Methods for the specification and verification of business processes

MPB (6 cfu, 295AA)

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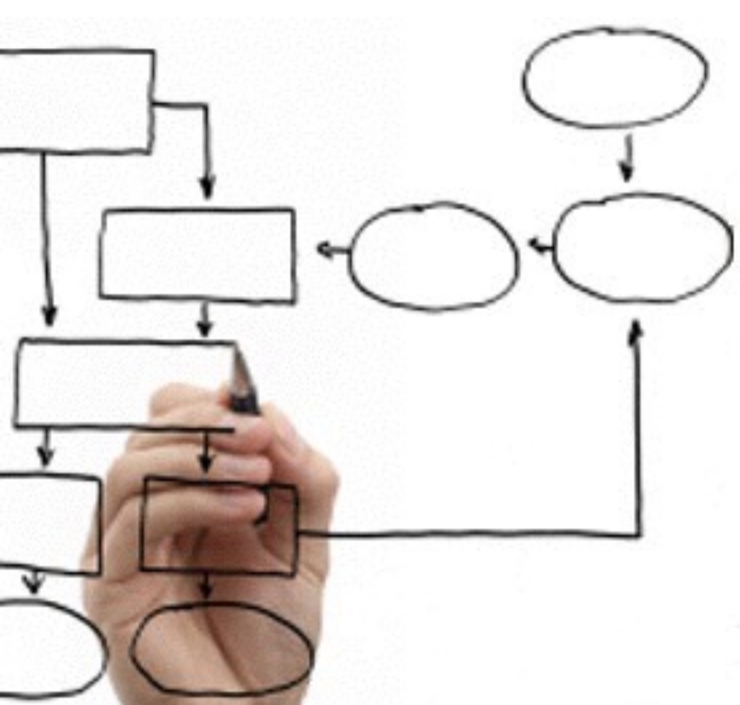
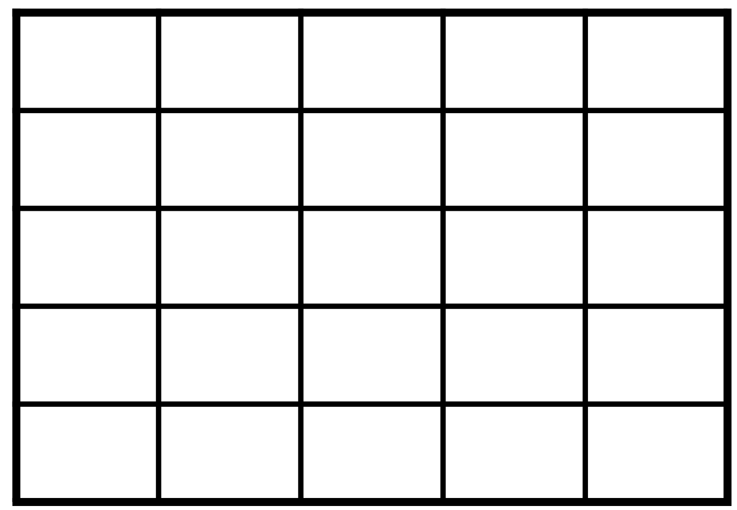
10 - Incidence matrices



Object



We give a matrix-based representation of Petri nets and their computations

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

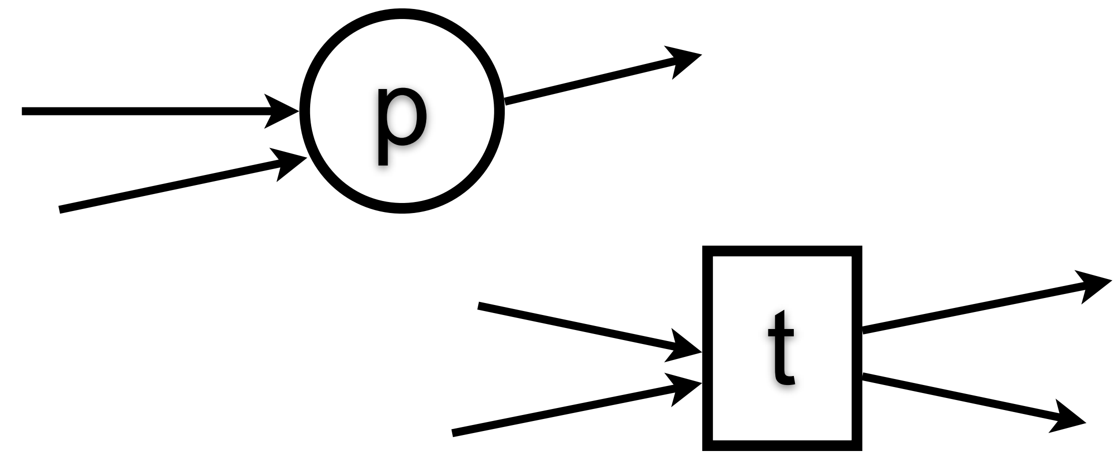
Key point

The change of the numbers of tokens on a place p caused by the firing of the transition t does not depend on the current marking

It is entirely determined by the net (i.e., by the flow relation)

Let us have a look at the relative changes for every place and transition...

How p relates to t

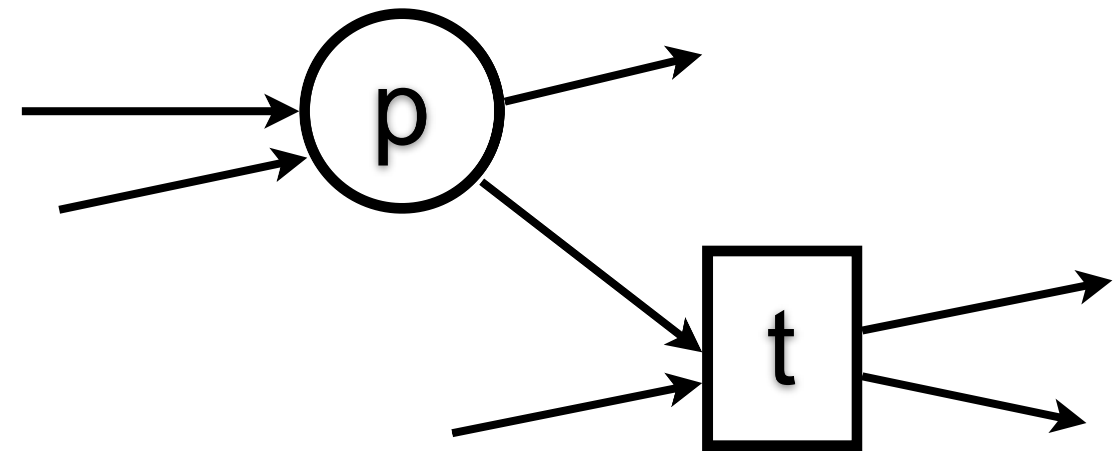


$(p, t) \notin F$ and $(t, p) \notin F$

Place p and transition t are completely unrelated:

- p has no influence on the enabling of t
- firing t does not change the number of tokens in p

