

# Business Processes Modelling

## MPB (6 cfu, 295AA)

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<http://www.di.unipi.it/~bruni>

10 - Petri nets properties



# Object

$$N \vdash \psi$$

We give a formal account of some key properties  
of Petri nets

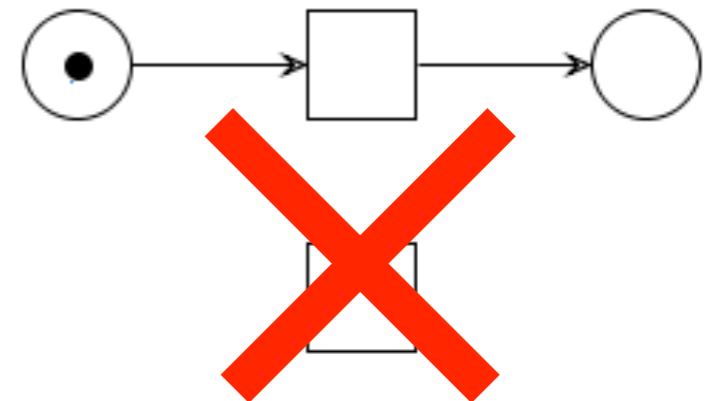
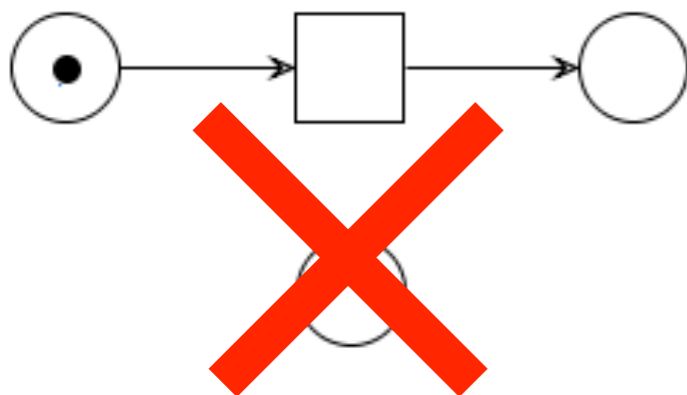
Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

# Disclaim

A node (place or transition) is called **isolated** if its pre- and post-sets are empty

In the following we only consider nets **without isolated nodes**



# Petri nets: behavioural properties

# Properties of Petri nets

We introduce some of the properties of Petri nets that can play an important role in the verification of business processes

Liveness  
Deadlock-freedom  
Boundedness  
Cyclicity (also Reversibility)

# Let's practice with some formula

$$N = (P, T, F, M_0)$$

for any markings  $M$  and  $M'$

it follows that  $M'$  is reachable from  $M_0$

$$\forall M, M', \quad M \in [M_0\rangle \wedge M' \in [M\rangle \quad \Rightarrow \quad M' \in [M_0\rangle$$

with  $M$  reachable from  $M_0$  and  $M'$  reachable from  $M$

any marking that is reachable from a  
reachable marking is also reachable

# Disclaim

When we say:  
for any reachable marking  $M$

we mean:  
for any marking  $M$  reachable from  $M_0$

# Liveness, intuitively

A transition  $t$  is **live** if  
from any reachable marking  $M$  another  
marking  $M'$  can be reached where  $t$  is enabled

