Business Processes Modelling

MPB (6 cfu, 295AA)

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09 - Petri nets properties

Object

$$N \vdash \psi$$

We give a formal account of some key properties of Petri nets

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Petri nets: behavioural properties

Properties of Petri nets

We introduce some of the properties of Petri nets that can play an important role in the verification of business processes

Liveness
Deadlock-freedom
Boundedness
Cyclicity (also Reversibility)

Liveness, intuitively

A transition t is **live** if
from any reachable marking M another
marking M’ can be reached where t is enabled

In other words:
at any point in time of the computation, we cannot exclude that t will fire in the future
or, equivalently,
at any point in time of the computation, it is still possible to enable t in the future

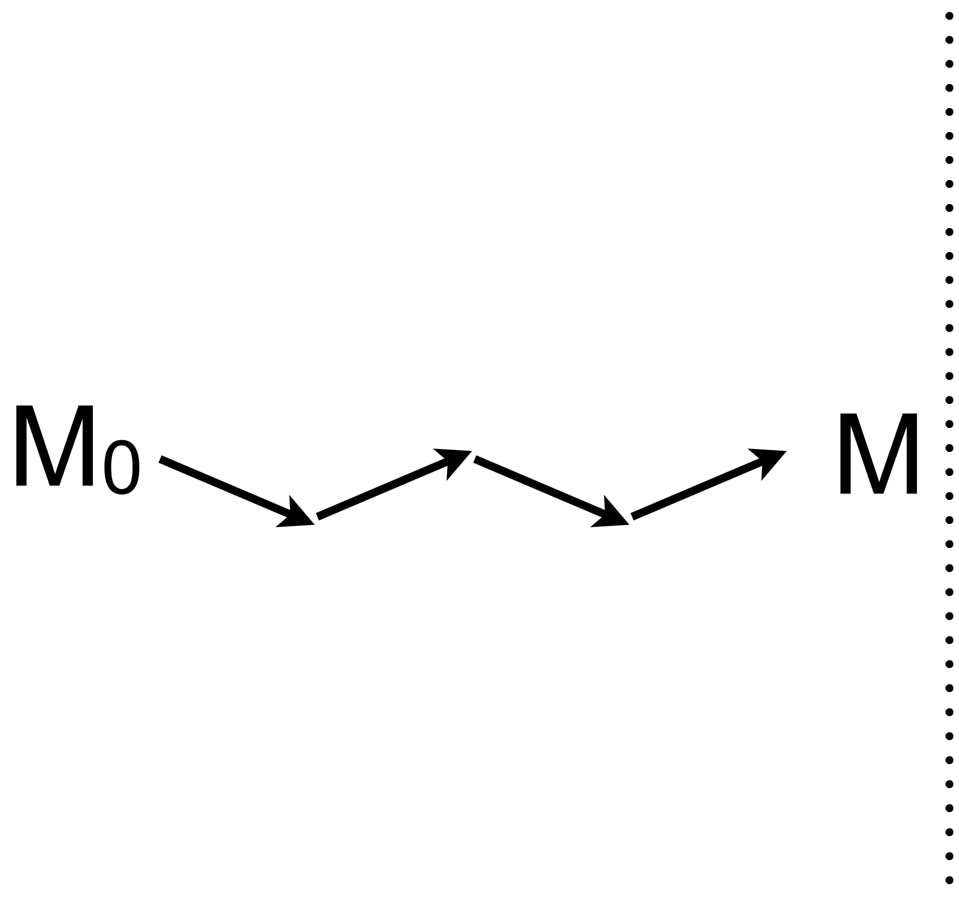
A Petri net is **live** if all of its transitions are live

Liveness illustrated

M_0

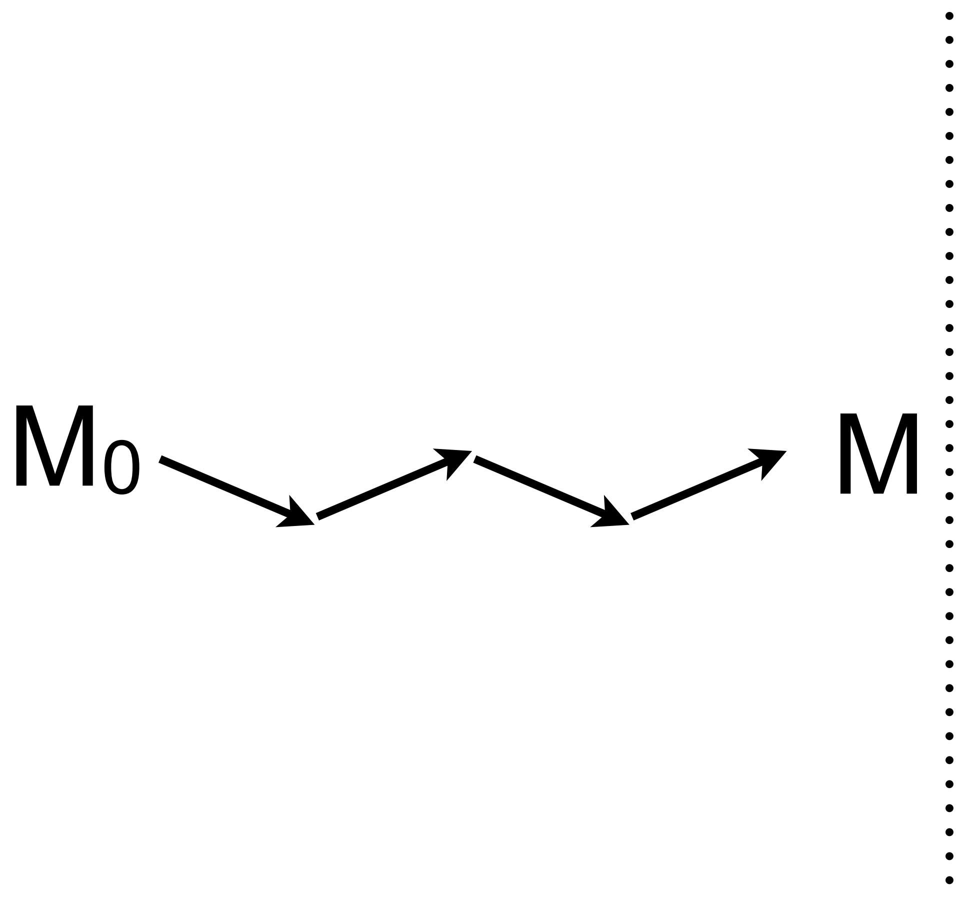
Liveness illustrated

For any reachable marking M



Liveness illustrated

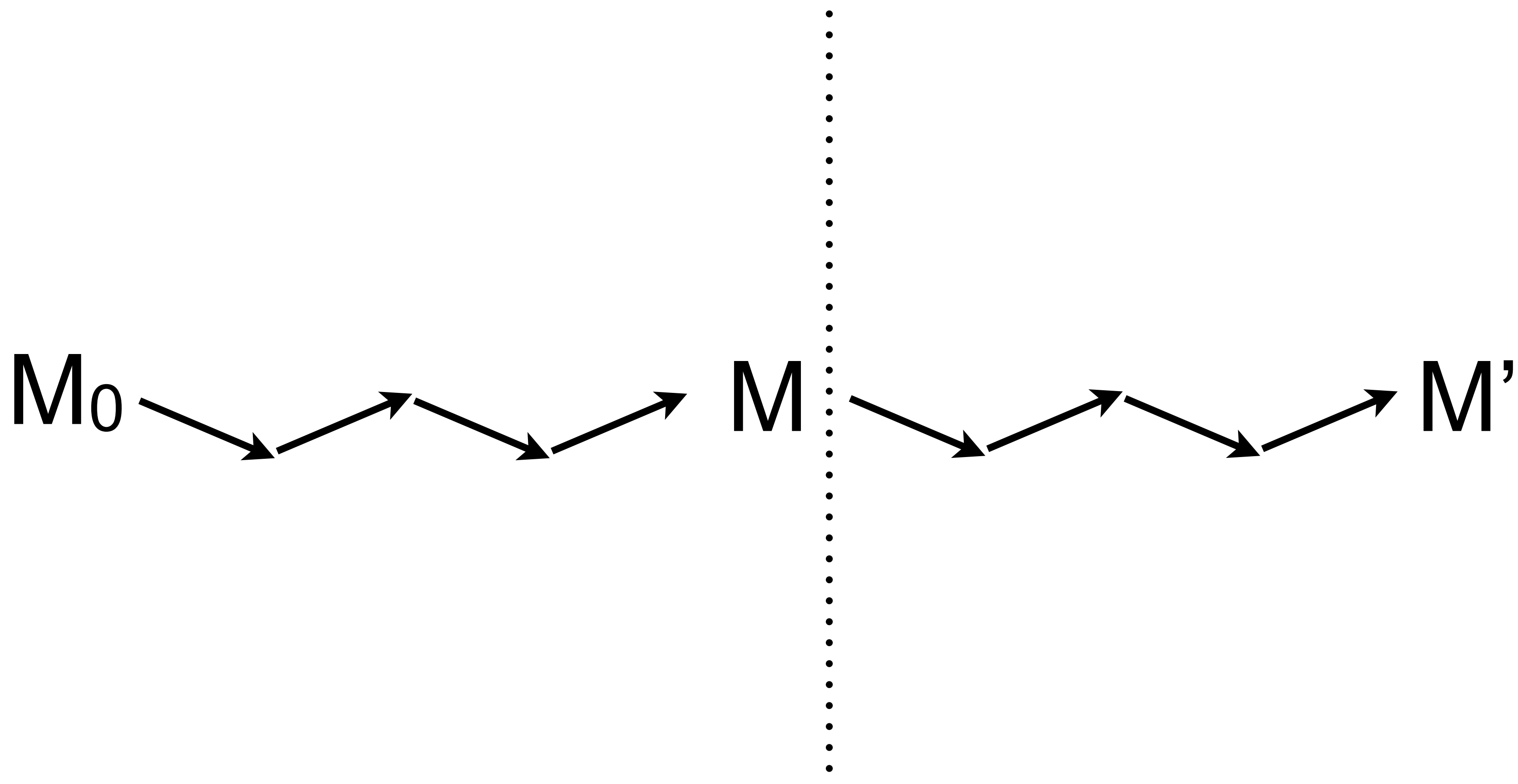
For any reachable marking M



Can we find a way to enable t?

