

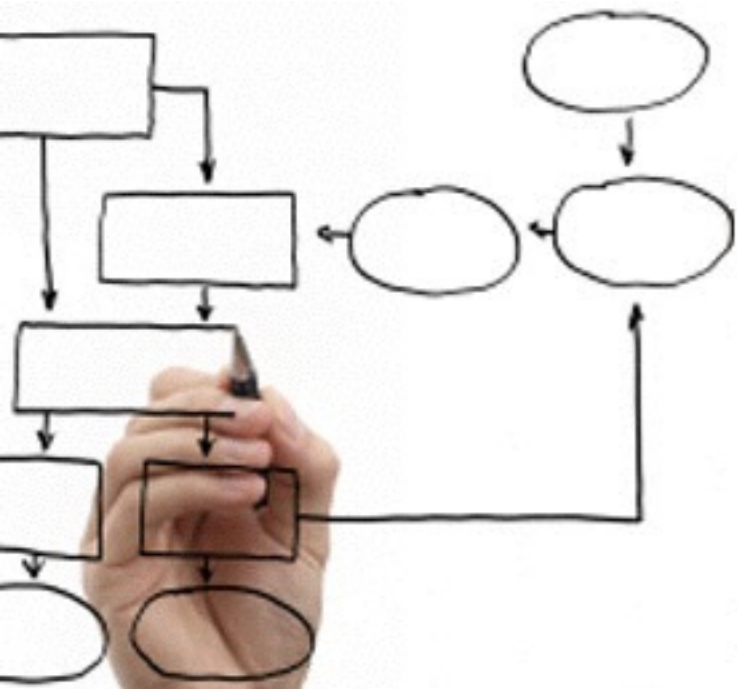
# Methods for the specification and verification of business processes

## MPB (6 cfu, 295AA)

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09 - Incidence matrices



# Object

We give a matrix-based representation of Petri nets and their computations

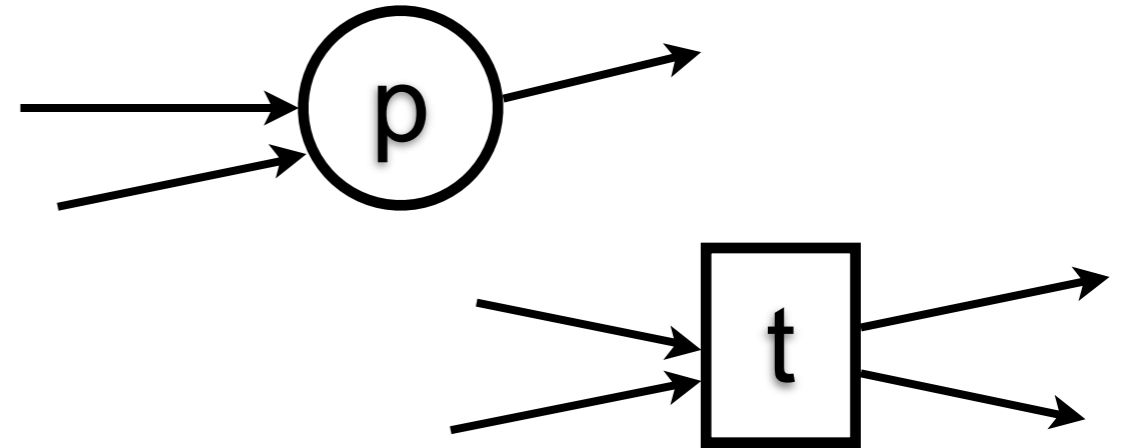
# Key point

The change of the numbers of tokens on a place  $p$  caused by the firing of the transition  $t$  does not depend on the current marking

It is entirely determined by the net  
(i.e., by the flow relation)

Let us have a look at the relative changes for every place and transition...

# How $p$ relates to $t$

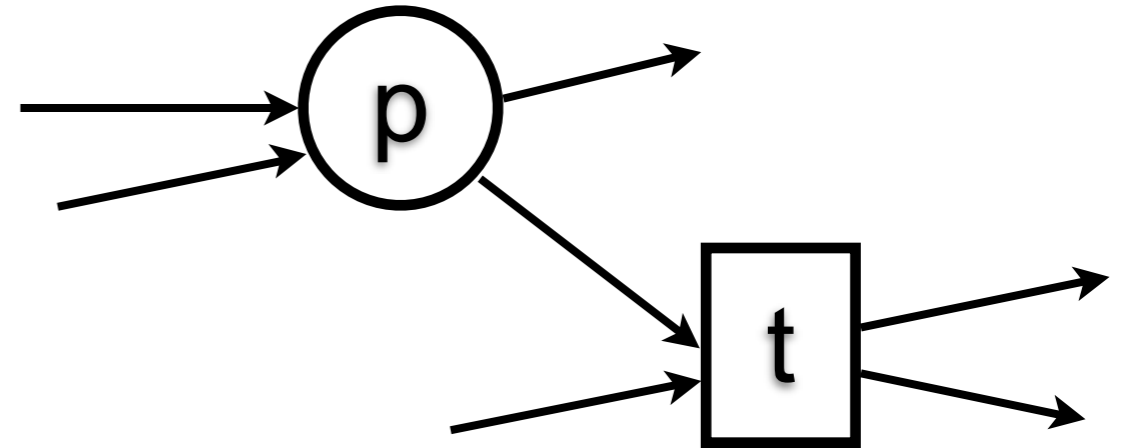


$$(p, t) \notin F \quad \text{and} \quad (t, p) \notin F$$

Place  $p$  and transition  $t$  are completely unrelated:

- $p$  has no influence on the enabling of  $t$
- firing  $t$  does not change the number of tokens in  $p$

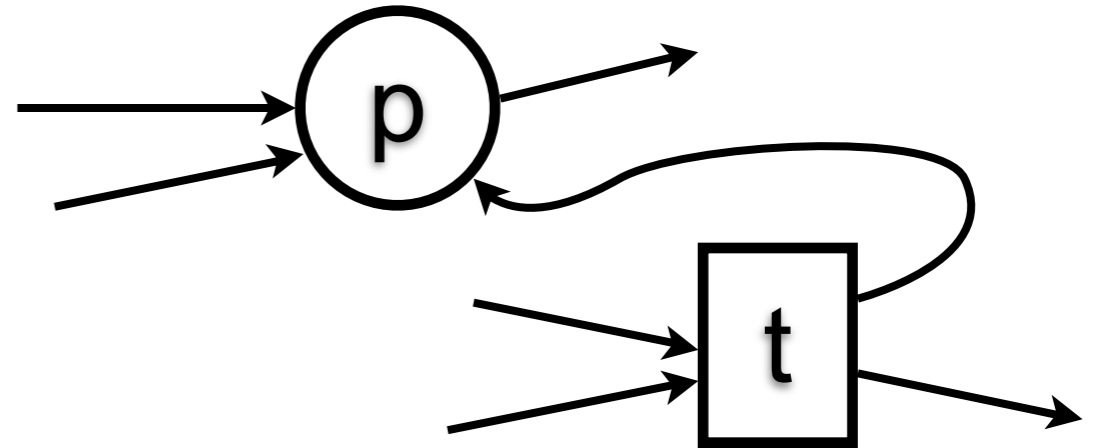
# How $p$ relates to $t$



$(p, t) \in F$  and  $(t, p) \notin F$

- one token in  $p$  is needed to enable  $t$
- firing  $t$  reduces by one the number of tokens in  $p$

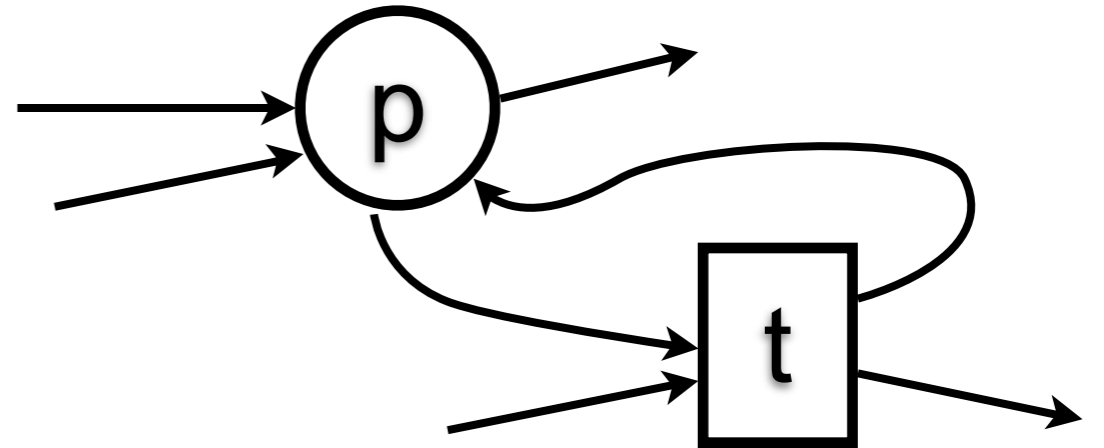
# How $p$ relates to $t$



$(p, t) \notin F$  and  $(t, p) \in F$

- firing  $t$  increases by one the number of tokens in  $p$

# How $p$ relates to $t$



$(p, t) \in F$  and  $(t, p) \in F$

- one token in  $p$  is needed to enable  $t$
- firing  $t$  does not change the number of tokens in  $p$

# Incidence matrix

Let  $N = (P, T, F)$  be a net.

Its **incidence matrix**  $\mathbf{N} : (P \times T) \rightarrow \{-1, 0, 1\}$  is defined as:

$$\mathbf{N}(p, t) = \begin{cases} 0 & \text{if } (p, t) \notin F \wedge (t, p) \notin F \text{ or } (p, t) \in F \wedge (t, p) \in F \\ -1 & \text{if } (p, t) \in F \wedge (t, p) \notin F \\ +1 & \text{if } (p, t) \notin F \wedge (t, p) \in F \end{cases}$$



# Matrix view

m columns, one for each transition

n rows,  
one for  
each place

	$t_1$	$t_2$	$t_3$					...					$t_m$
$p_1$		+1	-1										-1
$p_2$	-1		+1										
$p_3$		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
$p_n$		-1	+1										-1