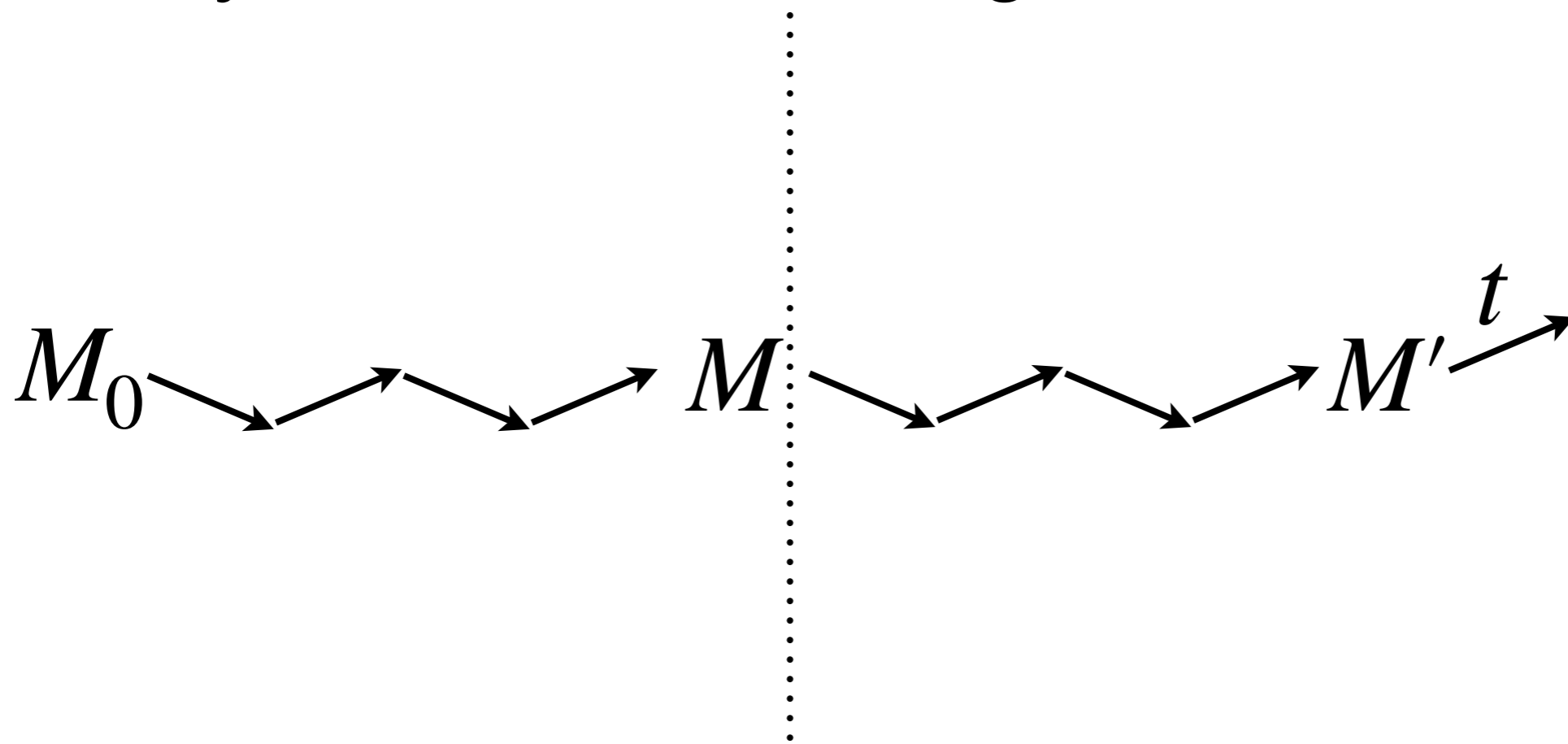
In other words:  
*at any point in time of the computation, we  
cannot exclude that  $t$  will fire in the future  
or, equivalently,  
at any point in time of the computation, it is still  
possible to enable  $t$  in the future*

A Petri net is **live** if all of its transitions are live



# Liveness illustrated

For any reachable marking  $M$ ...



...can we find a way to enable  $t$  ?

# Liveness, formally

$$(P, T, F, M_0)$$

for any transition  $t$

there is a marking  $M'$  reachable from  $M$

$$\forall t \in T, \quad \forall M \in [M_0 \rangle, \quad \exists M' \in [M \rangle, \quad M' \xrightarrow{t}$$