Liveness illustrated

For any reachable marking M



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Liveness illustrated

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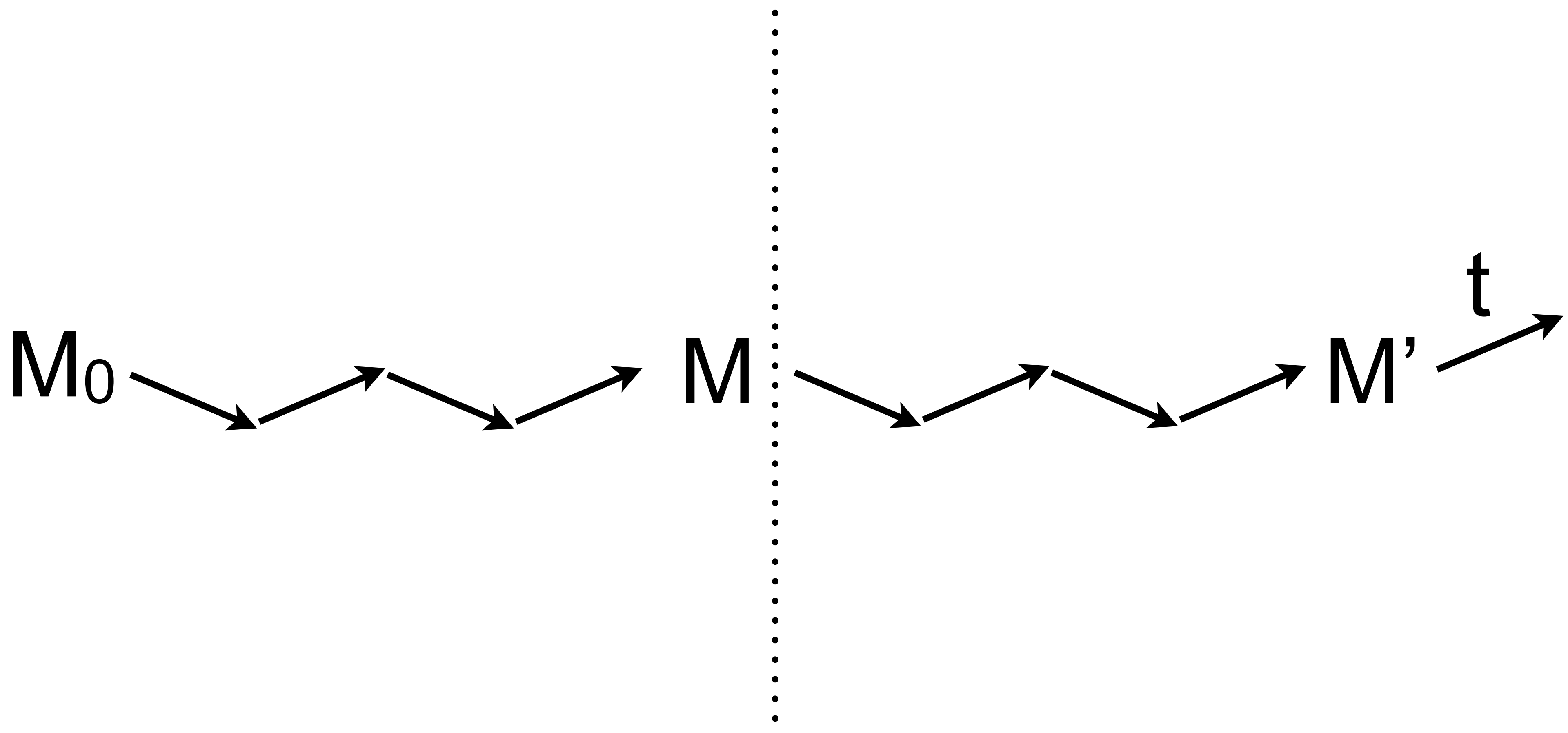


Can we find a way to enable t?

Liveness, formally

$$(P, T, F, M_0)$$

$$\forall t \in T, \quad \forall M \in [\, M_0 \,\rangle, \quad \exists M' \in [\, M \,\rangle, \quad M' \xrightarrow{t}$$

Digression

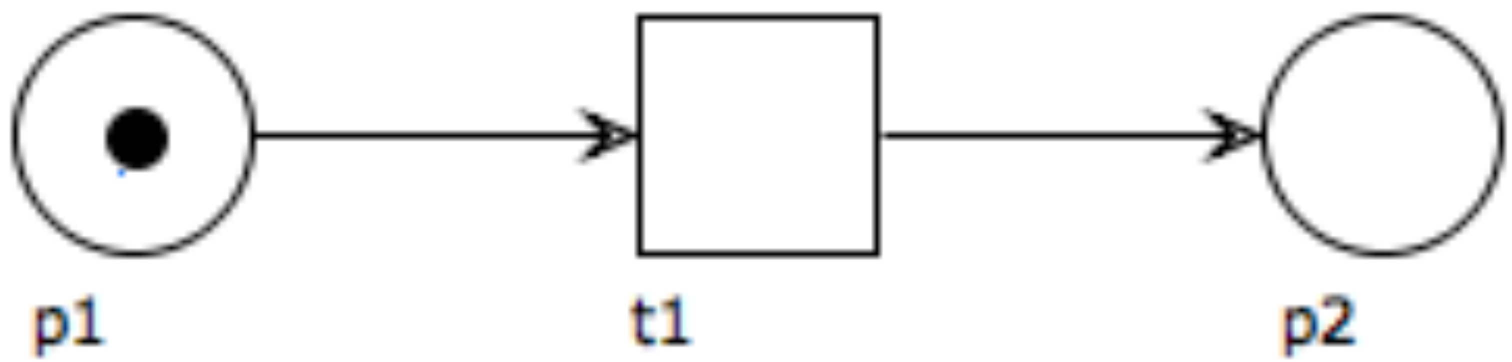
Order of quantifier is important:

quantification of the same kind can be switched

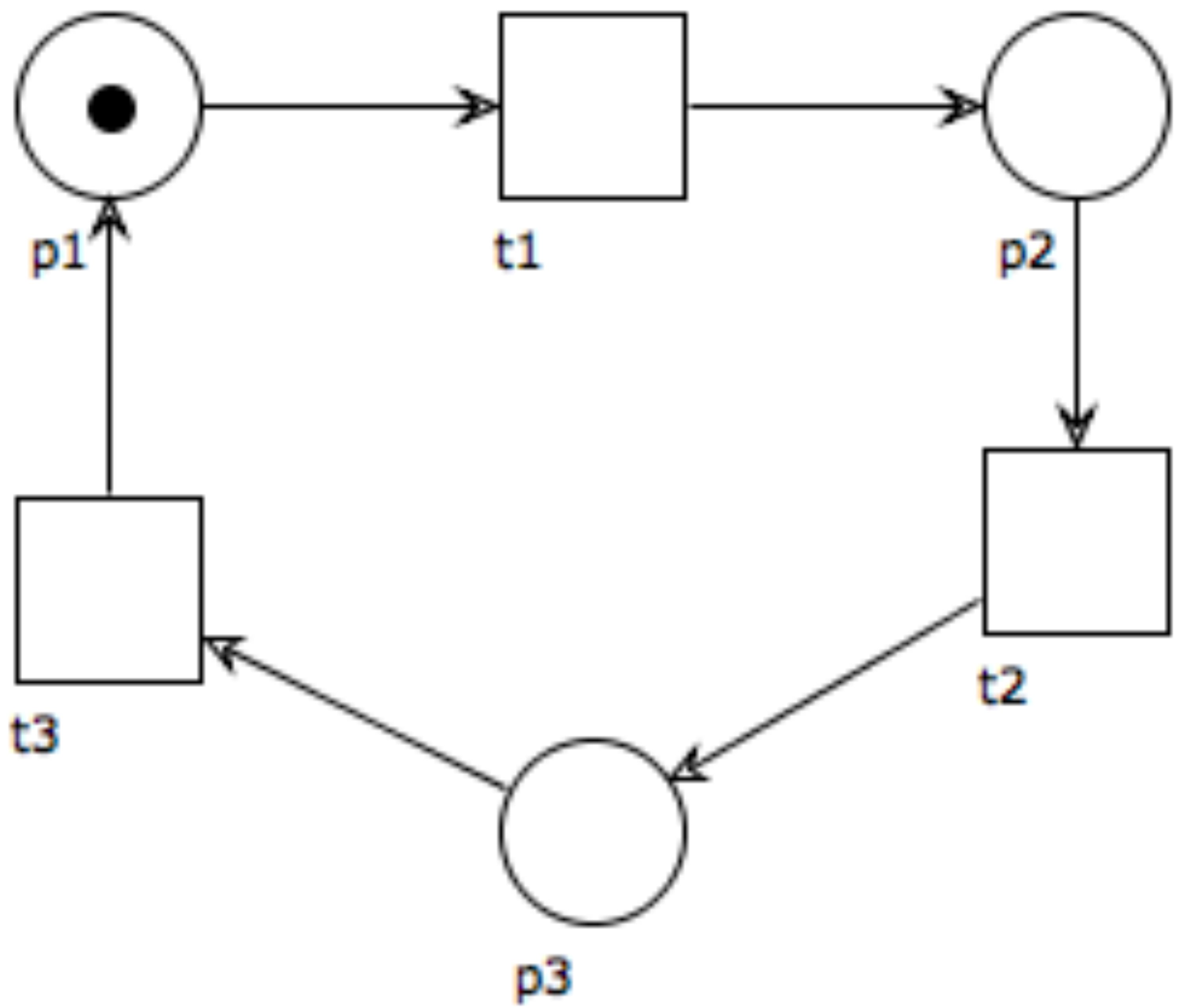
the order of universal and existential quantification is important

