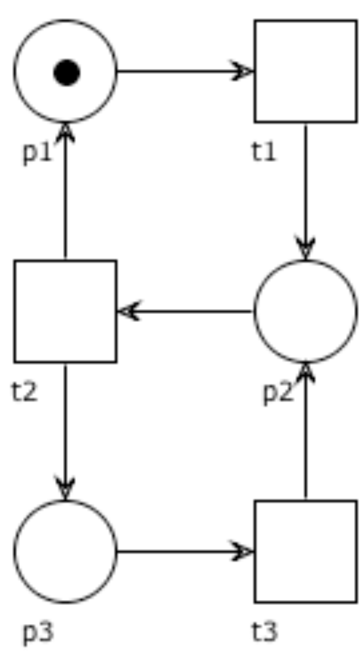


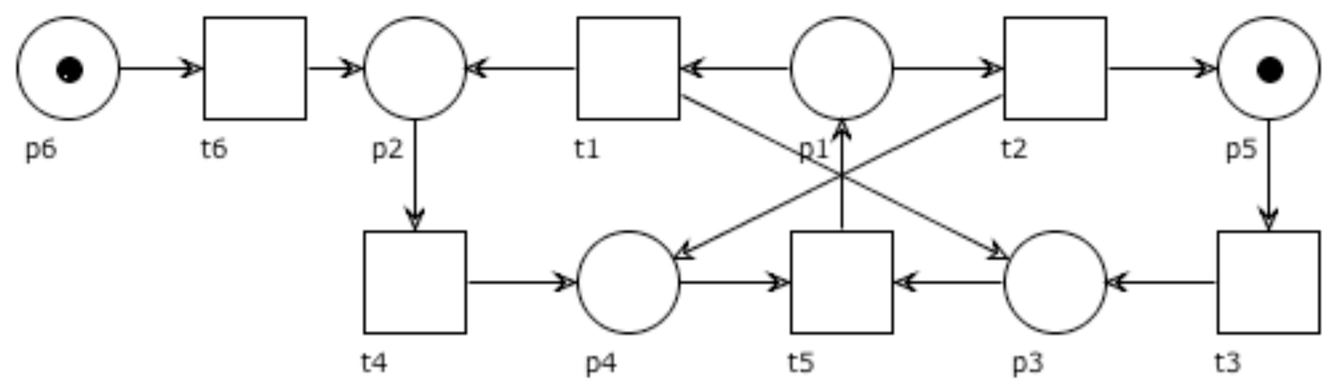


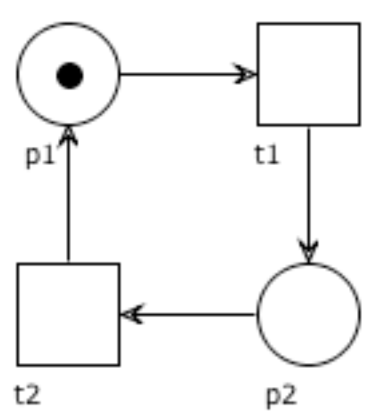


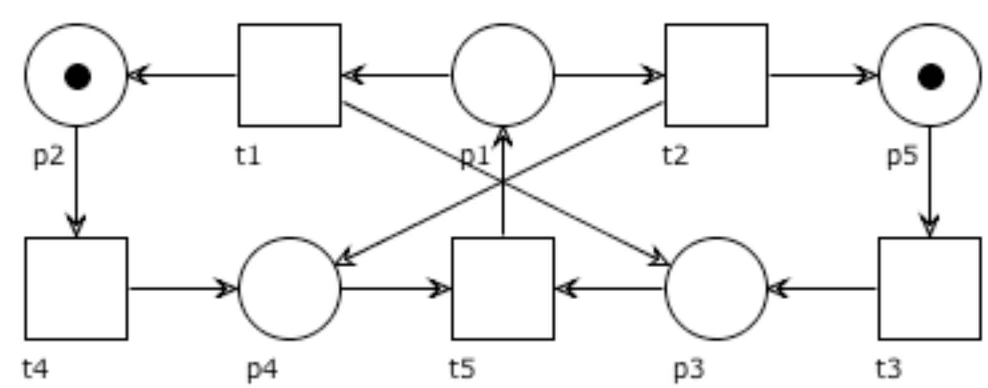
Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic









Petri nets: structural properties

Structural properties

All the properties we have seen so far are
behavioural (or **dynamic**)
(i.e. they depend on the initial marking and firing rules)

It is sometimes interesting to connect them to
structural properties
(i.e. the shape of the graph representing the net)

This way we can give **structural characterization** of
behavioural properties for a class of nets
(computationally less expensive to check)

A matter of terminology

To better reflect the above distinction, it is frequent:

to use the term **net system** for denoting a Petri net with a given initial marking
(we study behavioural properties of systems)

to use the term **net** for denoting a Petri net without specifying any initial marking
(we study structural properties of nets)