for any reachable marking  $M$

that enables  $t$

# Digression

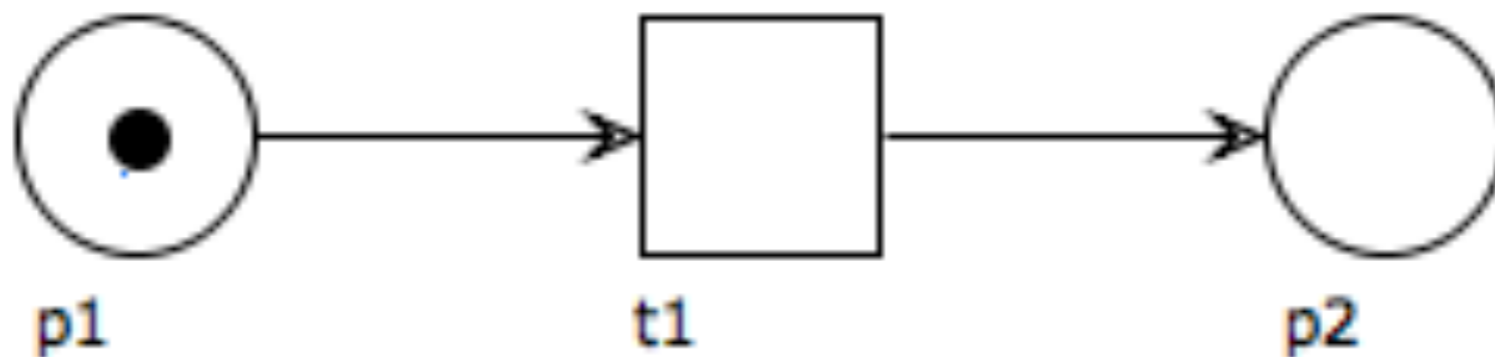
Order of quantifier is important:

quantification of the same kind can be  
switched

the order of universal and existential  
quantification is important

$$\forall n. \exists m. n < m \quad \neq \quad \exists m. \forall n. n < m$$

# Example: Non Live



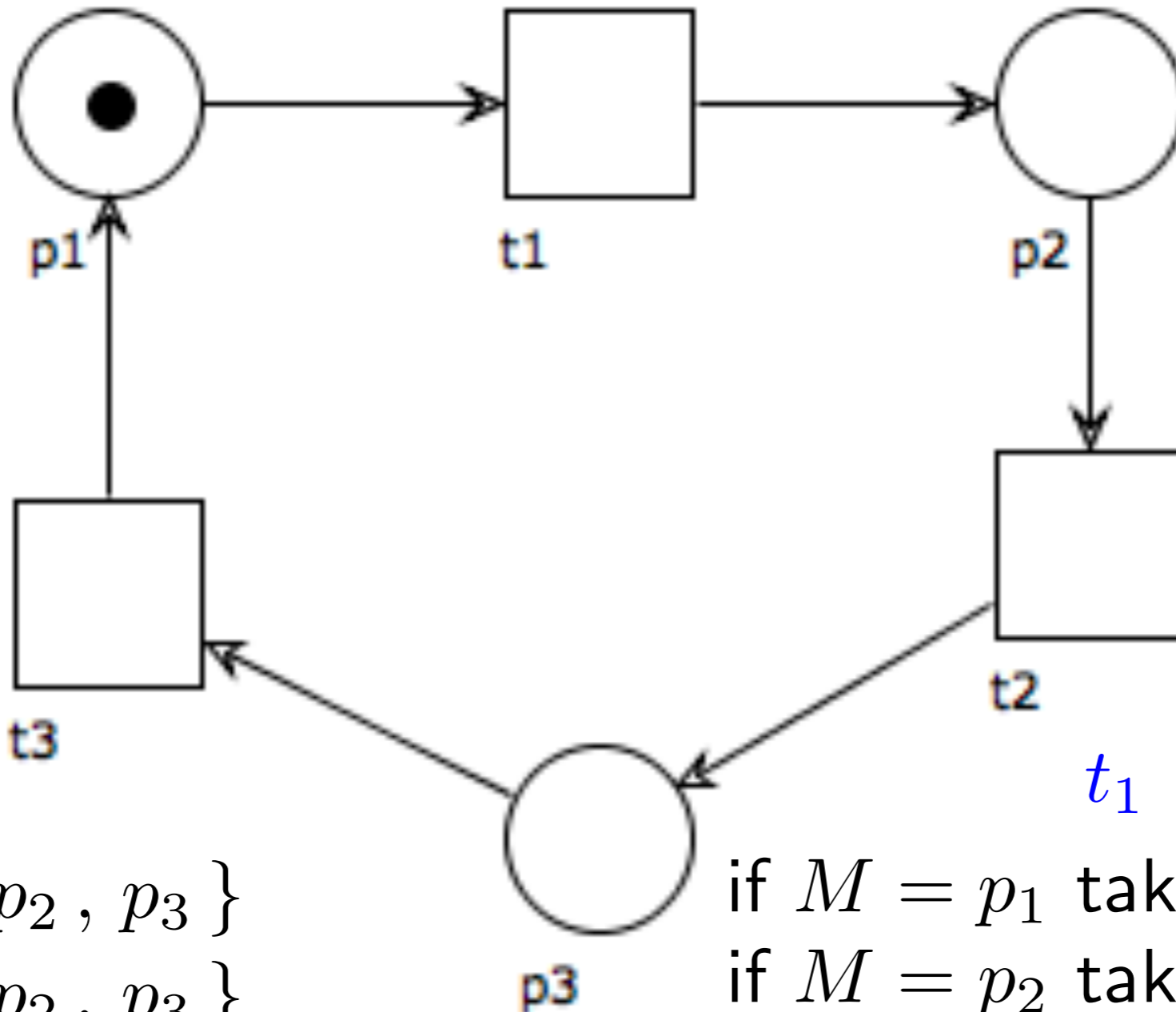
$$M_0 = p_1$$

$$[M_0] = \{ p_1, p_2 \}$$

$t_1$  non live!