$$\forall n.\ \exists m.\ n < m \qquad \not\equiv \qquad \exists m.\ \forall n.\ n < m$$

Example: Non Live



Example: Live



Liveness: pay attention!

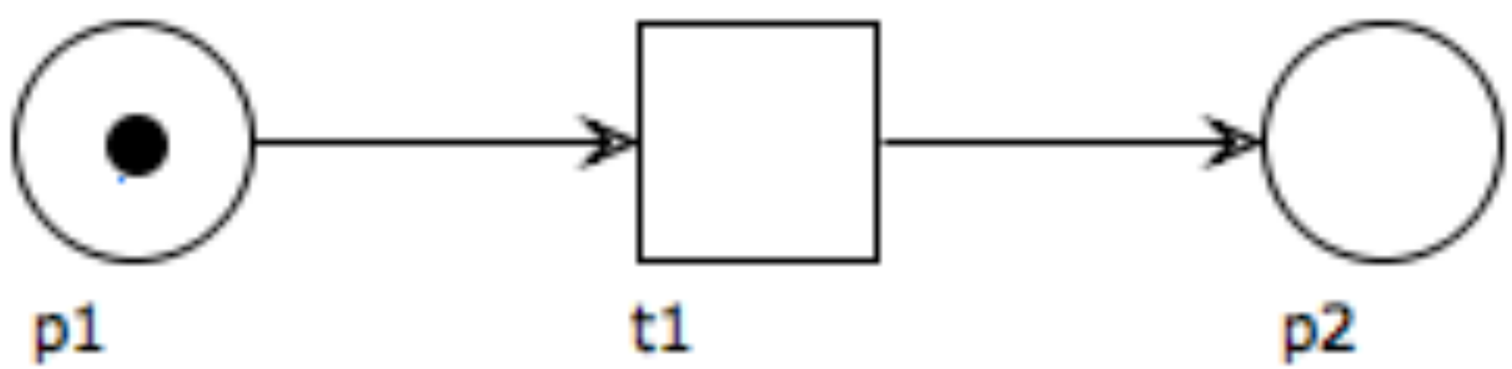
Liveness should not be confused with the following property:

starting from the initial marking M_0 it is possible to reach a marking M that enables t

$$\exists M \in [M_0\rangle .\ M \xrightarrow{t}$$

(this property just ensures that t is not "dead" at M_0)

Example: non live net (but t1 non dead!)

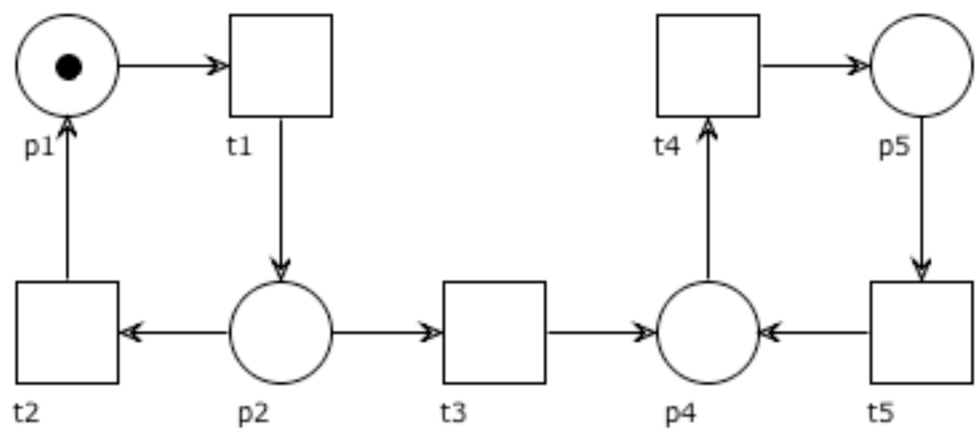


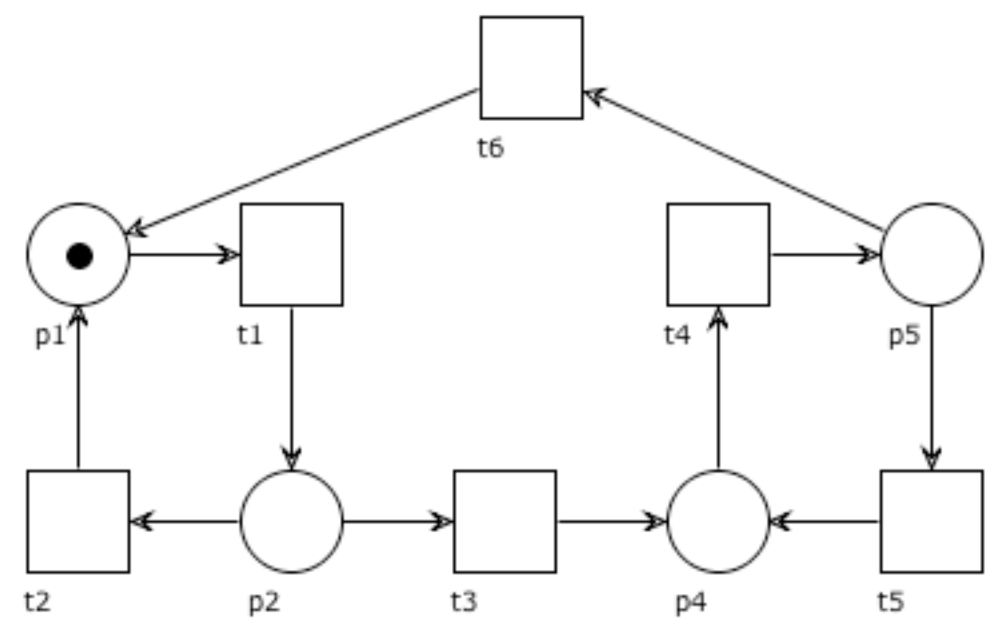
Dead transition

Given a marking M

A transition t is **dead** at M
if t will never be enabled in the future
(i.e., t is not enabled at any marking reachable from M)

Liveness: example





Which transitions are live?
Which are not?
Which are dead?
Is the net live?

Liveness on the occurrence graph

A transition t is live
iff
From any node of the occurrence graph we can reach a node with an outgoing arc labelled by t

A transition t is dead (at M_0)
iff
There is no t-labelled arc in the occurrence graph

Marked place

Given a marking M

We say that a place p is **marked** (at M)
if M(p) > 0
(i.e., there is a token in p in the marking M)

We say that p is **unmarked**
if M(p) = 0
(i.e., there is no token in p in the marking M)

Place-liveness, intuitively

A place p is **live** if
every time it becomes unmarked
there is still the possibility to be marked in the future
(or if it always stays marked)

A Petri net is **place-live** if all of its places are live

Live place

Definition: Let (P, T, F, M_0) be a net system.

A place $p \in P$ is **live** if $\forall M \in [\, M_0 \,\rangle .\, \exists M' \in [\, M \,\rangle .\, M'(p) > 0$

Place-liveness, formally

$$(P, T, F, M_0)$$

$$\forall p \in P. \quad \forall M \in [M_0\rangle. \quad \exists M' \in [M\rangle. \quad M'(p) > 0$$

Dead nodes, intuitively

Given a marking M

A transition t is **dead** at M
if t will never be enabled in the future
(i.e., t is not enabled in any marking reachable from M)

A place p is **dead** at M
if p will never be marked in the future
(i.e., p is unmarked in any marking reachable from M)

Dead nodes

Definition: Let (P, T, F) be a net

A transition $t \in T$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M' \not\xrightarrow{t}$

A place $p \in P$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M'(p) = 0$

Non-live vs Dead

If a transition is dead at some reachable marking M
then it is non-live

If a place is dead at some reachable marking M
then it is non-live

being non-live implies possibly becoming dead
(but not necessarily in the current marking)

Some obvious facts

If a system is not live, it must have a transition dead at some reachable marking

If a system is not place-live, it must have a place dead at some reachable marking

If a place / transition is dead at M, then it remains dead at any marking reachable from M
(the set of dead nodes can only increase during a run)

Every transition in the pre- or post-set of a dead place is also dead

True or false?