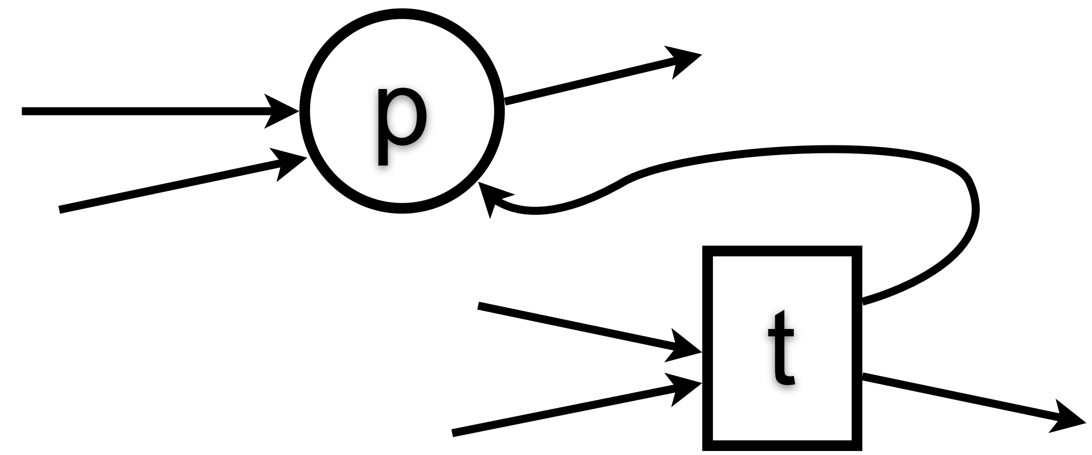
How p relates to t



$\underline{(p,t) \in F}$ and $(t,p) \notin F$

- one token in p is needed to enable t
- firing t reduces by one the number of tokens in p

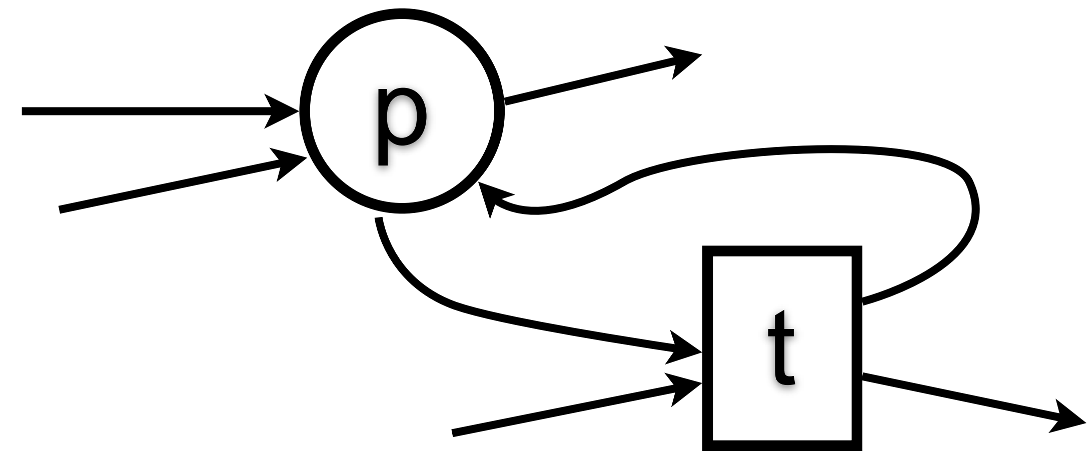
How p relates to t



$(p,t) \notin F$ and $\underline{(t,p) \in F}$

- firing t increases by one the number of tokens in p

How p relates to t



$\underline{(p, t) \in F}$ and $\underline{(t, p) \in F}$

- one token in p is needed to enable t
- firing t does not change the number of tokens in p

Incidence matrix

Let $N = (P, T, F)$ be a net.

Its **incidence matrix** $\mathbf{N} : (P \times T) \to \{-1, 0, 1\}$ is defined as:

$$\mathbf{N}(p,t) = \begin{cases} -1 & \text{if } (p,t) \in F \ \wedge \ (t,p) \notin F \\ +1 & \text{if } (p,t) \notin F \ \wedge \ (t,p) \in F \\ 0 & \text{otherwise} \\ & (\ (p,t) \notin F \ \wedge \ (t,p) \notin F \ \text{ or } \ (p,t) \in F \ \wedge \ (t,p) \in F\ ) \end{cases}$$

Matrix view

m columns, one for each transition

n rows, one for each place

	t_1	t_2	t_3					...					t_m
p_1		+1	-1										-1
p_2	-1		+1										
p_3		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
p_n		-1	+1										-1

Column vector $\mathbf{t_j}$

$\mathbf{t_j} : P \rightarrow \{-1, 0, 1\}$ such that $\mathbf{t_j}(p) = \mathbf{N}(p, t_j)$

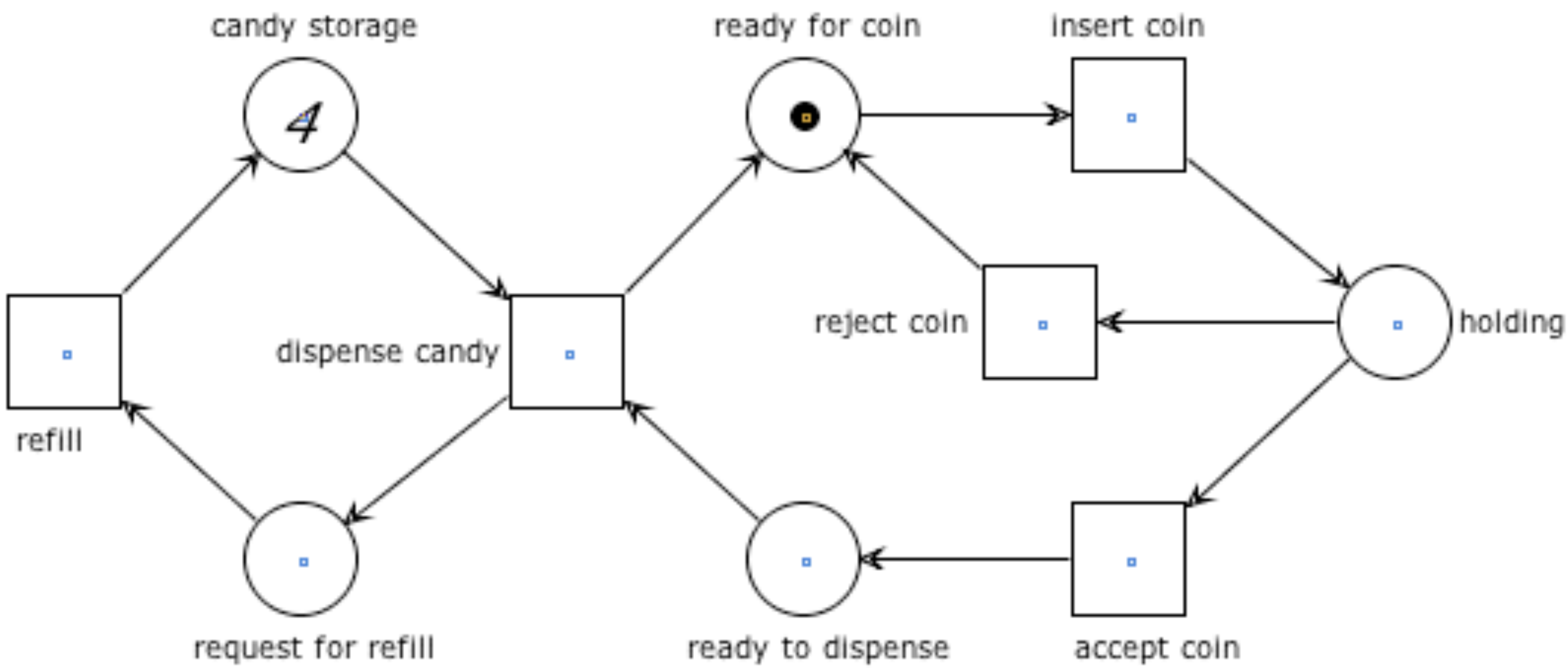
	t_1	t_2	t_3					...					t_m
p_1		+1	-1										-1
p_2	-1		+1										
p_3		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
p_n		-1	+1										-1