$$p_2 \not\rightarrow$$

# Example: Live



$$M_0 = p_1$$

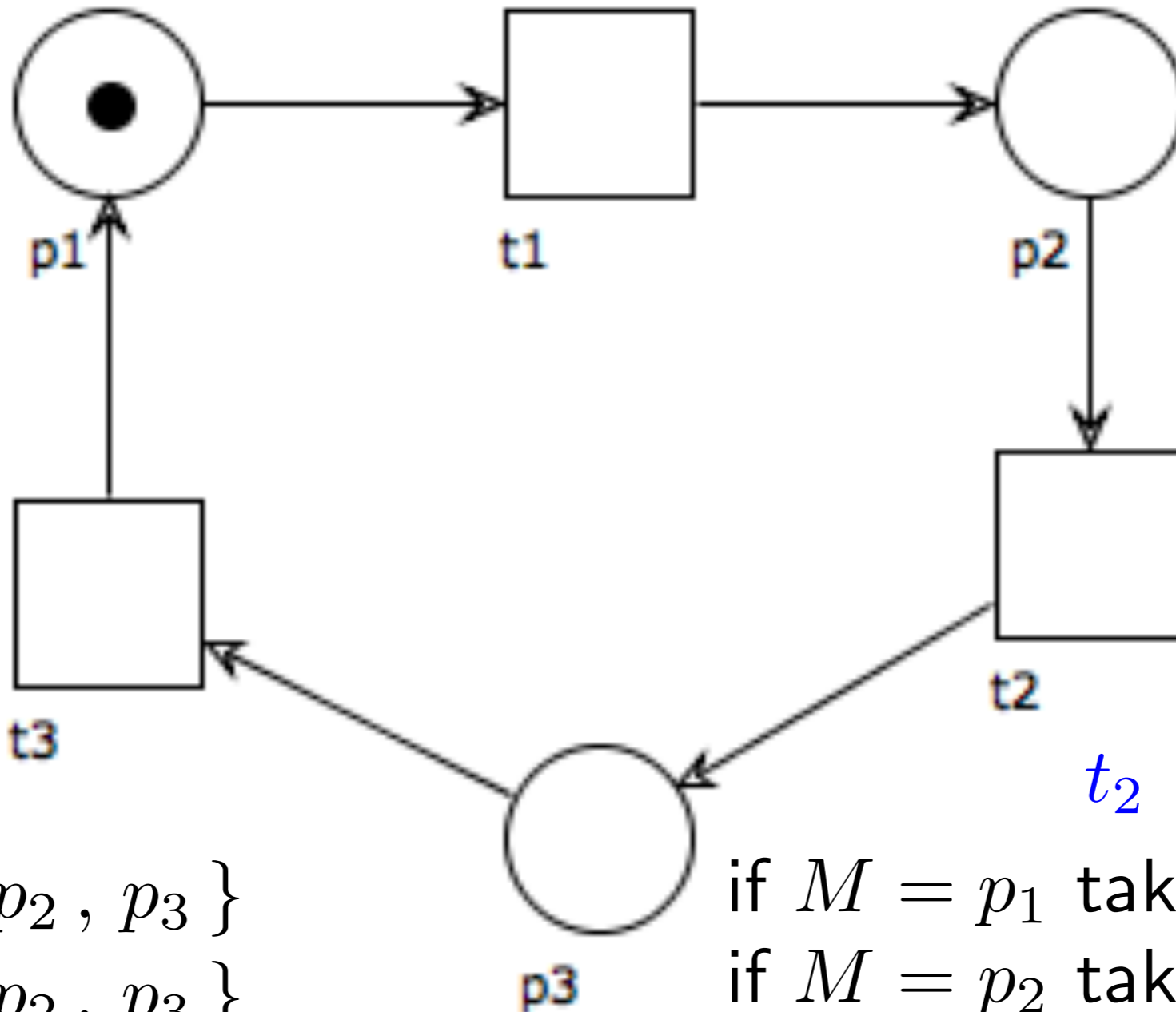
$$[M_0] = \{ p_1, p_2, p_3 \}$$

$$[p_2] = \{ p_1, p_2, p_3 \}$$

$$[p_3] = \{ p_1, p_2, p_3 \}$$

if  $M = p_1$  take  $M' = p_1 \xrightarrow{t_1}$   
 if  $M = p_2$  take  $M' = p_1 \xrightarrow{t_1}$   
 if  $M = p_3$  take  $M' = p_1 \xrightarrow{t_1}$