Every place in the pre- or post-set of a dead transition is also dead

Liveness implies place-liveness

Proposition: Live systems are also place-live

Liveness implies place-liveness

Proposition: Live systems are also place-live

Take any p and any $t \in \bullet p \cup p \bullet$

Let $M \in [\, M_0 \,\rangle$

By liveness: there is $M', M'' \in [\, M \,\rangle$ s.t. $M' \xrightarrow{t} M''$

Then $M'(p) > 0$ or $M''(p) > 0$

Place liveness on the occurrence graph

A place p is live
iff
From any node of the occurrence graph we can reach a node with a token in p

A place p is dead (at M_0)
iff
All the nodes of the occurrence graph have no token in p

Exercise

Draw a net that
is place-live but not live
(if you can)

Deadlock-freedom

A Petri net is **deadlock free**, if every reachable marking enables some transition

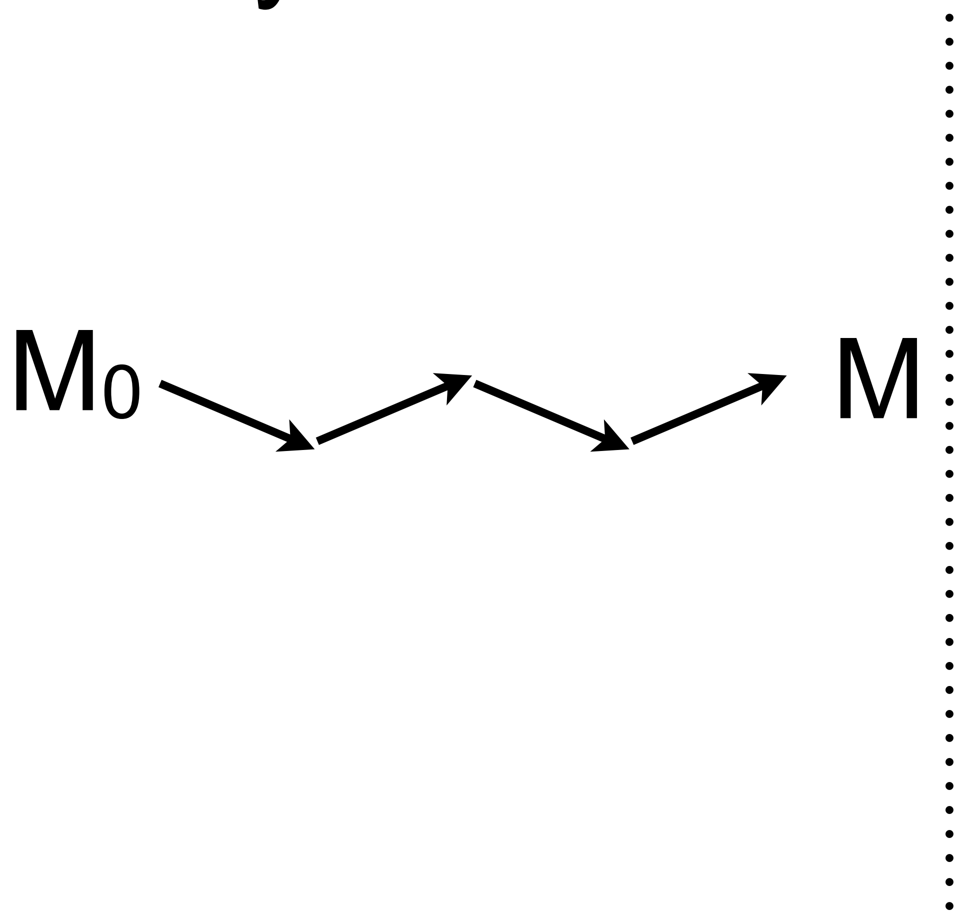
In other words, we are guaranteed that at any point in time of the computation, some transition can be fired