# Column vector $t_j$

$t_j : P \rightarrow \{-1, 0, 1\}$  such that  $t_j(p) = \mathbf{N}(p, t_j)$

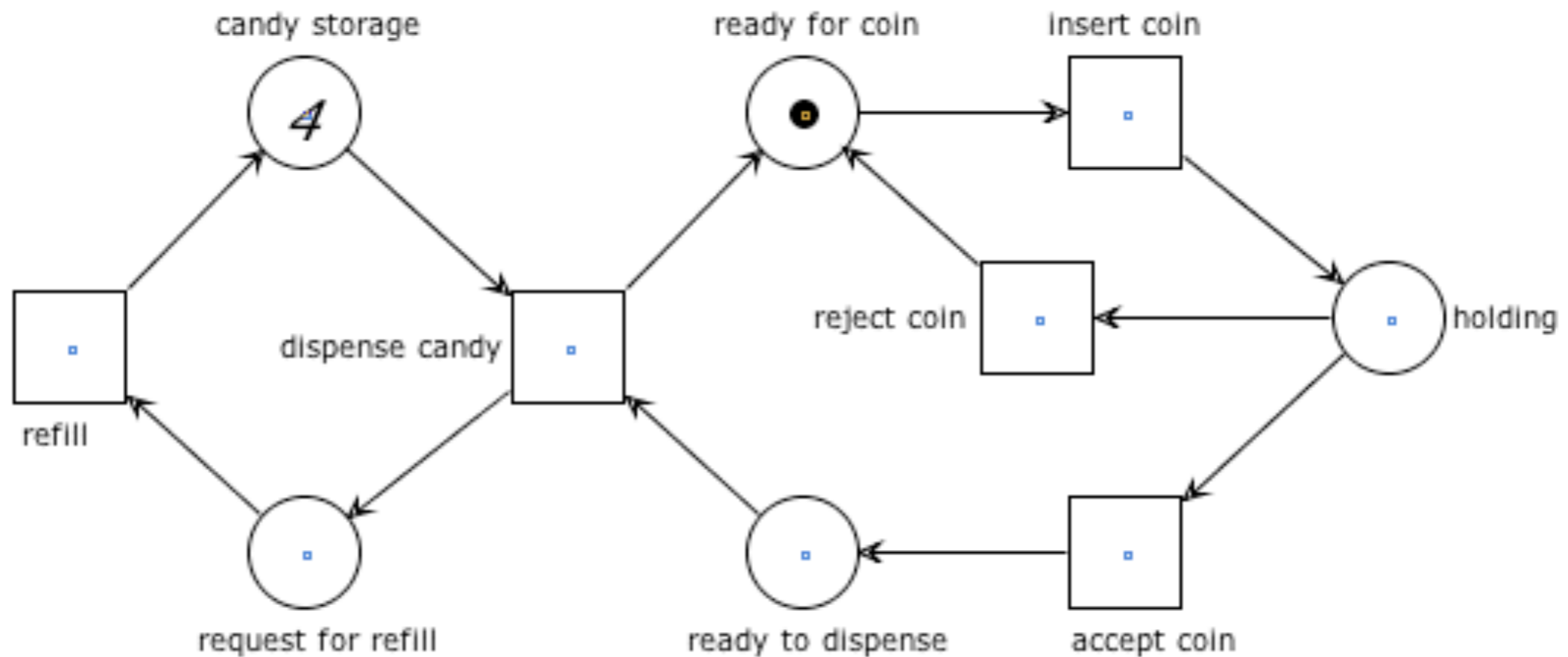
	$t_1$	$t_2$	$t_3$	...								$t_m$										
$p_1$		+1	-1																		-1	
$p_2$	-1		+1																			
$p_3$		+1																				
			+1																			
...										...												+1
	+1																					
			-1																			
																						+1
$p_n$		-1	+1																			-1

# Row vector $p_i$

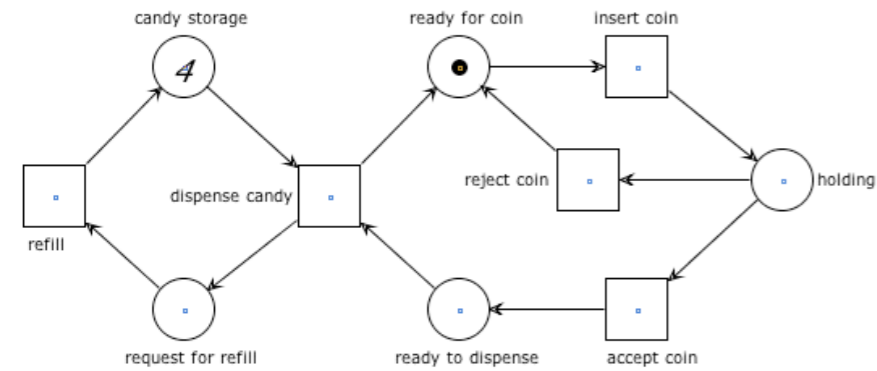
$\mathbf{p}_i : T \rightarrow \{-1, 0, 1\}$   
 such that  
 $\mathbf{p}_i(t) = \mathbf{N}(p_i, t)$

	$t_1$	$t_2$	$t_3$											$t_m$
$p_1$		+1	-1											-1
$p_2$	-1		+1											
$p_3$		+1												
			+1											
														+1
	+1													
			-1											
														+1
$p_n$		-1	+1											-1

# Example: vending machine



# Example: vending machine



	refill $t_1$	dispense candy $t_2$	insert coin $t_3$	accept coin $t_4$	reject coin $t_5$
candy storage $p_1$	1	-1			
request for refill $p_2$	-1	1			
ready for coin $p_3$		1	-1		1
holding $p_4$			1	-1	-1
ready to dispense $p_5$		-1		1	

# Vectors: notation

Let  $E = \{e_1, e_2, \dots, e_n\}$  be a finite set of elements.