# Example: Live



$$M_0 = p_1$$

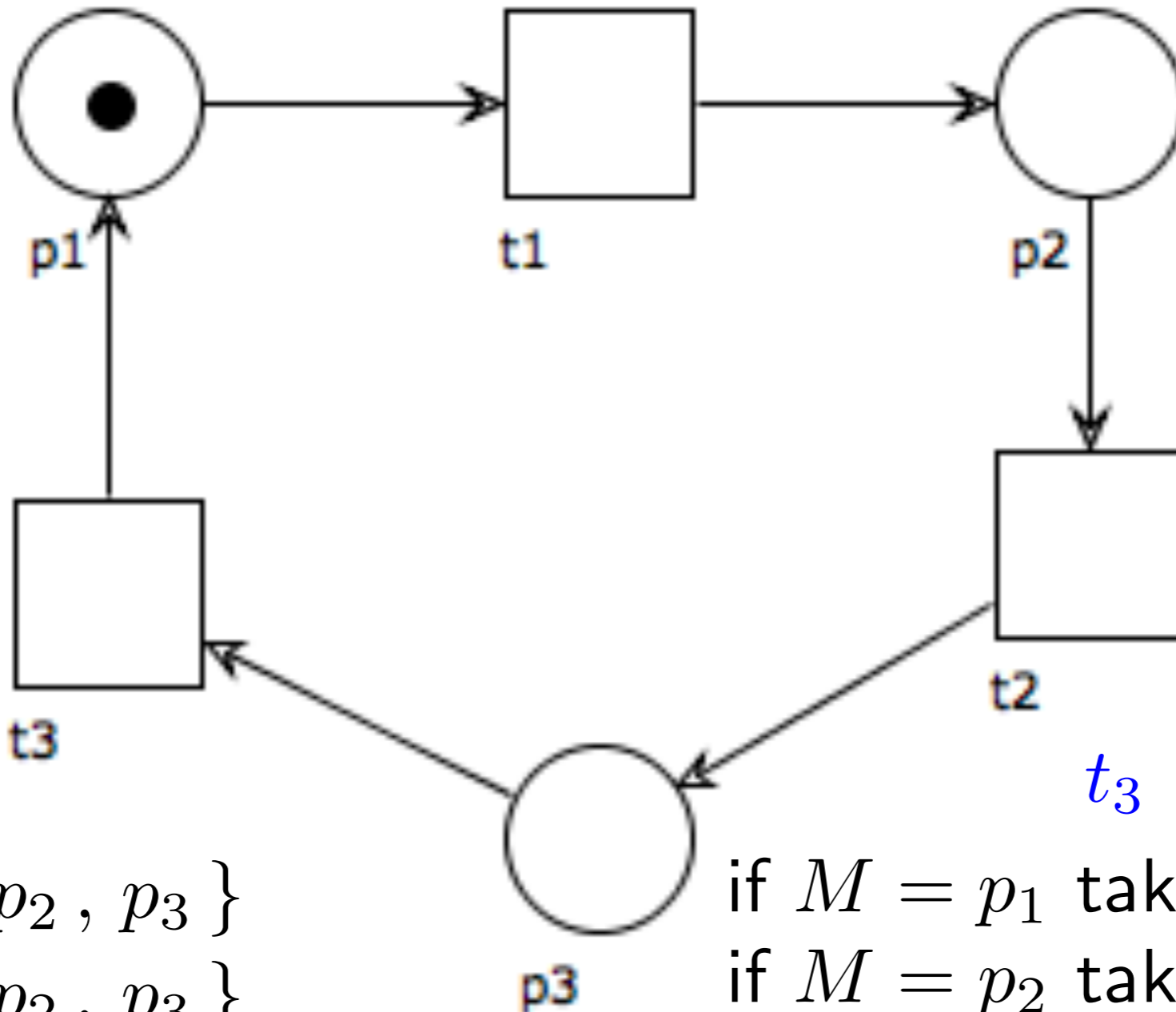
$$[M_0] = \{ p_1, p_2, p_3 \}$$

$$[p_2] = \{ p_1, p_2, p_3 \}$$

$$[p_3] = \{ p_1, p_2, p_3 \}$$

if  $M = p_1$  take  $M' = p_2 \xrightarrow{t_2}$   
 if  $M = p_2$  take  $M' = p_2 \xrightarrow{t_2}$   
 if  $M = p_3$  take  $M' = p_2 \xrightarrow{t_2}$