Deadlock-freedom illustrated

M_0

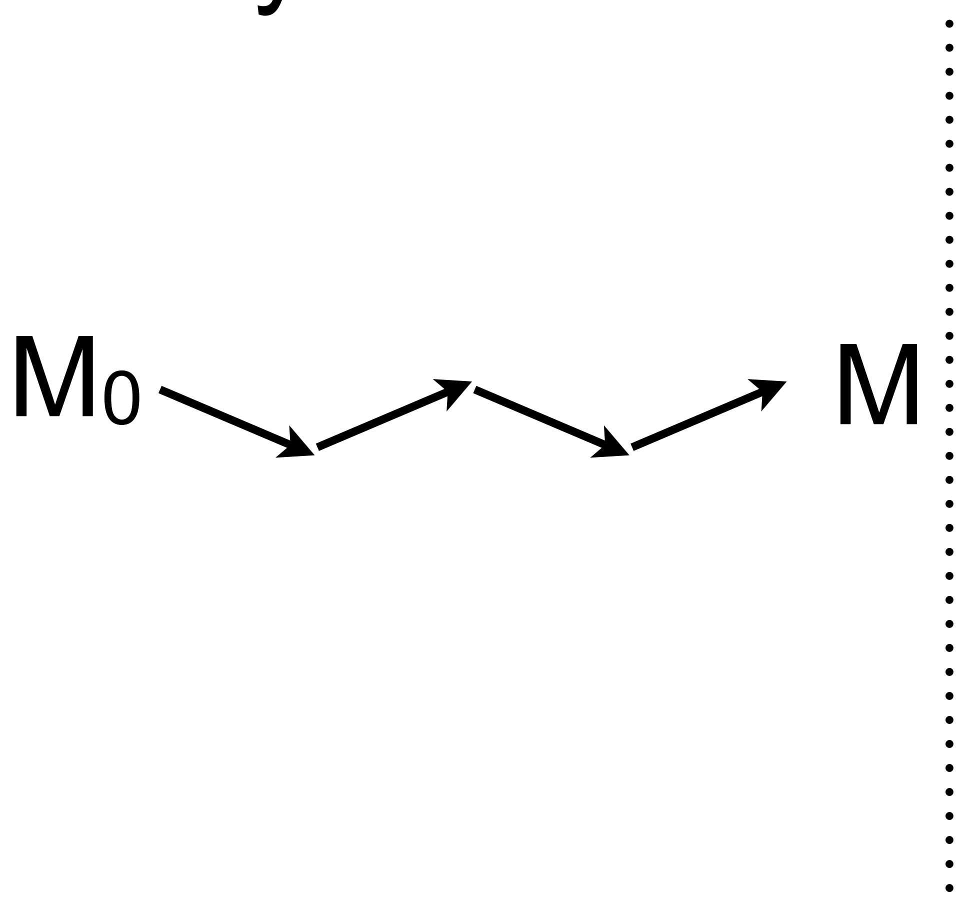
Deadlock-freedom illustrated

For any reachable marking M



Deadlock-freedom illustrated

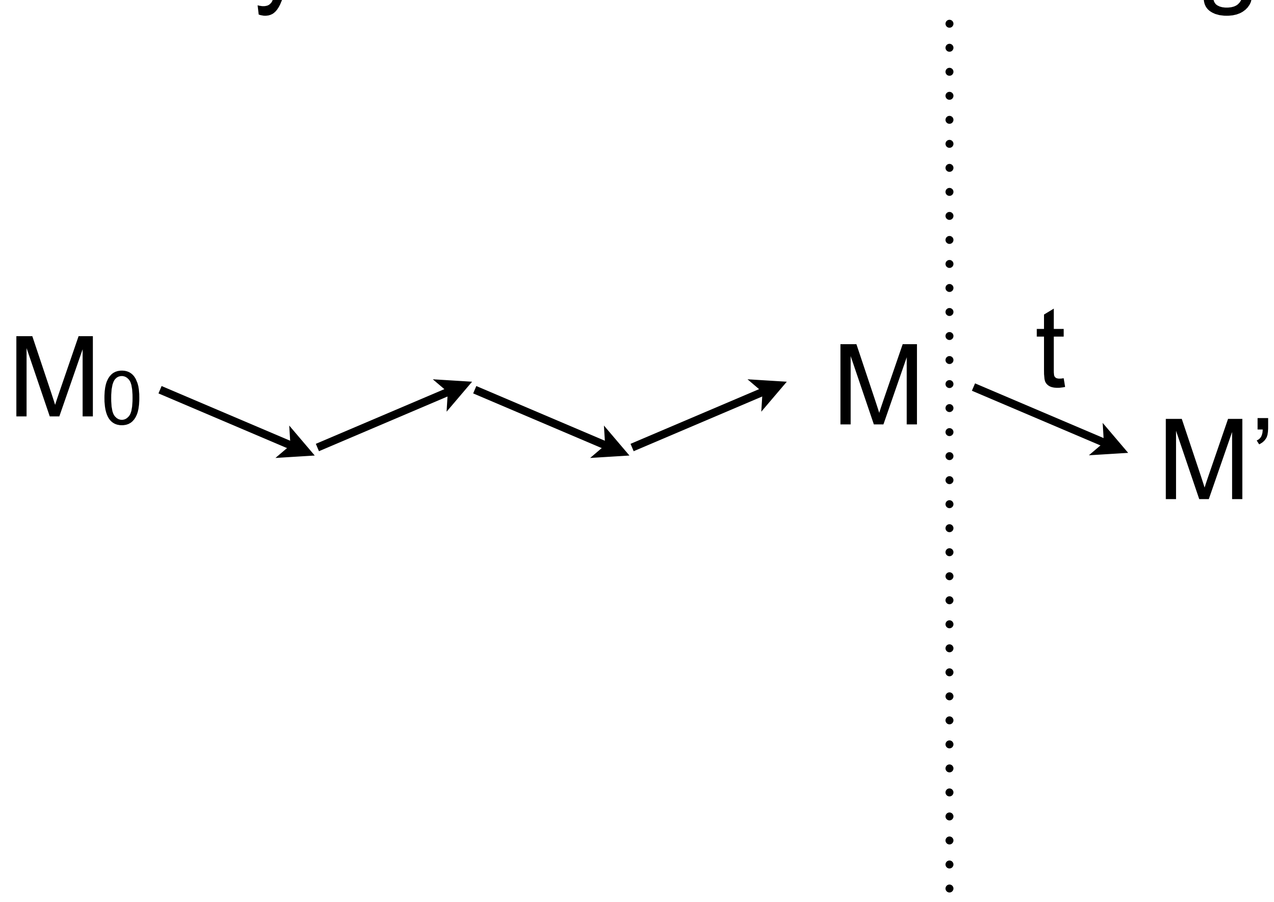
For any reachable marking M



Can we fire some transition?

Deadlock-freedom illustrated

For any reachable marking M



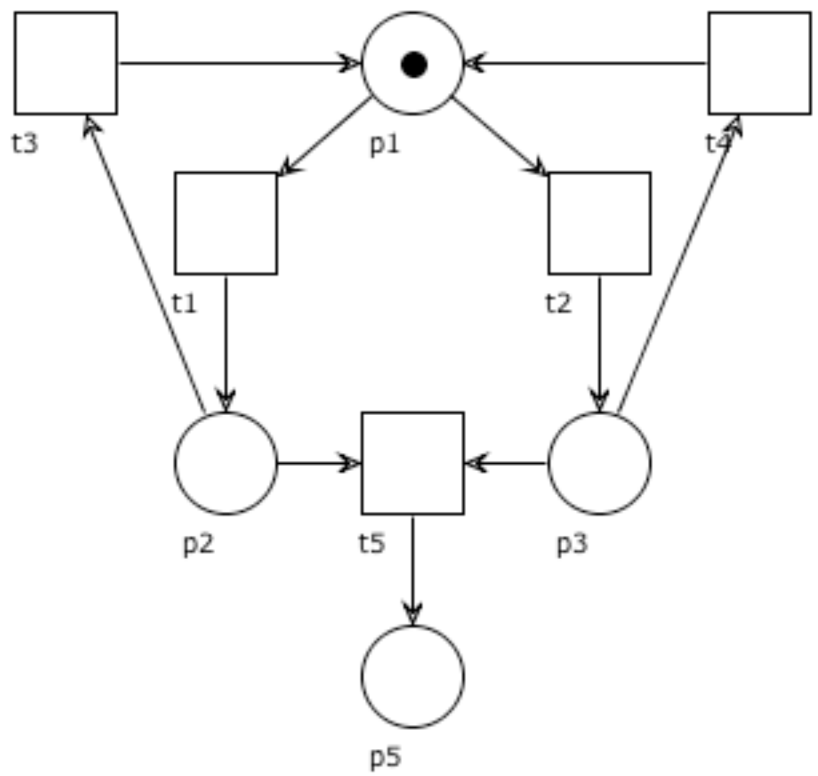
Can we fire some transition?

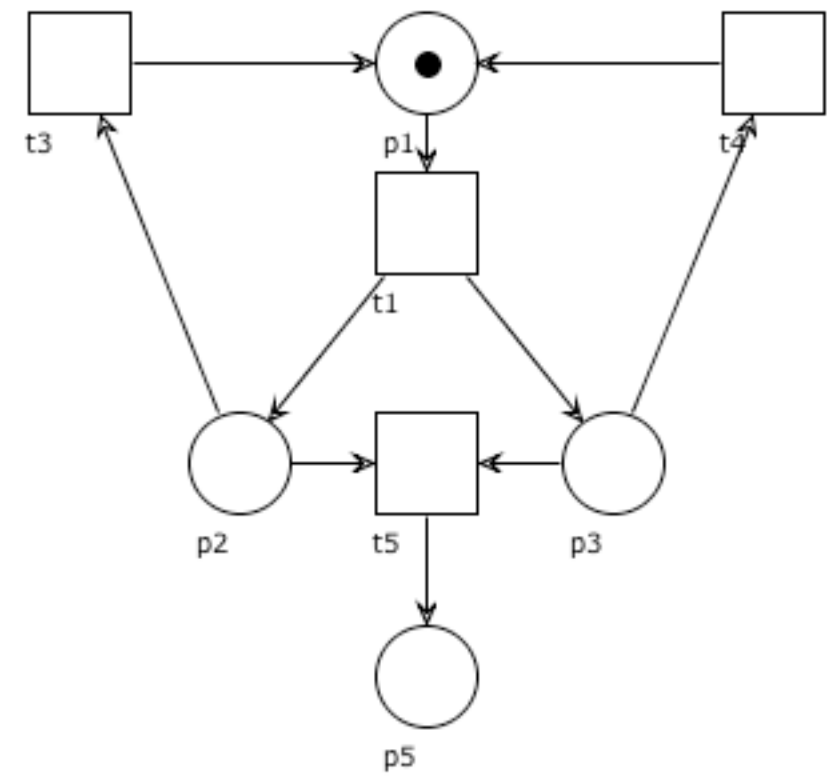
Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [\, M_0 \,\rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

Deadlock-freedom: example





Is the net deadlock-free?

Deadlock freedom on the occurrence graph

A net is deadlock free
iff
Every node of the occurrence graph has an outgoing arc

Question time

Does liveness imply deadlock-freedom?
(Can you exhibit a live Petri net that is not deadlock-free?)

Does deadlock-freedom imply liveness?
(Can you exhibit a deadlock-free net that is not live?)

Liveness implies deadlock freedom

Lemma If (P, T, F, M_0) is live, then it is deadlock-free

Liveness implies deadlock freedom

Lemma If (P, T, F, M_0) is live, then it is deadlock-free

By contradiction, let $M \in [\, M_0 \,\rangle$, with $M \not\rightarrow$

Let $t \in T$ (T cannot be empty).
By liveness, $\exists M' \in [\, M \,\rangle$ with $M' \xrightarrow{t}$.

Since M is dead, $[\, M \,\rangle = \{\, M \,\}$.
Therefore $M = M' \xrightarrow{t}$, which is absurd.

Digression

Contraposition

$$P \Rightarrow Q \qquad \equiv \qquad (\neg Q) \Rightarrow \neg P$$

Exercises

Prove each of the following properties
or give some counterexamples

If a system is not place-live, then it is not live

If a system is not live, then it is not place-live

If a system is place-live, then it is deadlock-free

If a system is deadlock-free, then it is place-live

k-Boundedness

Let k be a natural number

A place p is **k-bounded** if no reachable marking has more than k tokens in place p

A net is **k-bounded** if all of its places are k-bounded

In other words, if a net is k-bounded, then k is a capacity constraint that can be imposed over places without any risk of causing “overflow”

Safe nets

A place p is **safe** if it is 1-bounded

A net is **safe** if all of its places are safe

In other words, if the net is safe, then we know that, in any reachable marking, each place contains one token at most

Boundedness

A place p is **bounded** if it is k-bounded for some natural number k

A net is **bounded** if all of its places are bounded

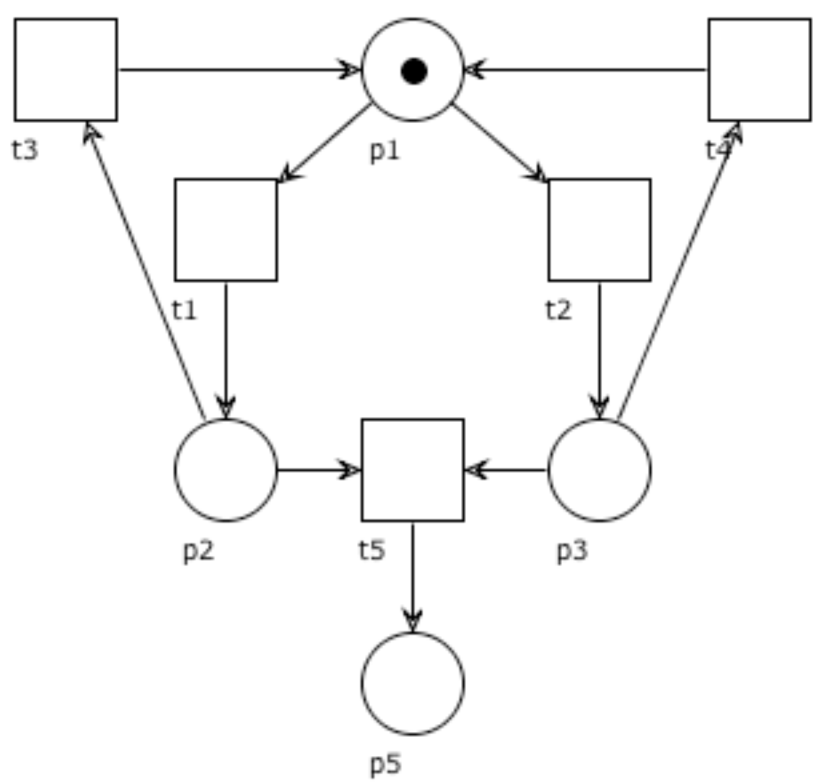
A net is **unbounded** if it is not bounded (i.e., there is at least one place in which any number of tokens can appear)