Row vector p_i

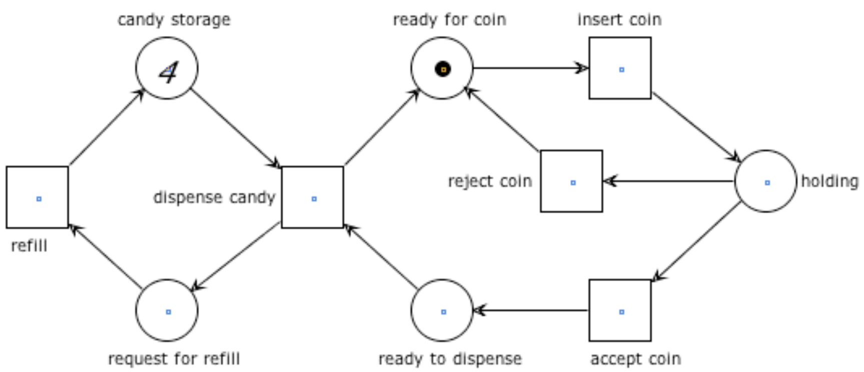
$$\mathbf{p_i} : T \rightarrow \{-1, 0, 1\}$$
such that
$$\mathbf{p_i}(t) = \mathbf{N}(p_i, t)$$

	t_1	t_2	t_3					…				t_m
p_1		+1	-1									-1
p_2	-1		+1									
p_3		+1										
			+1									
…								…				+1
	+1											
			-1									
												+1
p_n		-1	+1									-1

Example: vending machine



Example: vending machine



	refill t_1	dispense candy t_2	insert coin t_3	accept coin t_4	reject coin t_5
candy storage p_1	**1**	**-1**	**0**	**0**	**0**
request for refill p_2	**-1**	**1**	**0**	**0**	**0**
ready for coin p_3	**0**	**1**	**-1**	**0**	**1**
holding p_4	**0**	**0**	**1**	**-1**	**-1**
ready to dispense p_5	**0**	**-1**	**0**	**1**	**0**

Vectors: notation

Let $E = \{e_1, e_2, ..., e_n\}$ be a finite set of elements.

Any mapping $v : E \rightarrow \mathbb{Q}$ (or to $\mathbb{N}$, $\mathbb{Z}$,...) can be regarded as a vector:

$$\mathbf{v} = [\, v(e_1),\, v(e_2),\, ...,\, v(e_n)\,]$$

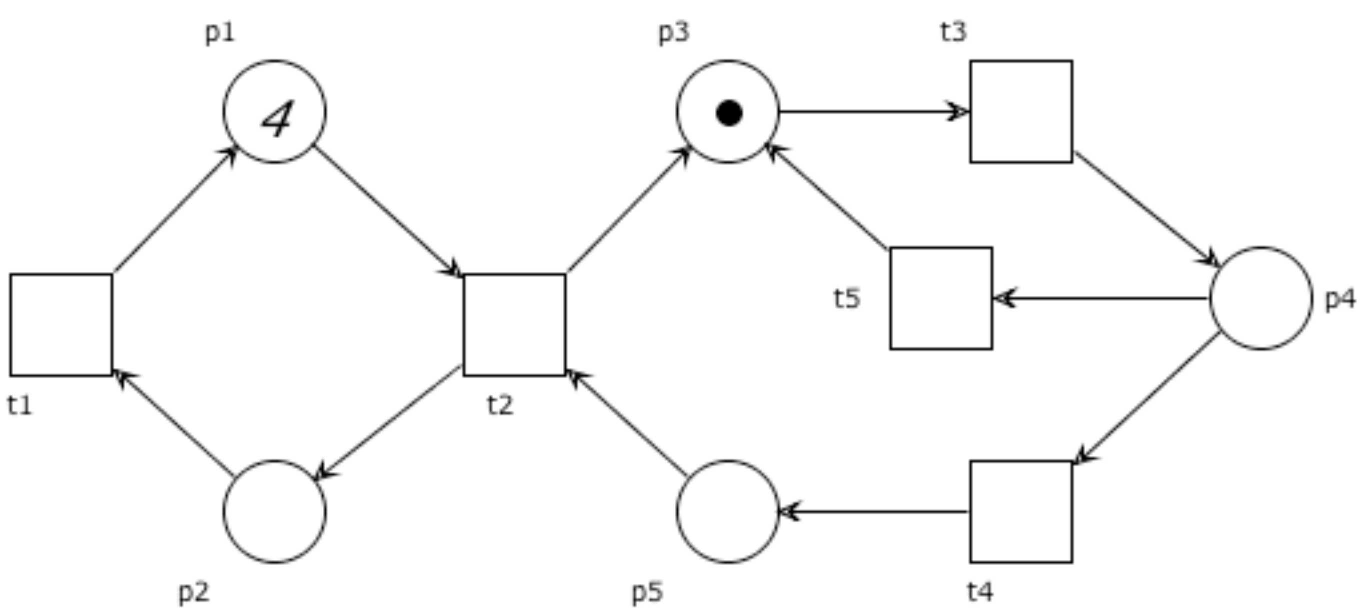
We **do not** use different symbols for row and columns vectors:

$$\mathbf{v} = \begin{bmatrix} v(e_1) \\ v(e_2) \\ \vdots \\ v(e_n) \end{bmatrix}$$