Any mapping  $v : E \rightarrow \mathbb{Q}$  (or to  $\mathbb{N}, \mathbb{Z}, \dots$ ) can be regarded as a vector:

$$\mathbf{v} = [v(e_1), v(e_2), \dots, v(e_n)]$$

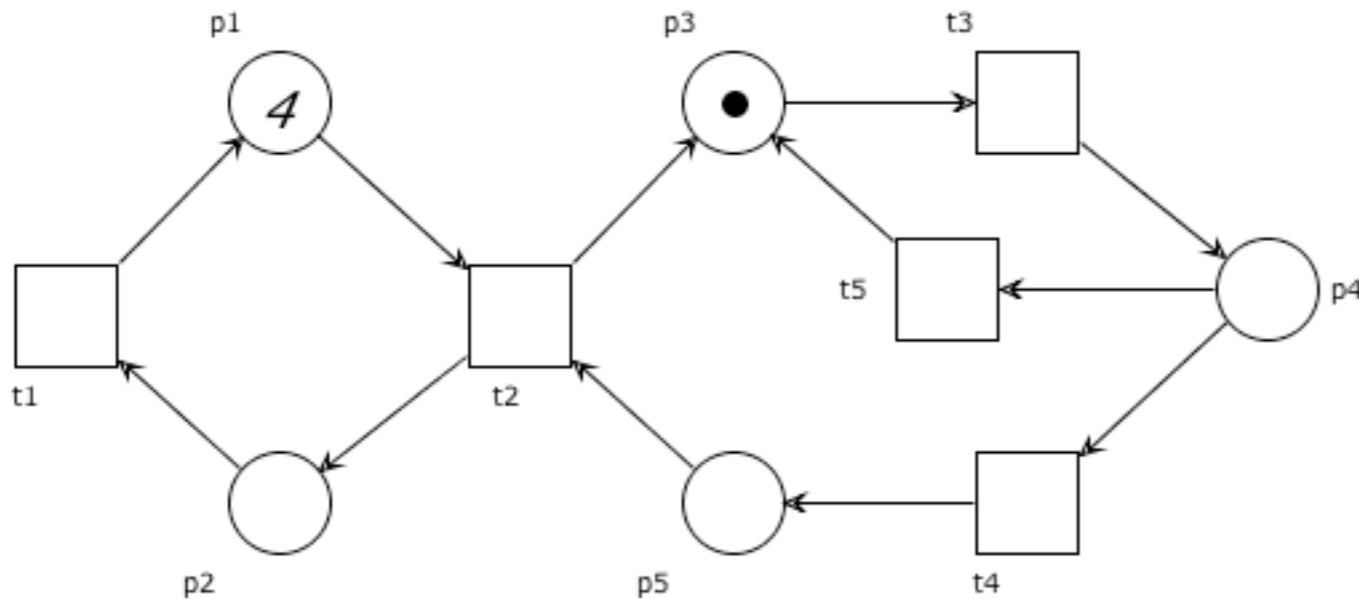
We **do not** use different symbols for row and columns vectors:

$$\mathbf{v} = \begin{bmatrix} v(e_1) \\ v(e_2) \\ \vdots \\ v(e_n) \end{bmatrix}$$

# Marking as a vector

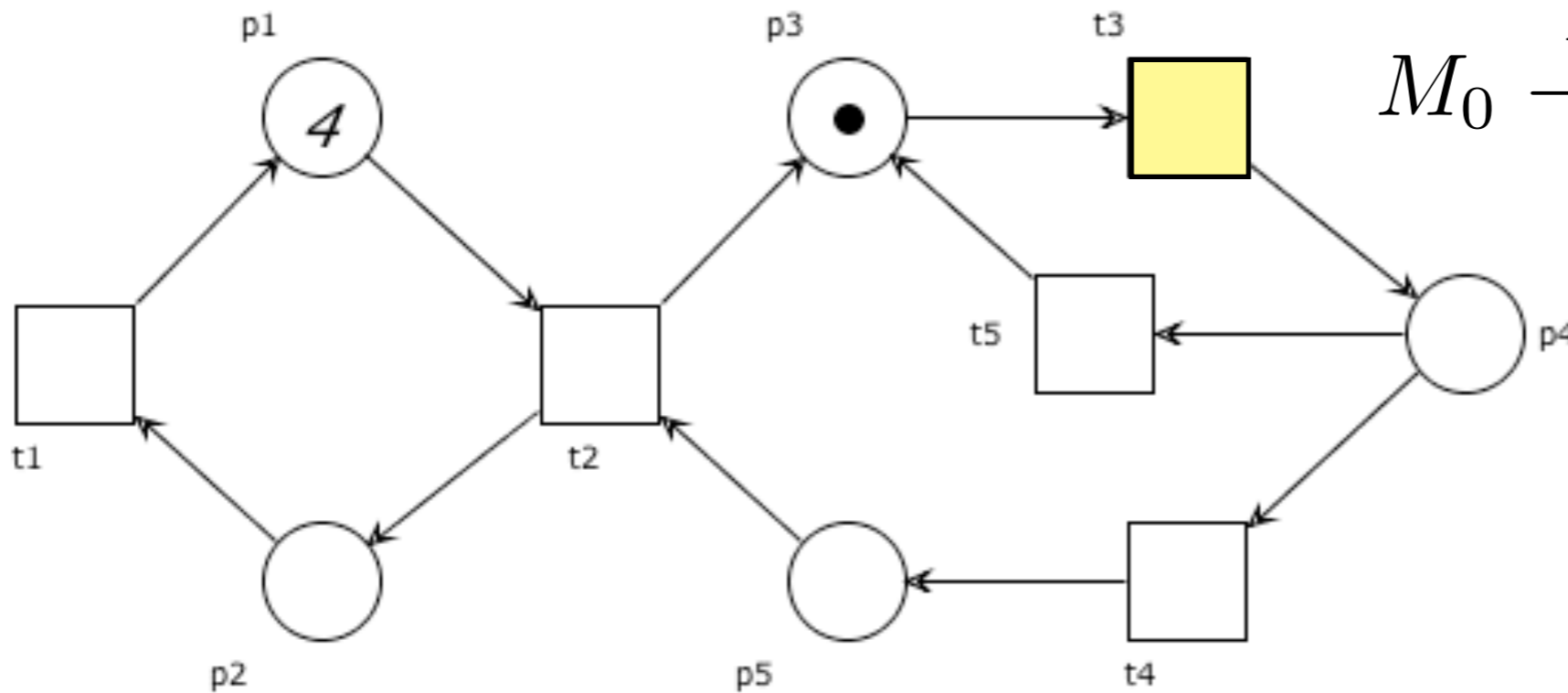
Any marking  $M : P \rightarrow \mathbb{N}$  corresponds to a vector:

$$M = [ M(p_1) \quad M(p_2) \quad \dots \quad M(p_n) ]$$



$$M_0 = [ 4 \quad 0 \quad 1 \quad 0 \quad 0 ]$$

# Firing, in vector notation

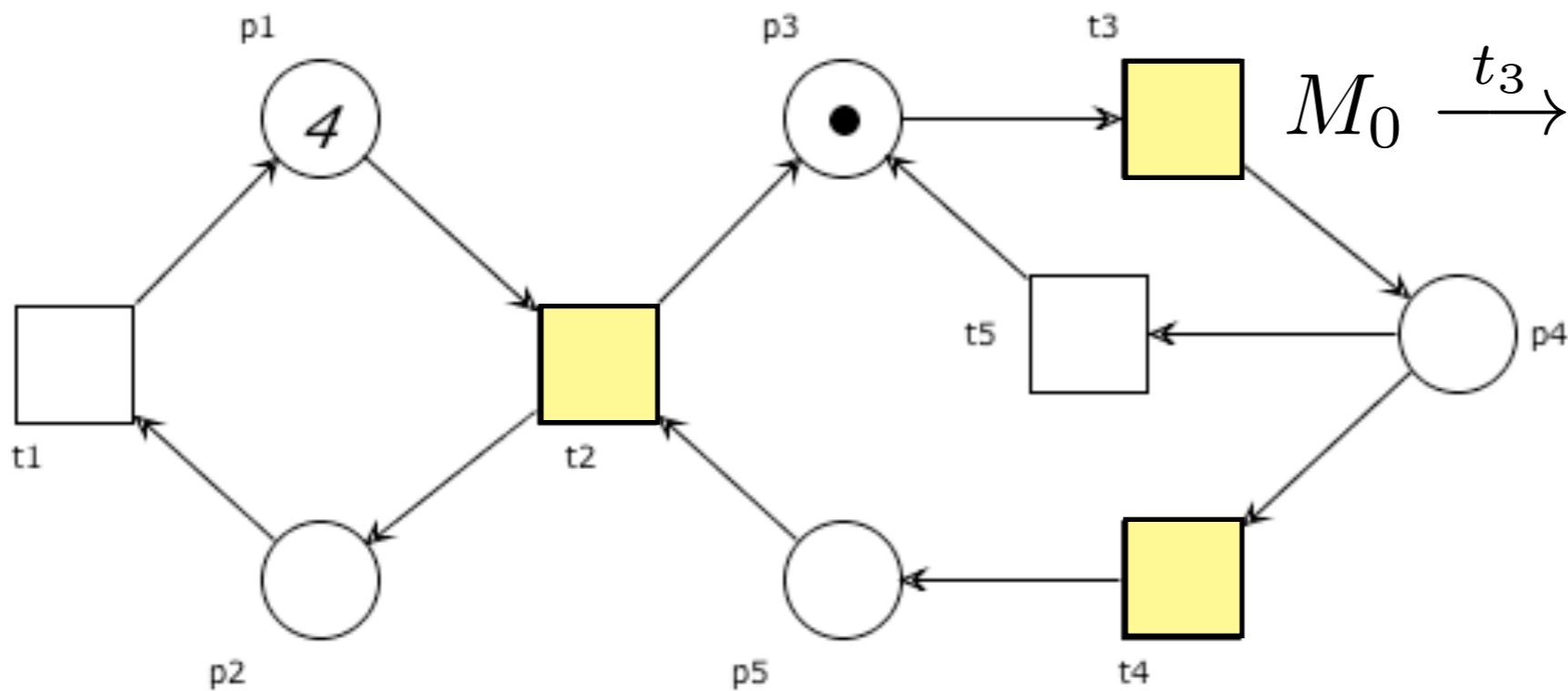


$$M_0 \xrightarrow{t_3} M_1 = 4p_1 + p_4$$

$$\begin{matrix} M_0 \\ \left[ \begin{array}{c} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{array} \right] \end{matrix}
 +
 \begin{matrix} t_3 \\ \left[ \begin{array}{c} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{array} \right] \end{matrix}
 =
 \begin{matrix} M_1 \\ \left[ \begin{array}{c} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{array} \right] \end{matrix}$$



# Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 \xrightarrow{t_4} M_2 \xrightarrow{t_2} M_3$$

$$\begin{matrix} M_0 \\ \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix} + \begin{matrix} t_3 \\ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \end{matrix} = \begin{matrix} M_1 \\ \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{matrix} + \begin{matrix} t_4 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{matrix} = \begin{matrix} M_2 \\ \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix} + \begin{matrix} t_2 \\ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \end{matrix} = \begin{matrix} M_3 \\ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

# Vectors: notation

Let  $\mathbf{v}, \mathbf{w}$  be two vectors over  $E$

We write  $\mathbf{v} \leq \mathbf{w}$  if  $v(e) \leq w(e)$  for any  $e \in E$

We write  $\mathbf{v} < \mathbf{w}$  if  $v \leq w$  and  $v(e) < w(e)$  for some  $e \in E$