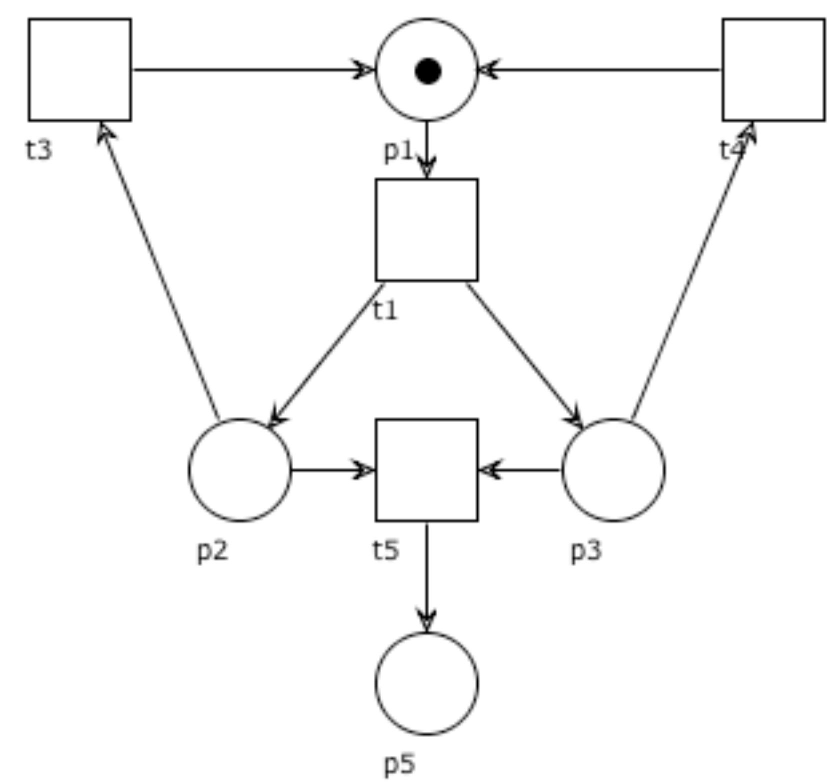
Boundedness, formally

$$(P, T, F, M_0)$$

$$\exists k \in \mathbb{N}, \quad \forall M \in [\, M_0 \,\rangle, \quad \forall p \in P, \quad M(p) \leq k$$

Boundedness: example





Which places are bounded?
Is the net bounded?
Which places are safe?
Is the net safe?

A puzzle about reachability

Theorem: If a system is...
then its reachability graph is finite

Theorem: A system is...
iff its reachability graph is finite

(fill the dots and the proofs)

A puzzle about reachability

Theorem: If a system is **bounded** then its reachability graph is finite

Theorem: A system is **bounded** iff its reachability graph is finite

(fill the dots and the proofs)

Boundedness implies finiteness

Theorem: If a system is **bounded** then its reachability graph is finite

Proof: if the system is bounded there exists k such that each place contains at most k tokens.

If there are n places it means that there are at most $(k+1)^n$ reachable markings.

Hence the occurrence graph has a finite number of nodes

Finiteness implies boundedness

Theorem: A system is **bounded**
if its reachability graph is finite

Proof: for each node M we take k_M be the maximum number of tokens in the same place.

Then we let k be the largest among all k_M
$k = \max \{k_M \mid M \text{ is a node of the graph}\}$
(k exists because the reachability graph is finite)

Clearly the system is k-bounded and thus bounded

Cyclicity (aka Reversibility)

A marking M is a **home marking** if it can be reached from every reachable marking

A net is **cyclic** (or **reversible**) if its initial marking is a home marking

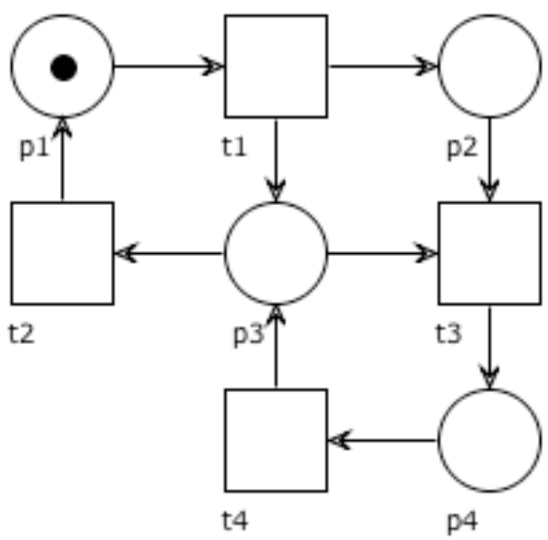
Orthogonal properties

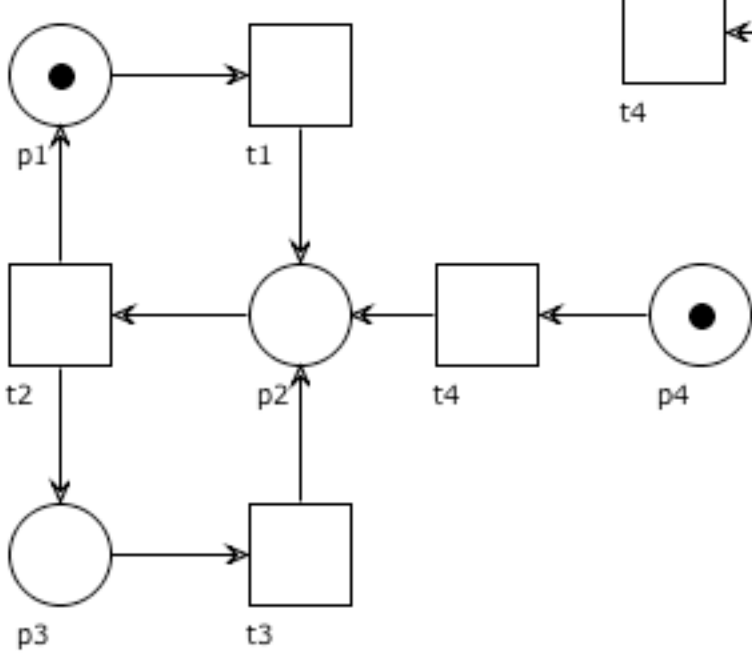
Liveness, boundedness and cyclicity are independent of each other

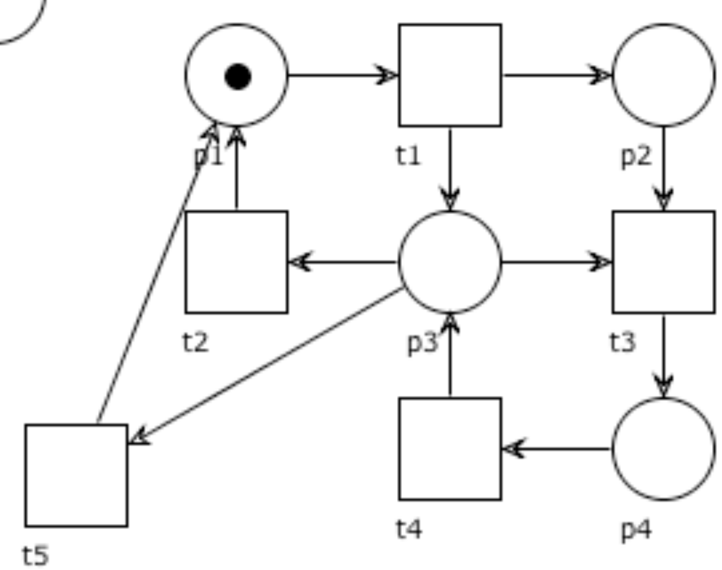
In other words, you can find nets that satisfy any arbitrary combination of the above three properties (and not the others)

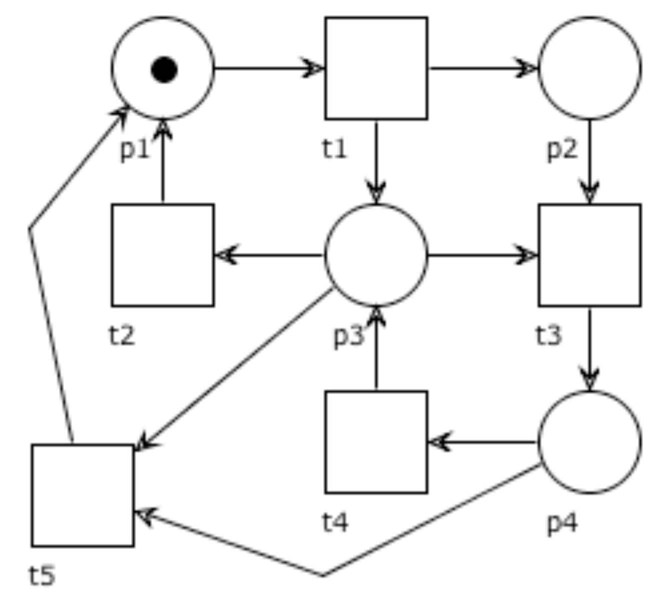
Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic



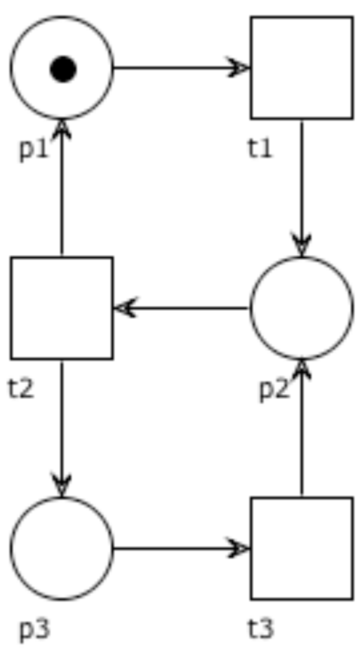


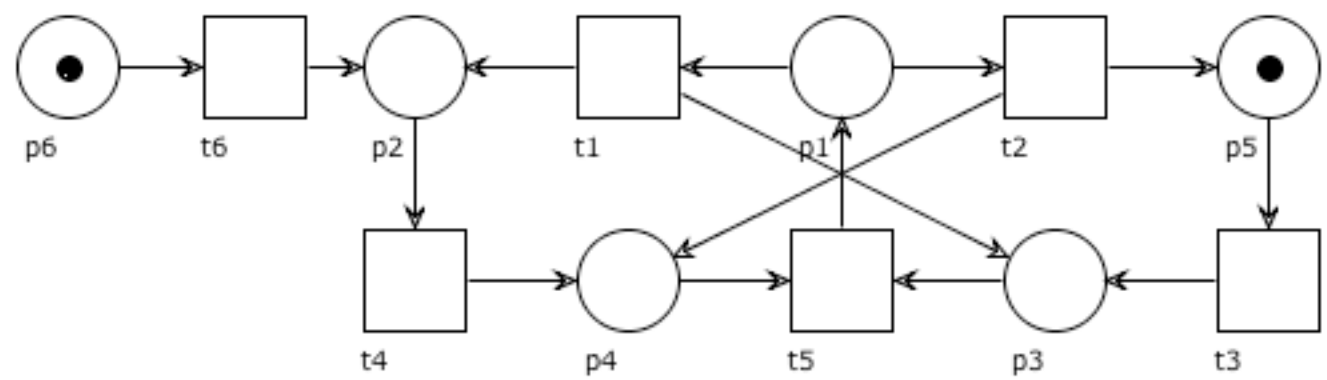


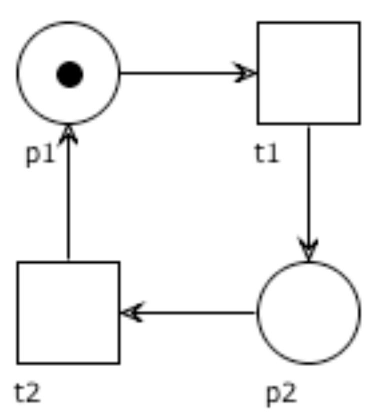


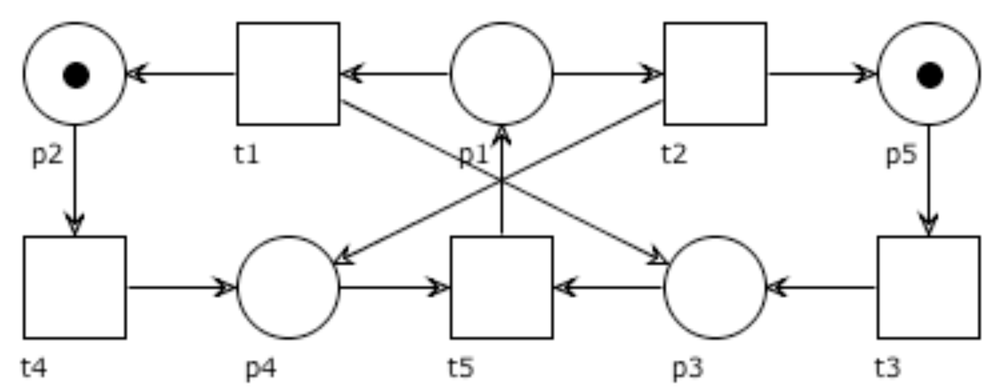
Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic









Petri nets:
structural properties

Structural properties

All the properties we have seen so far are
behavioural (or **dynamic**)
(i.e. they depend on the initial marking and firing rules)

It is sometimes interesting to connect them to
structural properties
(i.e. the shape of the graph representing the net)

This way we can give **structural characterization** of
behavioural properties for a class of nets
(computationally less expensive to check)

A matter of terminology

To better reflect the above distinction, it is frequent:

to use the term **net system** for denoting a Petri net **with** a given initial marking
(we study behavioural properties of systems)

to use the term **net** for denoting a Petri net **without** specifying any initial marking
(we study structural properties of nets)

Paths and circuits

A **path** of a net (P, T, F) is a non-empty sequence $x_1 x_2 ... x_k$ such that

$$(x_i, x_{i+1}) \in F \quad \text{for every } 1 \leq i < k$$

(and we say that it lead**s** from x_1 to x_k)

A path from x to y is called a **circuit** if:
no element occurs more than once in it and $(y, x) \in F$
(since for any x we have $(x, x) \notin F$, hence a circuit involves at least two nodes)

Connectedness

A net (P,T,F) is **weakly connected** iff it does not fall into (two or more) unconnected parts (i.e. no two subnets (P_1,T_1,F_1) and (P_2,T_2,F_2) with disjoint and non-empty sets of elements can be found that partition (P,T,F))

A weakly connected net is **strongly connected** iff for every arc (x,y) there is a path from y to x

Connectedness, formally

A net (P, T, F) is **weakly connected** if every two nodes x, y satisfy

$$(x, y) \in (F \cup F^{-1})^*$$

(here the * denotes
the reflexive and transitive
closure of a binary relation)

(i.e. if there is an <u>undirected</u> path from x to y)

It is **strongly connected** if $(x, y) \in F^*$

(here the * denotes
the reflexive and transitive
closure of a binary relation)

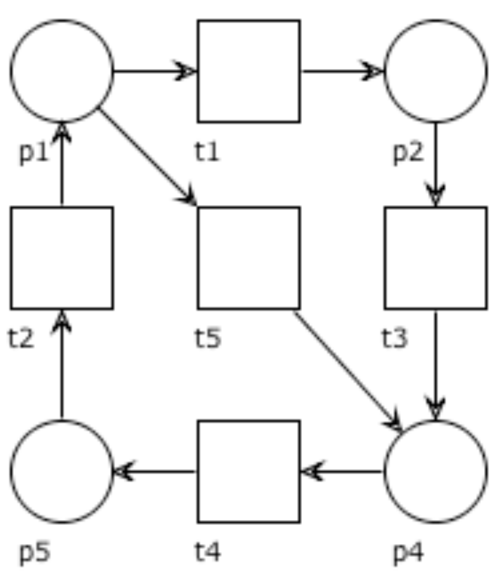
A note

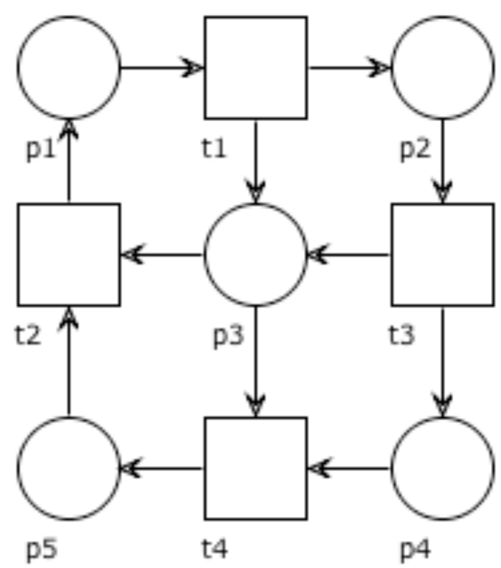
In the following we will consider (implicitly) weakly connected nets only

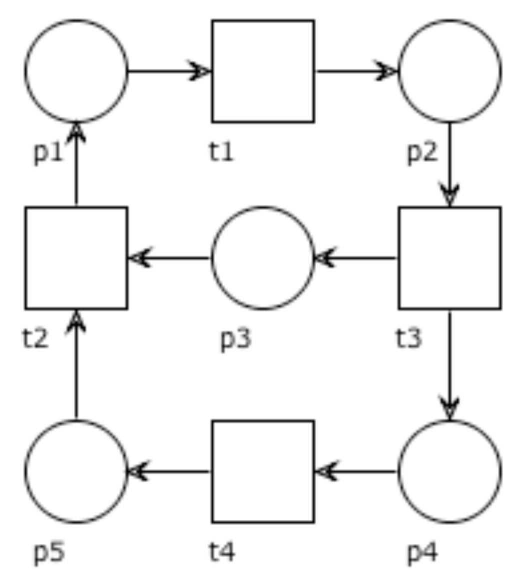
(if they are not, then we can study each of their subsystems separately)

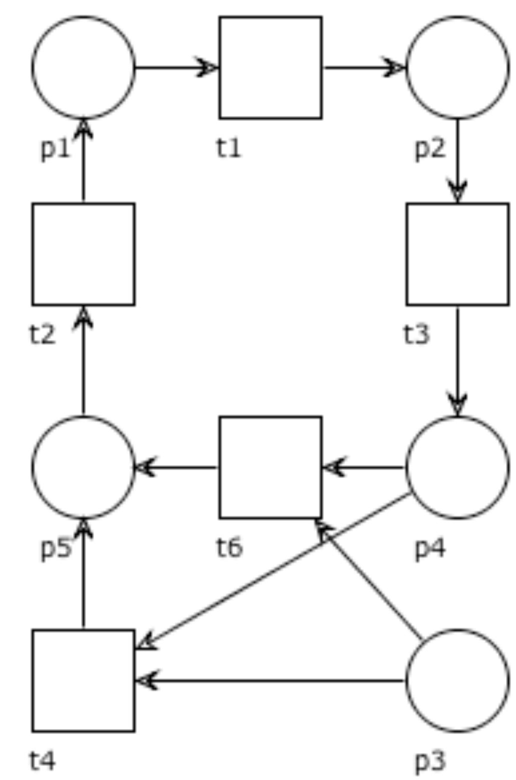
Question time

Is the net strongly connected?