Marking as a vector

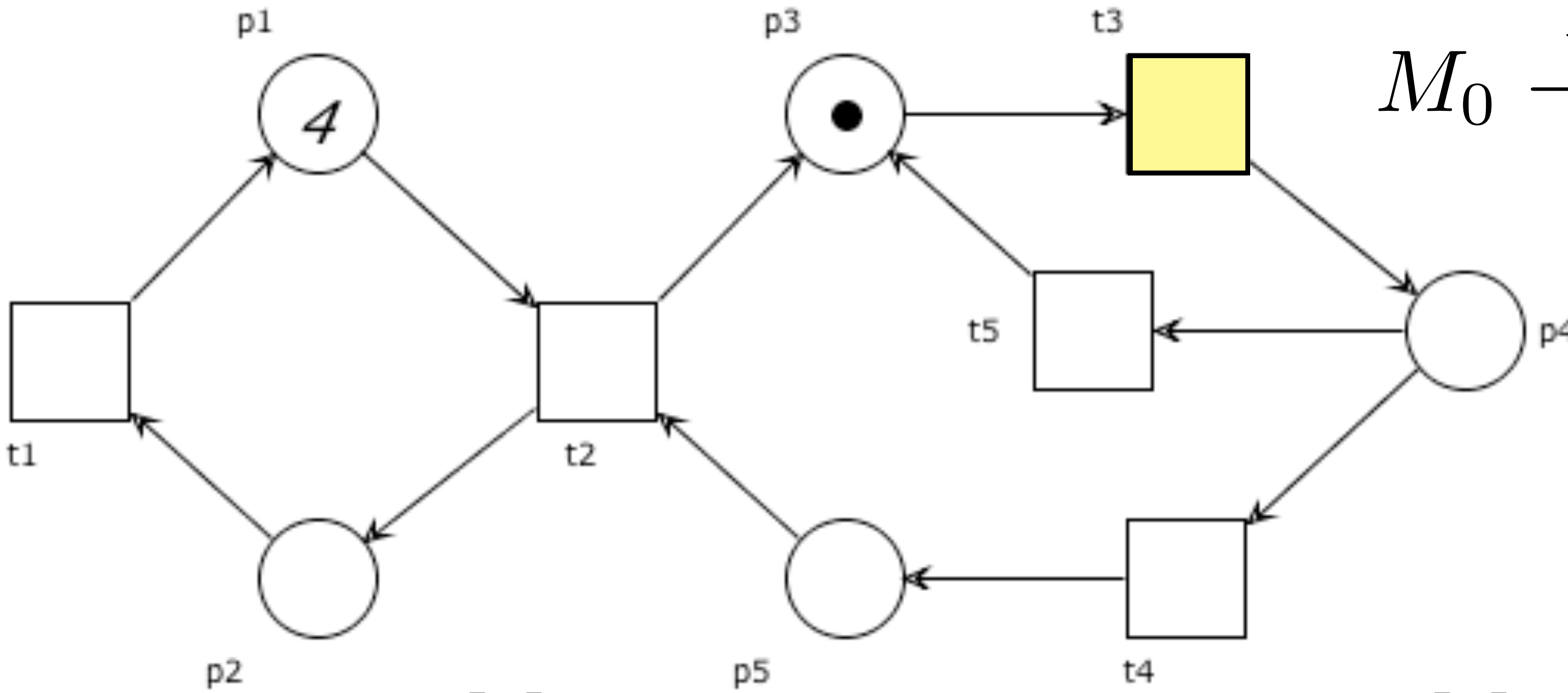
Any marking $M : P \to \mathbb{N}$ corresponds to a vector:

$$M = [\ M(p_1) \quad M(p_2) \quad \ldots \quad M(p_n)\ ]$$



$M_0 = [\ 4 \ \ 0 \ \ 1 \ \ 0 \ \ 0\ ]$

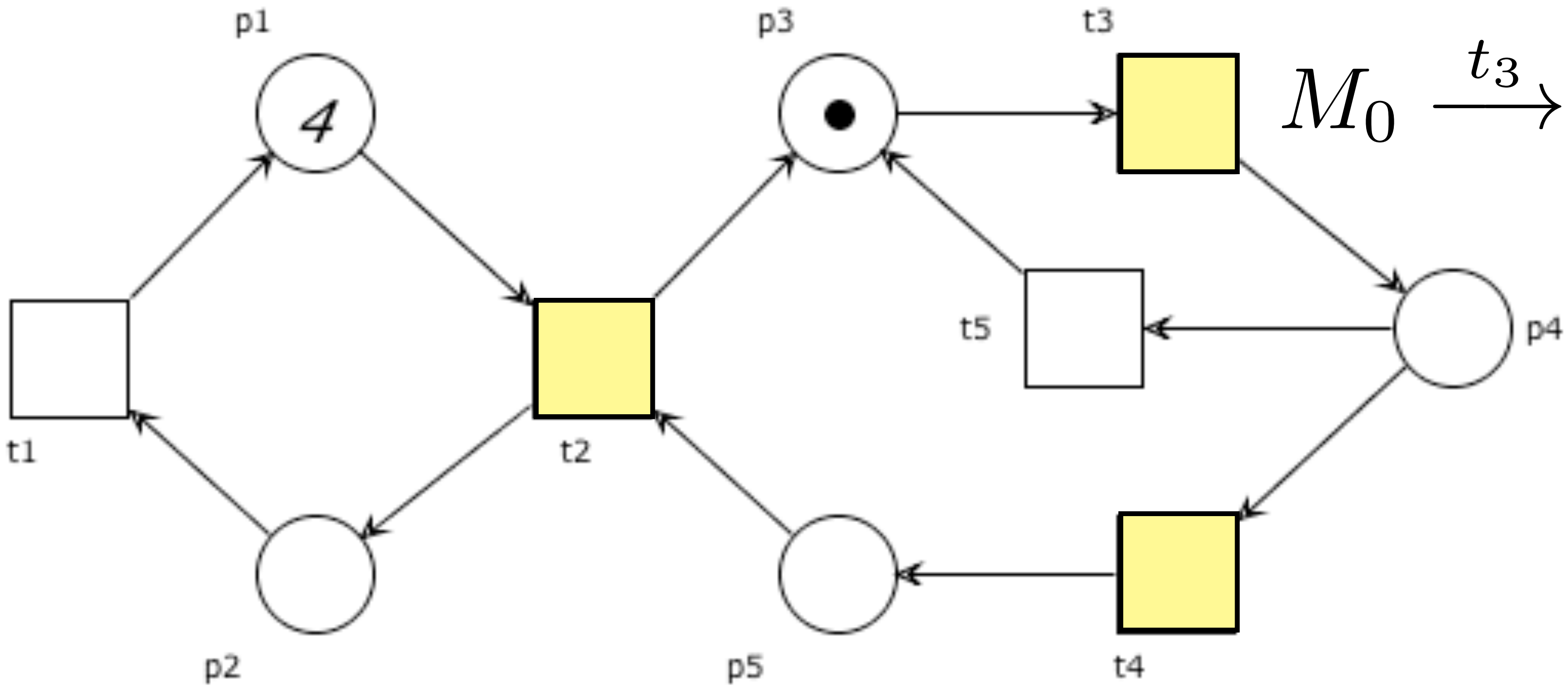
Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 = 4p_1 + p_4$$

$$\overset{M_0}{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{t_3}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} = \overset{M_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