We write  $\mathbf{v} \prec \mathbf{w}$  if  $v(e) < w(e)$  for any  $e \in E$

We let  $\mathbf{0}$  denote any vector of any length whose entries are all 0

# Products

Let  $\mathbf{x}, \mathbf{y}$  be two vectors of equal length  $n$  (written  $|\mathbf{x}| = |\mathbf{y}| = n$ )

We define their scalar product by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

# Products: example

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) + (0 \cdot 0) + (1 \cdot 1) = 0 + 1 - 2 + 0 + 1 = 0$$

# Products

Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{y}$  be all vectors of equal length

Let  $X$  be a  $(k \times n)$ -matrix whose  $i$ -th row is  $\mathbf{x}_i$

We define the product  $X \cdot \mathbf{y}$  as the (column) vector where

$$(X \cdot \mathbf{y})_i = \mathbf{x}_i \cdot \mathbf{y}$$

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_k \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{y} \\ \mathbf{x}_2 \cdot \mathbf{y} \\ \vdots \\ \mathbf{x}_k \cdot \mathbf{y} \end{bmatrix}$$

# Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1 \\ -1 + 1 \\ 1 - 2 + 1 \\ 2 - 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

# Products

Let  $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k$  be all vectors of equal length

Let  $Y$  be a  $(n \times k)$ -matrix whose  $i$ -th column is  $\mathbf{y}_i$

We define the product  $\mathbf{x} \cdot Y$  as the (row) vector where

$$(\mathbf{x} \cdot Y)_i = \mathbf{x} \cdot \mathbf{y}_i$$

$$[x_1 \ x_2 \ \dots \ x_n] \cdot [\mathbf{y}_1 \ \mathbf{y}_2 \ \dots \ \mathbf{y}_k] = [\mathbf{x} \cdot \mathbf{y}_1 \quad \mathbf{x} \cdot \mathbf{y}_2 \quad \dots \quad \mathbf{x} \cdot \mathbf{y}_k]$$

# Products

Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_h$  be all vectors of equal length

Let  $X$  be a  $(k \times n)$ -matrix whose  $i$ -th row is  $\mathbf{x}_i$

Let  $Y$  be a  $(n \times h)$ -matrix whose  $j$ -th column is  $\mathbf{y}_j$

We define the product  $X \cdot Y$  as the  $(k \times h)$ -matrix where

$$(X \cdot Y)_{i,j} = \mathbf{x}_i \cdot \mathbf{y}_j$$

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_k \end{bmatrix} \cdot [\mathbf{y}_1 \ \mathbf{y}_2 \ \dots \ \mathbf{y}_h] = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \mathbf{x}_1 \cdot \mathbf{y}_2 & \dots & \mathbf{x}_1 \cdot \mathbf{y}_h \\ \mathbf{x}_2 \cdot \mathbf{y}_1 & \mathbf{x}_2 \cdot \mathbf{y}_2 & \dots & \mathbf{x}_2 \cdot \mathbf{y}_h \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_k \cdot \mathbf{y}_1 & \mathbf{x}_k \cdot \mathbf{y}_2 & \dots & \mathbf{x}_k \cdot \mathbf{y}_h \end{bmatrix}$$



# Vector perspective

Let  $P = \{ p_1, \dots, p_n \}$  and  $T = \{ t_1, \dots, t_m \}$

The net  $(P, T, F)$  can be seen as a matrix  $(n \times m)$

A marking is a vector of length  $n$

But we miss an ingredient:

can a firing be seen as a vector?

can a firing sequence be seen as a vector?  
(of limited length)

# Parikh vectors of transition sequences

Let  $N = (P, T, F)$  be a net and  $\sigma \in T^*$  a finite sequence of transitions.

The **Parikh vector**

$$\vec{\sigma} : T \rightarrow \mathbb{N}$$

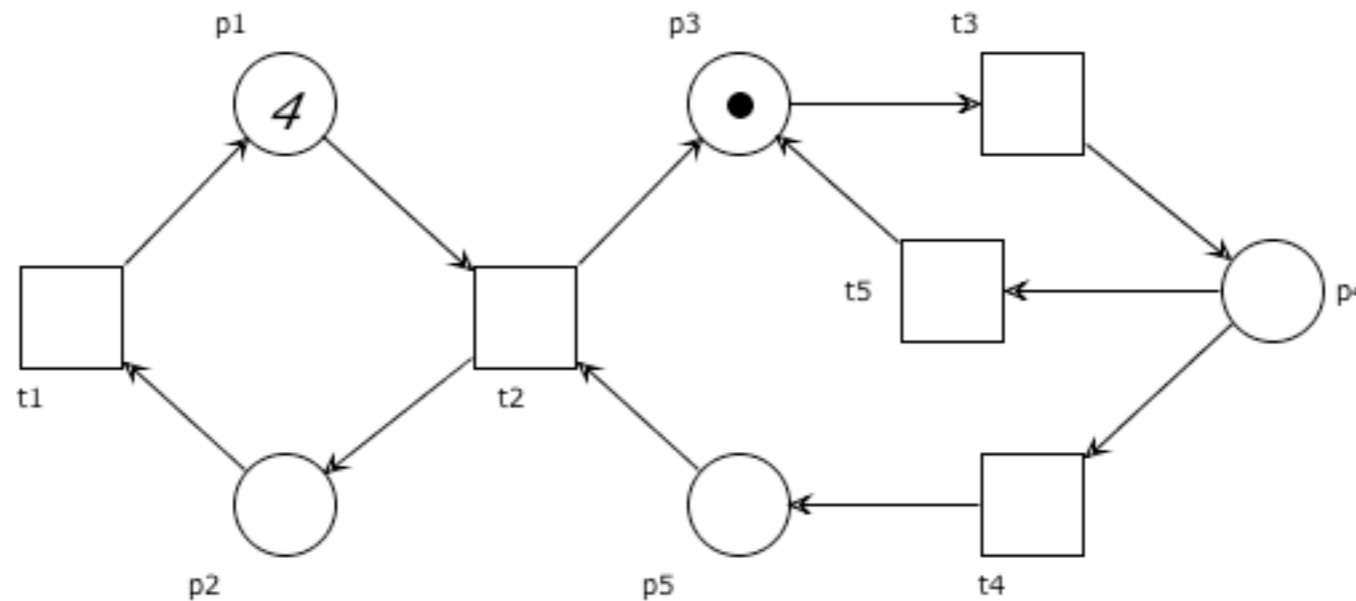
of  $\sigma$  maps every  $t \in T$  to the number of its occurrences in  $\sigma$ .