# Example: Live



$$M_0 = p_1$$

$$[M_0] = \{ p_1, p_2, p_3 \}$$

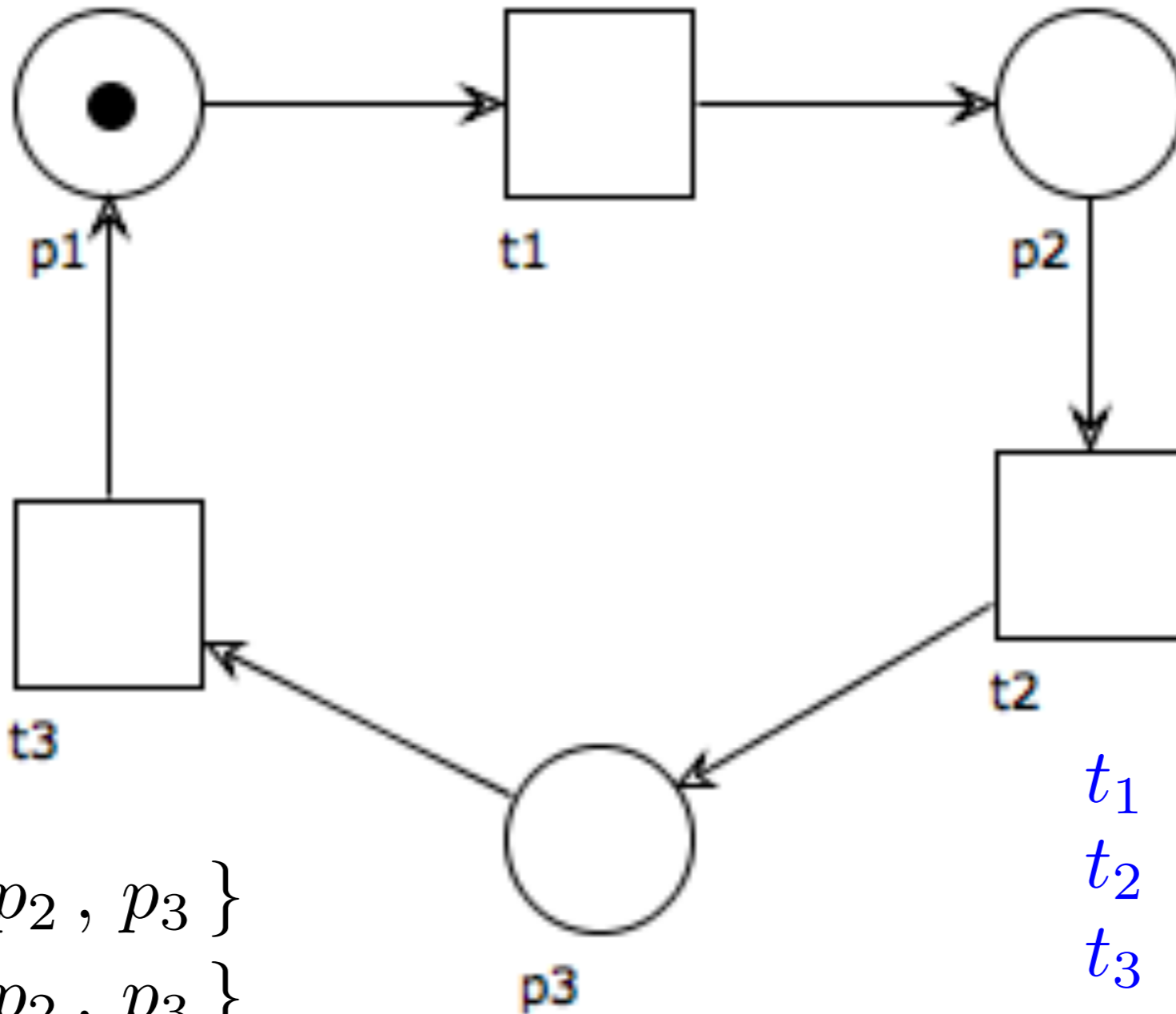
$$[p_2] = \{ p_1, p_2, p_3 \}$$

$$[p_3] = \{ p_1, p_2, p_3 \}$$

$t_3$  live

if  $M = p_1$  take  $M' = p_3 \xrightarrow{t_3}$   
 if  $M = p_2$  take  $M' = p_3 \xrightarrow{t_3}$   
 if  $M = p_3$  take  $M' = p_3 \xrightarrow{t_3}$

# Example: Live



$$M_0 = p_1$$

$$[M_0] = \{ p_1, p_2, p_3 \}$$

$$[p_2] = \{ p_1, p_2, p_3 \}$$

$$[p_3] = \{ p_1, p_2, p_3 \}$$

$t_1$  live  
 $t_2$  live  
 $t_3$  live  
 $N$  live



# Liveness: pay attention!

Liveness of  $t$  should not be confused with the following property:

*starting from the initial marking  $M_0$  it is possible to reach a marking  $M$  that enables  $t$*

$$\exists M \in [M_0\rangle. M \xrightarrow{t}$$

(this property just ensures that  $t$  is not "dead" at  $M_0$ )