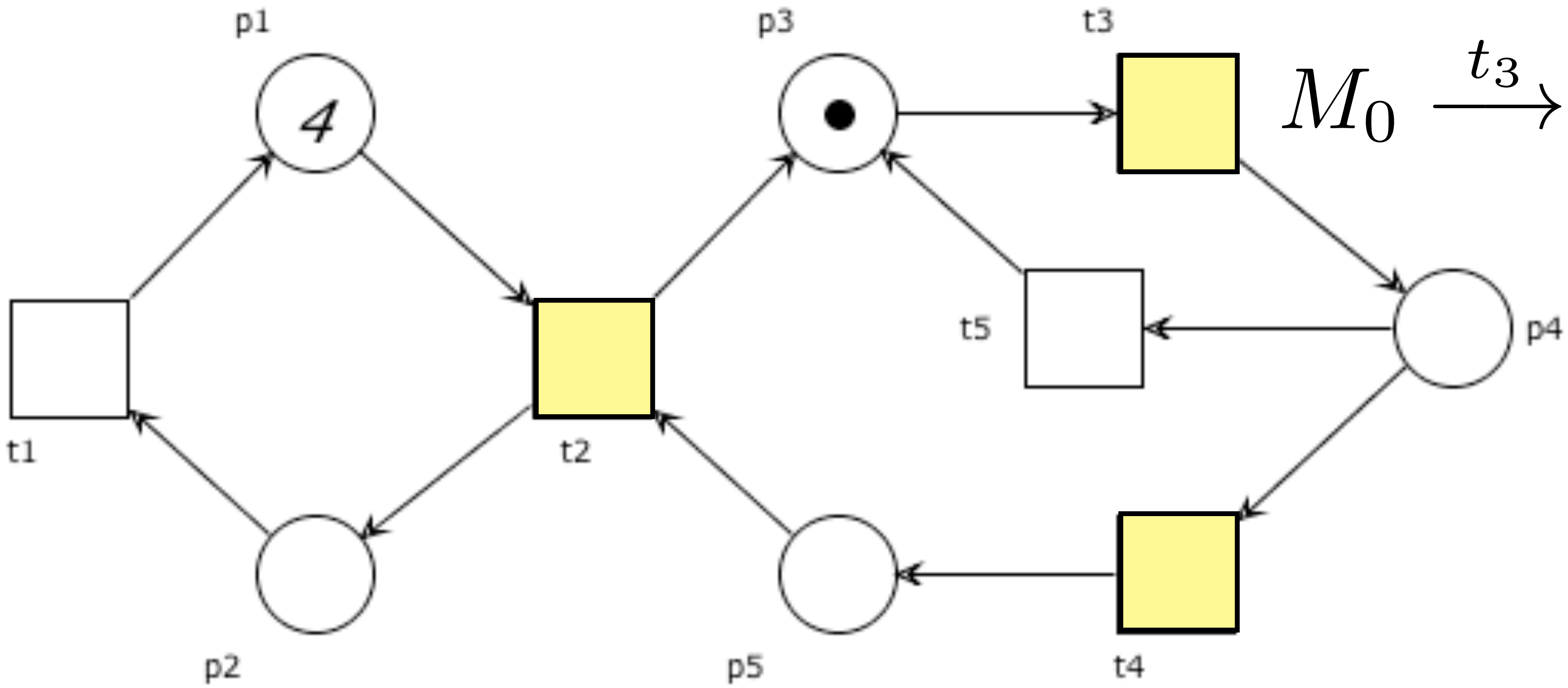
Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 \xrightarrow{t_4} M_2 \xrightarrow{t_2} M_3$$

$$\overset{M_0}{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{t_3}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} = \overset{M_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

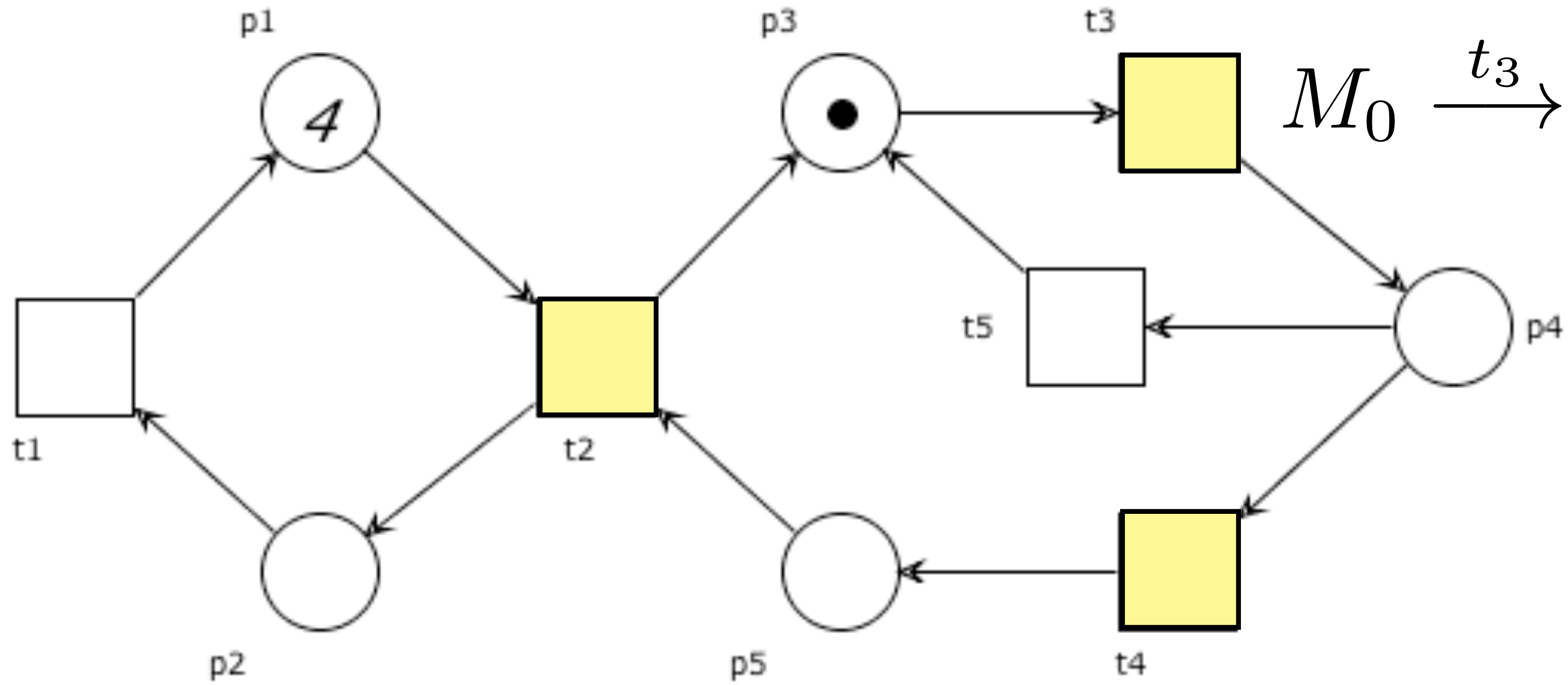
Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 \xrightarrow{t_4} M_2 \xrightarrow{t_2} M_3$$

$$\overset{M_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + \overset{t_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}} = \overset{M_2}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 \xrightarrow{t_4} M_2 \xrightarrow{t_2} M_3$$

$$\overset{M_2}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} + \overset{t_2}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \overset{M_3}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