# Parikh vector of a firing

As a special case, for a sequence  $\sigma = t$  (one single transition):

$$\vec{t} = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ t_1 & & & t & & & t_m \end{bmatrix}$$

# Parikh vector: example



$$M_0 = 4p_1 + p_3$$

$$M_0 \xrightarrow{\sigma = t_3 t_5 t_3 t_4 t_2} 3p_1 + p_2 + p_3 \quad \vec{\sigma} = [0 \quad 1 \quad 2 \quad 1 \quad 1]$$

$$M_0 \xrightarrow{\sigma' = t_3 t_4 t_2 t_3 t_4 t_2 t_3 t_5 t_3} 2p_1 + 2p_2 + p_4 \quad \vec{\sigma}' = [0 \quad 2 \quad 4 \quad 2 \quad 1]$$

# First fact

$$\mathbf{N} \cdot \vec{t}_j = t_j$$

	$t_1$	$t_j$	$t_m$	
$p_1$				0
				0
				1
				0
$p_n$				0

$\vec{t}_j$

0	$t_1$
0	
1	$t_j$
0	
0	$t_m$

# Second fact

$$\mathbf{N} \cdot \vec{t}_j = t_j$$

If  $M \xrightarrow{t} M'$  then  $M' = M + \mathbf{t}$

# Third fact

$$\mathbf{N} \cdot \vec{t}_j = t_j$$

If  $M \xrightarrow{t} M'$  then  $M' = M + \mathbf{t}$

If  $M \xrightarrow{t} M'$  then  $M' = M + \mathbf{N} \cdot \vec{t}$

# Marking equation lemma

**Lemma:** If  $M \xrightarrow{\sigma} M'$  then  $M' = M + \mathbf{N} \cdot \vec{\sigma}$

The proof is by induction on the length of  $\sigma$

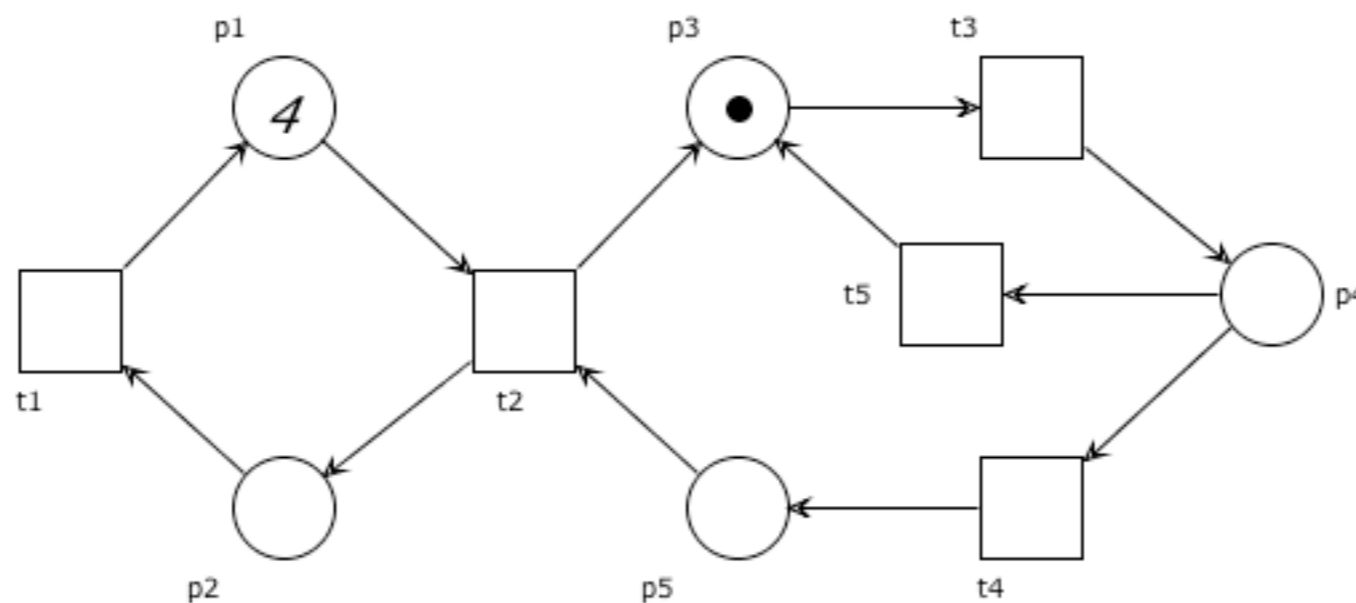
**base** ( $\sigma = \epsilon$ ): and therefore  $M' = M$ . The equality holds trivially, because  $\vec{\sigma} = \mathbf{0}$