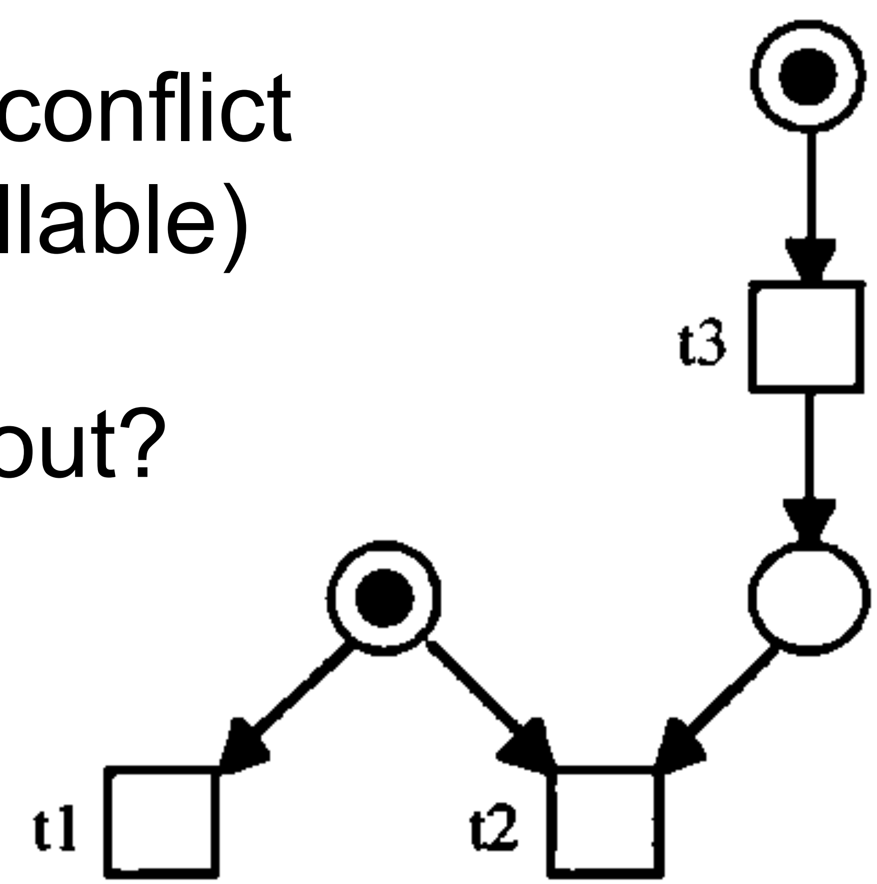
Interference of conflicts and synch

Typical situation:

initially t1 and t2 are not in conflict

but when t3 fires they are in conflict
(the firing of t3 is not controllable)

How to rule this situation out?



S-systems / S-nets

A Petri net is called **S-system** if every transition has one input place and one output place
(S comes from *Stellen*, the German word for place)

This way any synchronization is ruled out

The theory of S-systems is very simple

T-systems / T-nets

A Petri net is called **T-system** if every place has one input transition and one output transition

This way all choices/conflicts are ruled out

T-systems are concurrent but essentially deterministic

T-systems have been studied extensively since the early Seventies

Free-choice nets

The aim is to avoid that a choice between transitions
is influenced by the rest of the system

Easiest way:
keep places with more than one output transition apart
from transitions with more than one input place

In other words, if (p,t) is an arc, then it means that
t is the only output transition of p (no conflict)
OR
p is the only input place of t (no synch)

Free-choice systems / nets

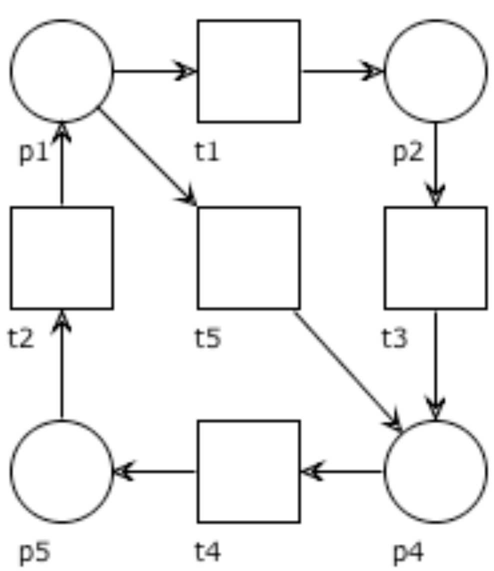
But we can study a slightly more general class of nets by requiring a weaker constraint

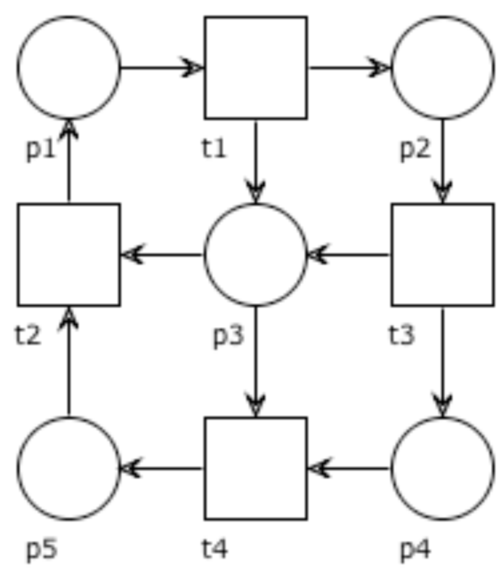
A Petri net is **free-choice** if
for any pair of transitions
their pre-sets are either disjoint or equal

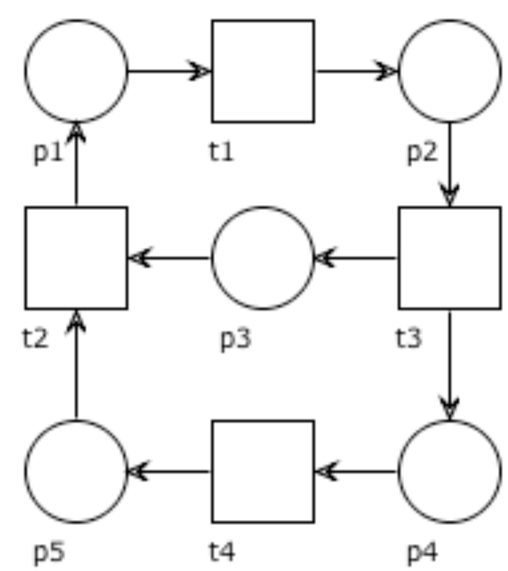
or, equivalently, if
for any pair of places
their post-sets are either disjoint or equal

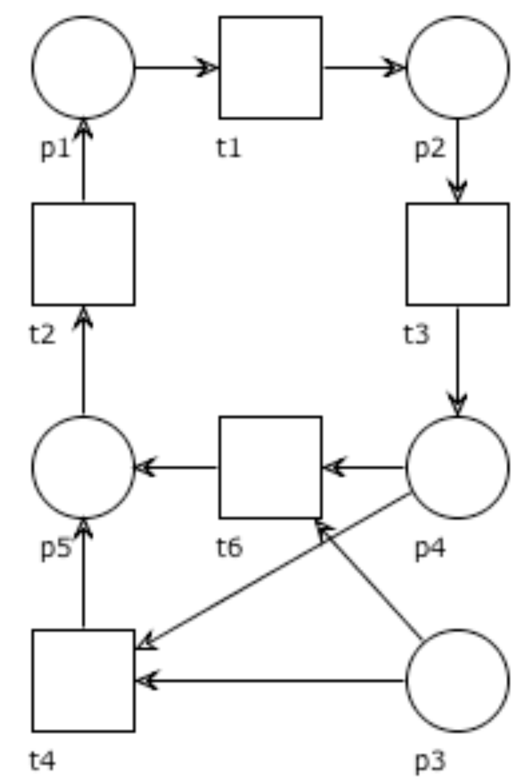
Question time

Is the net an S-net, a T-net, free-choice?









Exercises

Prove that every S-net is free-choice

Prove that every T-net is free-choice

Show a free-choice net that is neither an S-net nor a T-net