Vectors: notation

Let $\mathbf{v}, \mathbf{w}$ be two vectors over E

We write $\mathbf{v} \leq \mathbf{w}$ if $v(e) \leq w(e)$ for any $e \in E$

We write $\mathbf{v} < \mathbf{w}$ if $v \leq w$ and $v(e) < w(e)$ for some $e \in E$

We write $\mathbf{v} \prec \mathbf{w}$ if $v(e) < w(e)$ for any $e \in E$

We let $\mathbf{0}$ denote any vector of any length whose entries are all 0

Products

Let $\mathbf{x}, \mathbf{y}$ be two vectors of equal length n (written $|\mathbf{x}| = |\mathbf{y}| = n$)

We define their scalar product by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

$$\left[\, x_1 \; x_2 \; ... \; x_n \,\right] \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

Products: example

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} =$$

Products: example

$$\begin{bmatrix} \boxed{0} & 1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \boxed{1} \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) +$$

Products: example

$$\begin{bmatrix} 0 & \boxed{1} & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \boxed{1} \\ 2 \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) +$$

Products: example

$$\begin{bmatrix} 0 & 1 & \boxed{-1} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \boxed{2} \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) +$$

Products: example

$$\begin{bmatrix} 0 & 1 & -1 & \boxed{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ \boxed{0} \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) + (0 \cdot 0) +$$

Products: example

$$\begin{bmatrix} 0 & 1 & -1 & 0 & \boxed{1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ \boxed{1} \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) + (0 \cdot 0) + (1 \cdot 1)$$

Products: example

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) + (0 \cdot 0) + (1 \cdot 1) = 0 + 1 - 2 + 0 + 1 = 0$$

Products

Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}, \mathbf{y}$ be all vectors of equal length

Let X be a $(k \times n)$-matrix whose i-th row is $\mathbf{x_i}$

We define the product $X \cdot \mathbf{y}$ as the (column) vector where

$$(X \cdot \mathbf{y})_i = \mathbf{x_i} \cdot \mathbf{y}$$

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_k} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mathbf{x_1} \cdot \mathbf{y} \\ \mathbf{x_2} \cdot \mathbf{y} \\ \vdots \\ \mathbf{x_k} \cdot \mathbf{y} \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Products

Let $\mathbf{x}, \mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_k}$ be all vectors of equal length

Let Y be a $(n \times k)$-matrix whose i-th column is $\mathbf{y_i}$

We define the product $\mathbf{x} \cdot Y$ as the (row) vector where

$$(\mathbf{x} \cdot Y)_i = \mathbf{x} \cdot \mathbf{y_i}$$

$$[\, x_1 \; x_2 \; ... \; x_n \,] \cdot [\, \mathbf{y_1} \; \mathbf{y_2} \; ... \; \mathbf{y_k} \,] = [\, \mathbf{x} \cdot \mathbf{y_1} \quad \mathbf{x} \cdot \mathbf{y_2} \quad ... \quad \mathbf{x} \cdot \mathbf{y_k} \,]$$

Products

Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}, \mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_h}$ be all vectors of equal length

Let X be a $(k \times n)$-matrix whose i-th row is $\mathbf{x_i}$

Let Y be a $(n \times h)$-matrix whose j-th column is $\mathbf{y_j}$

We define the product $X \cdot Y$ as the $(k \times h)$-matrix where

$$(X \cdot Y)_{i,j} = \mathbf{x_i} \cdot \mathbf{y_j}$$

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_k} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{y_1} \; \mathbf{y_2} \; ... \; \mathbf{y_h} \end{bmatrix} = \begin{bmatrix} \mathbf{x_1} \cdot \mathbf{y_1} & \mathbf{x_1} \cdot \mathbf{y_2} & ... & \mathbf{x_1} \cdot \mathbf{y_h} \\ \mathbf{x_2} \cdot \mathbf{y_1} & \mathbf{x_2} \cdot \mathbf{y_2} & ... & \mathbf{x_2} \cdot \mathbf{y_h} \\ \vdots & \vdots & & \vdots \\ \mathbf{x_k} \cdot \mathbf{y_1} & \mathbf{x_k} \cdot \mathbf{y_2} & ... & \mathbf{x_k} \cdot \mathbf{y_h} \end{bmatrix}$$

Vector perspective

Let $P = \{ p_1, ..., p_n \}$ and $T = \{ t_1, ..., t_m \}$

The net (P,T,F) can be seen as a matrix (n x m)

A marking is a vector of length n

But we miss an ingredient:

can a firing sequence be seen as a vector?
(of limited length)

Parikh vectors of transition sequences

Let $N = (P, T, F)$ be a net and $\sigma \in T^*$ a finite sequence of transitions.

The **Parikh vector**

$$\vec{\sigma} : T \to \mathbb{N}$$

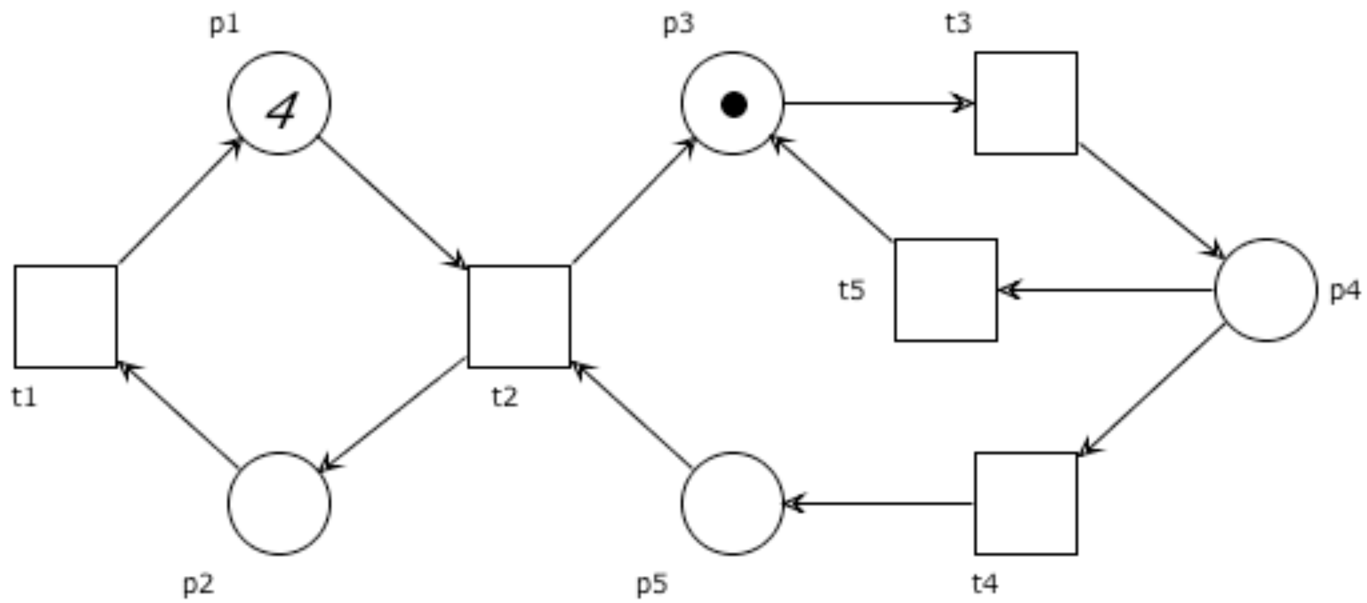
of σ maps every $t \in T$ to the number of its occurrences in σ.