**induction** ( $\sigma = \sigma' t$  for some sequence  $\sigma'$  and transition  $t$ ):

$$\begin{aligned} \text{Let } M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'. \text{ We have: } M' &= M'' + \mathbf{t} \\ &= M'' + \mathbf{N} \cdot \vec{t} \\ &= M + \mathbf{N} \cdot \vec{\sigma}' + \mathbf{N} \cdot \vec{t} \\ &= M + \mathbf{N} \cdot (\vec{\sigma}' + \vec{t}) \\ &= M + \mathbf{N} \cdot \vec{(\sigma' t)} \\ &= M + \mathbf{N} \cdot \vec{\sigma} \end{aligned}$$



# Marking equation: example



$$M_0 = [4 \quad 0 \quad 1 \quad 0 \quad 0] \quad \sigma = t_3 t_5 t_3 t_4 t_2 \quad \vec{\sigma} = [0 \quad 1 \quad 2 \quad 1 \quad 1]$$

$$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

# Marking equation lemma: consequences

The marking reached by any occurrence sequence only depends on the number of occurrences of each transition

It does not depend on the order in which transitions occur

**Every fireable permutation of the same transitions leads to the same marking**

# Monotonicity lemma (1)

**Lemma:** If  $M \xrightarrow{\sigma} M'$  then  $M + L \xrightarrow{\sigma} M' + L$  for any  $L$

The proof is by induction on the length of  $\sigma$

**base** ( $\sigma = \epsilon$ ): the empty sequence is always enabled, at any marking

**induction** ( $\sigma = \sigma' t$  for some sequence  $\sigma'$  and transition  $t$ ):

Let  $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$ .

By the marking equation lemma:  $M' = M'' + \mathbf{N} \cdot \vec{t}$

By the induction hypothesis  $M + L \xrightarrow{\sigma'} M'' + L$

Moreover,  $M'' + L \xrightarrow{t}$  because  $M'' \xrightarrow{t}$ .

By the marking equation lemma:  $M'' + L \xrightarrow{t} M'' + L + \mathbf{N} \cdot \vec{t} = M' + L$

(from previous slides)

# Enabledness

**Proposition:**  $M \xrightarrow{\sigma}$  iff  $M \xrightarrow{\sigma'}$  for every prefix  $\sigma'$  of  $\sigma$

( $\Rightarrow$ ) immediate from definition

( $\Leftarrow$ ) trivial if  $\sigma$  is finite ( $\sigma$  itself is a prefix of  $\sigma$ )

When  $\sigma$  is infinite: taken any  $i \in \mathbb{N}$  we need to prove that  $t_i = \sigma(i)$  is enabled after the firing of the prefix  $\sigma' = t_1 t_2 \dots t_{i-1}$  of  $\sigma$ .

But this is obvious, because

$$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_{i-1}} M_{i-1} \xrightarrow{t_i} M_i$$

is also a finite prefix of  $\sigma$  and therefore  $M_{i-1} \xrightarrow{t_i}$

# Monotonicity lemma (2)

**Lemma:** If  $M \xrightarrow{\sigma}$  then  $M + L \xrightarrow{\sigma}$  for any  $L$

If  $\sigma$  is finite then the thesis follows from monotonicity lemma 1

If  $\sigma$  is infinite, then it suffices to prove that:

$M + L \xrightarrow{\sigma'}$  for any finite prefix  $\sigma'$  of  $\sigma$

Take any such prefix  $\sigma'$ . Then,  $M \xrightarrow{\sigma'}$  (because  $M \xrightarrow{\sigma}$ )

By the marking equation lemma,  $M \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma}'$ .

By monotonicity lemma 1,  $M + L \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma}' + L$

Hence  $M + L \xrightarrow{\sigma'}$

# Boundedness Lemma

**Lemma:** If a system is bounded and  $M \in [M_0\rangle$  with  $M \supseteq M_0$ , then  $M = M_0$ .

Let  $M_0 \xrightarrow{\sigma} M$ .

By  $M \supseteq M_0$ , there exists a marking  $L$  with  $M = M_0 + L$ .

Let  $M_k = M_0 + k \cdot L$  for every  $k \in \mathbb{N}$ .

By the Monotonicity Lemma, we have:

$$M_0 \xrightarrow{\sigma} M_1 \xrightarrow{\sigma} M_2 \cdots$$

i.e.,  $M_k \in [M_0\rangle$  for any  $k \in \mathbb{N}$ .

Since the system is bounded, it must be  $L = \emptyset$ .

# Repetition Lemma

**Lemma:** If  $M \xrightarrow{\sigma} M'$  and  $M \xrightarrow{\sigma\sigma\cdots}$ , then  $M \subseteq M'$ .

We proceed by contradiction.

Suppose  $M \not\subseteq M'$ , i.e., there exist  $k > 0, p \in P$  such that  $M'(p) = M(p) - k$ .

By the Marking Equation Lemma we have  $M' = M + \mathbf{N} \cdot \vec{\sigma}$ .

Therefore  $(\mathbf{N} \cdot \vec{\sigma})(p) = -k$ .

Let  $n = M(p) + 1$  and  $\sigma' = \underbrace{\sigma \cdots \sigma}_n$ .

By hypothesis we have  $M \xrightarrow{\sigma'} M''$ ,

and by the Marking Equation Lemma  $M''(p) = M(p) - nk < 0$ , which is absurd.