Paths and circuits

A **path** of a net (P, T, F) is a non-empty sequence $x_1 x_2 ... x_k$ such that

$$(x_i, x_{i+1}) \in F \quad \text{for every } 1 \leq i < k$$

(and we say that it leads from x_1 to x_k)

A path from x to y is called a **circuit** if:
no element occurs more than once in it and $(y, x) \in F$
(since for any x we have $(x, x) \notin F$, hence a circuit involves at least two nodes)

Connectedness

A net (P,T,F) is **weakly connected** iff it does not fall into (two or more) unconnected parts (i.e. no two subnets (P_1,T_1,F_1) and (P_2,T_2,F_2) with disjoint and non-empty sets of elements can be found that partition (P,T,F))

A weakly connected net is **strongly connected** iff for every arc (x,y) there is a path from y to x

Connectedness, formally

A net (P, T, F) is **weakly connected** if every two nodes x, y satisfy

$$(x, y) \in (F \cup F^{-1})^*$$

(i.e. if there is an undirected path from x to y)

It is **strongly connected** if $(x, y) \in F^*$

A note

In the following we will consider (implicitly) weakly connected nets only

(if they are not, then we can study each of their subsystems separately)

S-systems / S-nets

A Petri net is called **S-system** if every transition has one input place and one output place
(S comes from *Stellen*, the German word for place)

This way any synchronization is ruled out

The theory of S-systems is very simple

T-systems / T-nets

A Petri net is called **T-system** if every place has one input transition and one output transition

This way all choices/conflicts are ruled out

T-systems have been studied extensively since the early Seventies

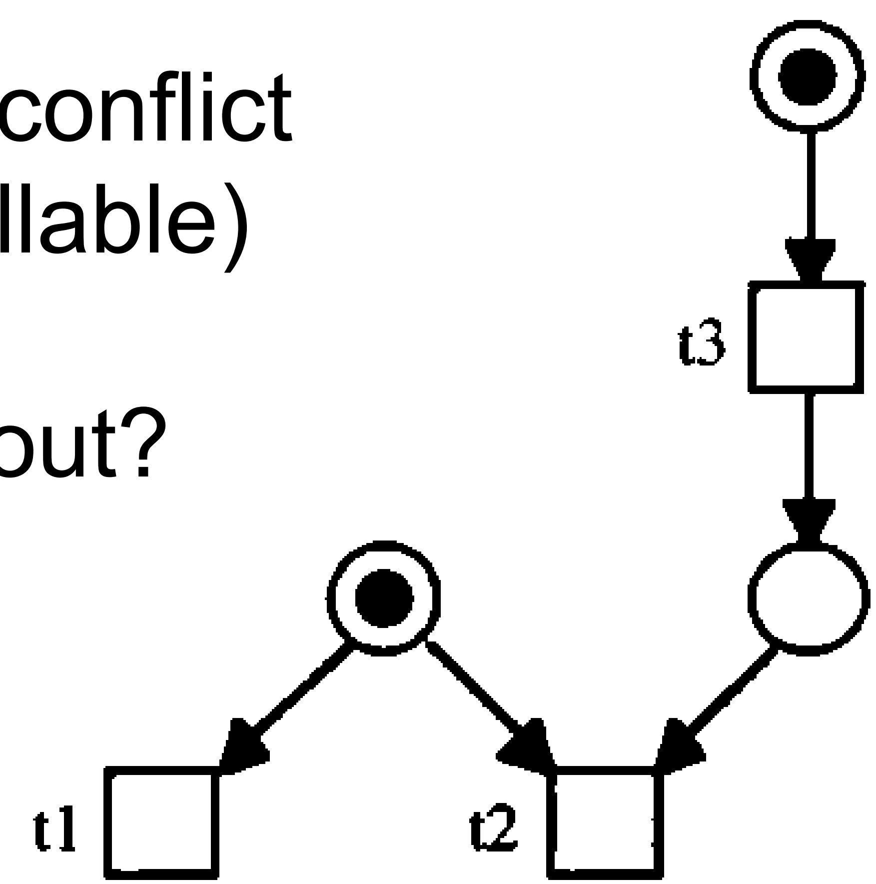
Interference of conflicts and synch

Typical situation:

initially t1 and t2 are not in conflict

but when t3 fires they are in conflict
(the firing of t3 is not controllable)

How to rule this situation out?



Free-choice nets

The aim is to avoid that a choice between transitions
is influenced by the rest of the system

Easiest way:
keep places with more than one output transition apart
from transitions with more than one input place

In other words, if (p,t) is an arc, then it means that
t is the only output transition of p (no conflict)
OR
p is the only input place of t (no synch)

Free-choice systems / nets

But we can study a slightly more general class of nets
by requiring a weaker constraint

A Petri net is **free-choice** if
whenever there is an arc (p,t), then there is an arc
from any input place of t
to any output transition of p

Question time

Is the net an S-net, a T-net, free-choice?