Parikh vector of a firing

As a special case, for a sequence $\sigma = t$ (one single transition):

$$\vec{t} = \begin{bmatrix} \underset{t_1}{0} & \ldots & 0 & \underset{t}{1} & 0 & \ldots & \underset{t_m}{0} \end{bmatrix}$$

Parikh vector: example



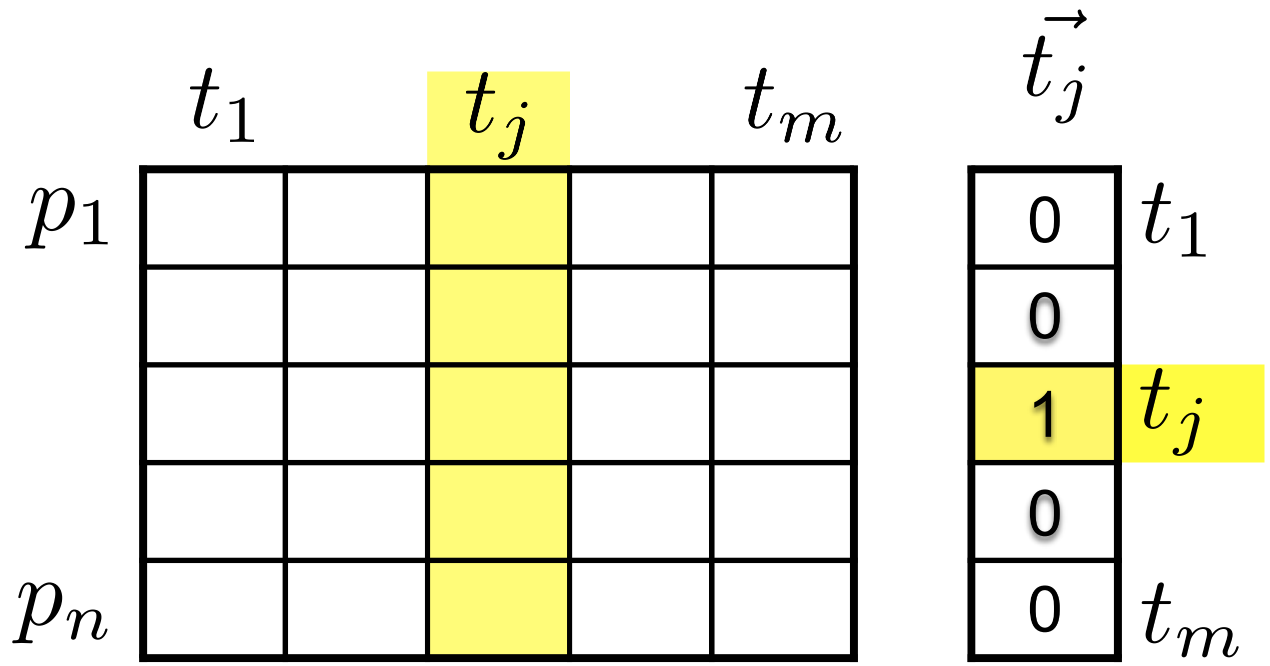
$$M_0 = 4p_1 + p_3$$

$$M_0 \xrightarrow{\sigma = t_3 t_5 t_3 t_4 t_2} 3p_1 + p_2 + p_3 \qquad \vec{\sigma} = [\,0 \quad 1 \quad 2 \quad 1 \quad 1\,]$$

$$M_0 \xrightarrow{\sigma' = t_3 t_4 t_2 t_3 t_4 t_2 t_3 t_5 t_3} 2p_1 + 2p_2 + p_4 \qquad \vec{\sigma'} = [\,0 \quad 2 \quad 4 \quad 2 \quad 1\,]$$

First fact

$$\mathbf{N} \cdot \vec{t_j} = \mathbf{t_j}$$



Second fact

$$\mathbf{N} \cdot \vec{t}_j = \mathbf{t_j}$$

If $M \stackrel{t}{\longrightarrow} M'$ then $M' = M + \mathbf{t}$

Third fact

$$\mathbf{N} \cdot \vec{t}_j = \mathbf{t_j}$$

If $M \xrightarrow{t} M'$ then $M' = M + \mathbf{t}$

If $M \xrightarrow{t} M'$ then $M' = M + \mathbf{N} \cdot \vec{t}$

Marking equation lemma

Lemma: If $M \xrightarrow{\sigma} M'$ then $M' = M + \mathbf{N} \cdot \vec{\sigma}$

Marking equation lemma

Lemma: If $M \xrightarrow{\sigma} M'$ then $M' = M + \mathbf{N} \cdot \vec{\sigma}$

The proof is by induction on the length of σ

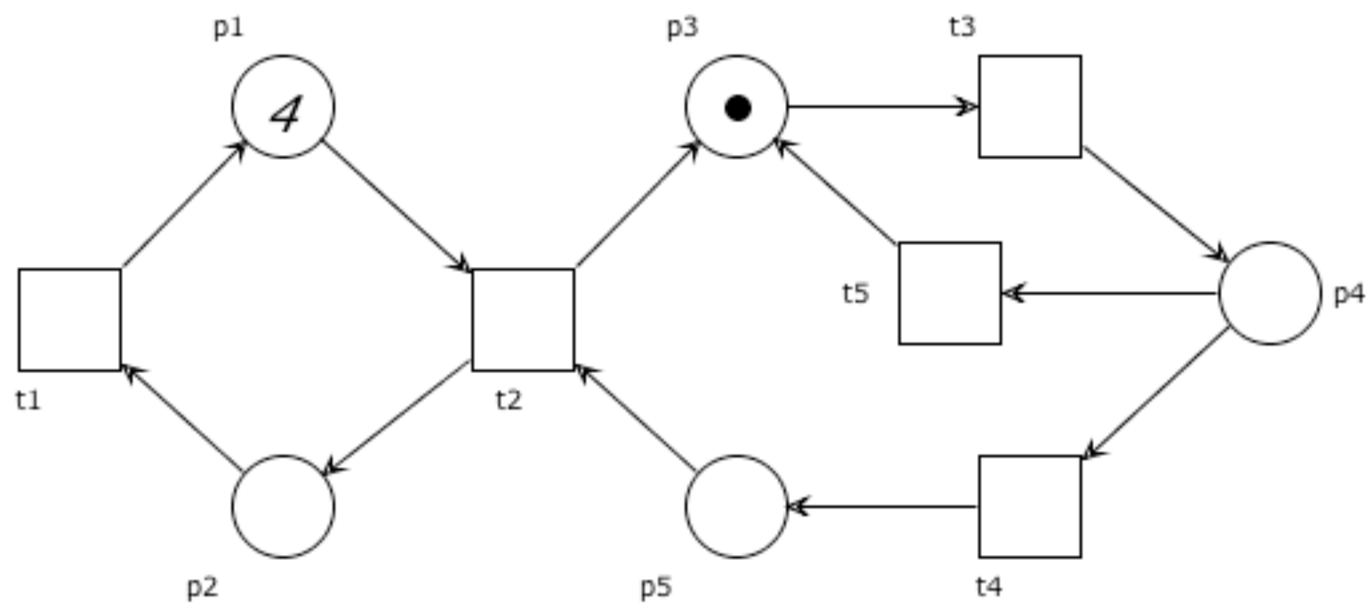
$\mathbf{base}$ $(\sigma = \epsilon)$**:** and therefore $M' = M$. The equality holds trivially, because $\vec{\sigma} = \mathbf{0}$

induction $(\sigma = \sigma' t$ **for some sequence** σ' **and transition** $t)$**:**

Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$. We have:

$$
\begin{aligned}
M' &= M'' + \mathbf{t} \\
&= M'' + \mathbf{N} \cdot \vec{t} \\
&= M + \mathbf{N} \cdot \vec{\sigma'} + \mathbf{N} \cdot \vec{t} \\
&= M + \mathbf{N} \cdot (\vec{\sigma'} + \vec{t}) \\
&= M + \mathbf{N} \cdot (\overrightarrow{\sigma' t}) \\
&= M + \mathbf{N} \cdot \vec{\sigma}
\end{aligned}
$$

Marking equation: example



$M_0 = [\,4 \quad 0 \quad 1 \quad 0 \quad 0\,]$ $\quad \sigma = t_3 t_5 t_3 t_4 t_2$ $\quad \vec{\sigma} = [\,0 \quad 1 \quad 2 \quad 1 \quad 1\,]$

$$\overset{M_0}{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{N}{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}} \cdot \overset{\vec{\sigma}}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Marking equation lemma: consequences

The marking reached by any occurrence sequence only depends on the number of occurrences of each transition

It does not depend on the order in which transitions occur

Every fireable permutation of the same transitions leads to the same marking

Monotonicity lemma (1)

Lemma: If $M \xrightarrow{\sigma} M'$ then $M + L \xrightarrow{\sigma} M' + L$ for any L

The proof is by induction on the length of σ

base $(\sigma = \epsilon)$**:** the empty sequence is always enabled, at any marking

induction $(\sigma = \sigma' t$ **for some sequence** σ' **and transition** $t)$**:**

Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By the marking equation lemma: $M' = M'' + \mathbf{N} \cdot \vec{t}$

By the induction hypothesis $M + L \xrightarrow{\sigma'} M'' + L$

Moreover, $M'' + L \xrightarrow{t}$ because $M'' \xrightarrow{t}$.

By the marking equation lemma: $M'' + L \xrightarrow{t} M'' + L + \mathbf{N} \cdot \vec{t} = M' + L$

Recall: Enabledness proposition

Proposition: $M \xrightarrow{\sigma}$ iff $M \xrightarrow{\sigma'}$ for every prefix σ' of σ

($\Rightarrow$) immediate from definition

($\Leftarrow$) trivial if σ is finite (σ itself is a prefix of σ)

When σ is infinite: taken any $i \in \mathbb{N}$ we need to prove that $t_i = \sigma(i)$ is enabled after the firing of the prefix $\sigma' = t_1 t_2 ... t_{i-1}$ of σ.

But this is obvious, because

$$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} ... \xrightarrow{t_{i-1}} M_{i-1} \xrightarrow{t_i} M_i$$

is also a finite prefix of σ and therefore $M_{i-1} \xrightarrow{t_i}$

Monotonicity lemma (2)

Lemma: If $M \xrightarrow{\sigma}$ then $M + L \xrightarrow{\sigma}$ for any L

If σ is finite then the thesis follows from monotonicity lemma 1

If σ is infinite, then it suffices to prove that:
$M + L \xrightarrow{\sigma'}$ for any finite prefix σ' of σ

Take any such prefix σ'. Then, $M \xrightarrow{\sigma'}$ (because $M \xrightarrow{\sigma}$)
By the marking equation lemma, $M \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma'}$.
By monotonicity lemma 1, $M + L \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma'} + L$
Hence $M + L \xrightarrow{\sigma'}$

Monotonicity lemma, intuitively

If some activities can be done with less (resources),
then the same activities can be done with more (resources)

If we perform activities with more resources than needed,
then the additional resources are preserved

Boundedness Lemma

Lemma: If a system is bounded and $M \in [M_0\rangle$ with $M \supseteq M_0$, then $M = M_0$.

Let $M_0 \xrightarrow{\sigma} M$.

By $M \supseteq M_0$, there exists a marking L with $M = M_0 + L$.

Let $M_k = M_0 + k \cdot L$ for every $k \in \mathbb{N}$.

By the Monotonicity Lemma, we have:
$M_0 \xrightarrow{\sigma} M_1 \xrightarrow{\sigma} M_2 \cdots$
i.e., $M_k \in [M_0\rangle$ for any $k \in \mathbb{N}$.

Since the system is bounded, it must be $L = \emptyset$.

Boundedness lemma: consequences

If we show that a marking M is reachable with

$$M \supset M_0$$

then the system is not bounded

Repetition Lemma

Lemma: If $M \xrightarrow{\sigma} M'$ and $M \xrightarrow{\sigma\sigma\cdots}$, then $M \subseteq M'$.

We proceed by contradiction.
Suppose $M \not\subseteq M'$, i.e., there exist $k > 0, p \in P$ such that $M'(p) = M(p) - k$.

By the Marking Equation Lemma we have $M' = M + \mathbf{N} \cdot \vec{\sigma}$.
Therefore $(\mathbf{N} \cdot \vec{\sigma})(p) = -k$.

Let $n = M(p) + 1$ and $\sigma' = \underbrace{\sigma \cdots \sigma}_{n}$.

By hypothesis we have $M \xrightarrow{\sigma'} M''$,
and by the Marking Equation Lemma $M''(p) = M(p) - nk < 0$, which is absurd.