# Dead transition

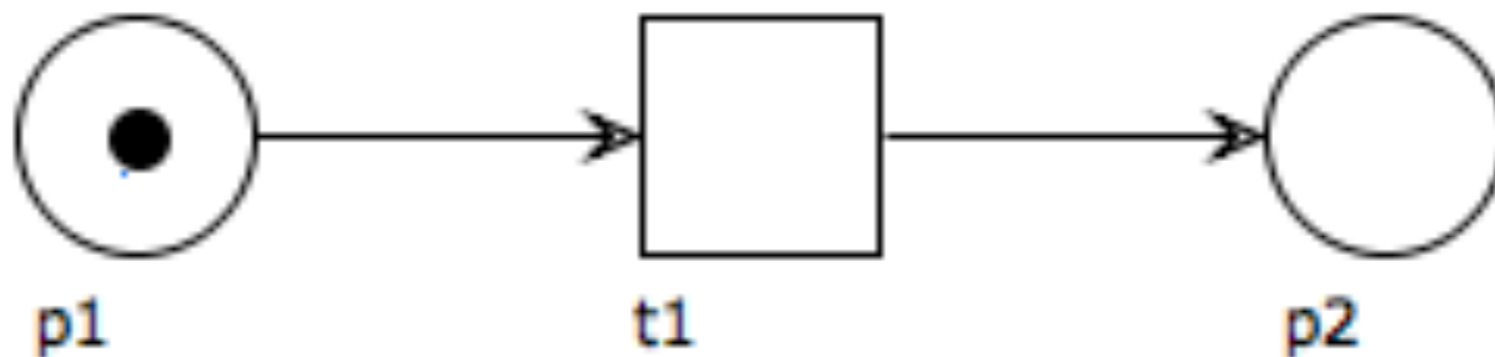
Given a marking  $M$

A transition  $t$  is **dead** at  $M$

if  $t$  will never be enabled in the future

(i.e.,  $t$  is not enabled at any marking reachable from  $M$ )

Example: t1 non live  
(but t1 non dead!)



# Liveness, a formal recap

$$N = (P, T, F, M_0) \quad t \in T$$

$$\text{Live}(t, N) \equiv \forall M \in [M_0\rangle, \exists M' \in [M\rangle, M' \xrightarrow{t}$$

$$\text{NonLive}(t, N) \equiv \neg \text{Live}(t, N)$$

$$\text{Dead}(t, N) \equiv \forall M \in [M_0\rangle, M \not\xrightarrow{t}$$

$$\text{NonDead}(t, N) \equiv \neg \text{Dead}(t, N)$$

$$\text{Live}(N) \equiv \forall t \in T, \text{Live}(t, N)$$

$$\text{NonLive}(N) \equiv \neg \text{Live}(N)$$

# Liveness, a formal recap

$$N = (P, T, F, M_0) \quad t \in T$$

$$\text{Live}(t, N) \equiv \forall M \in [M_0], \exists M' \in [M], M' \xrightarrow{t}$$

$$\begin{aligned} \text{NonLive}(t, N) &\equiv \neg \text{Live}(t, N) \\ &\equiv \neg(\forall M \in [M_0], \exists M' \in [M], M' \xrightarrow{t}) \\ &\equiv \exists M \in [M_0], \forall M' \in [M], M' \not\xrightarrow{t} \end{aligned}$$

# Liveness, a formal recap

$$N = (P, T, F, M_0) \quad t \in T$$

$$\text{Dead}(t, N) \equiv \forall M \in [M_0], \quad M \not\stackrel{t}{\rightarrow}$$

$$\begin{aligned} \text{NonDead}(t, N) &\equiv \neg \text{Dead}(t, N) \\ &\equiv \neg(\forall M \in [M_0], \quad M \not\stackrel{t}{\rightarrow}) \\ &\equiv \exists M \in [M_0], \quad M \stackrel{t}{\rightarrow} \end{aligned}$$

# Liveness, a formal recap

$$N = (P, T, F, M_0)$$

$$\begin{aligned} \text{NonLive}(N) &\equiv \neg \text{Live}(N) \\ &\equiv \neg(\forall t \in T, \text{Live}(t, N)) \\ &\equiv \exists t \in T, \text{NonLive}(t, N) \\ &\equiv \exists t \in T, \exists M \in [M_0\rangle, \forall M' \in [M\rangle, M' \not\xrightarrow{t} \end{aligned}$$

# Non-live vs Dead

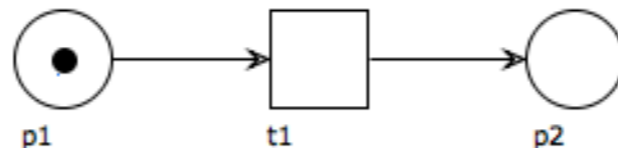
$$\forall M \in [M_0], \quad M \not\stackrel{t}{\rightarrow}$$

$$\text{Dead}(t, N) \quad \Rightarrow \quad \text{NonLive}(t, N)$$

$$\exists M \in [M_0], \quad \forall M' \in [M], \quad M' \not\stackrel{t}{\rightarrow}$$

just take  $M = M_0$

$$\text{NonLive}(t, N) \quad \not\Rightarrow \quad \text{Dead}(t, N)$$





# Digression

How to disprove an implication?

$$P \not\Rightarrow Q$$

$$P \wedge \neg Q$$

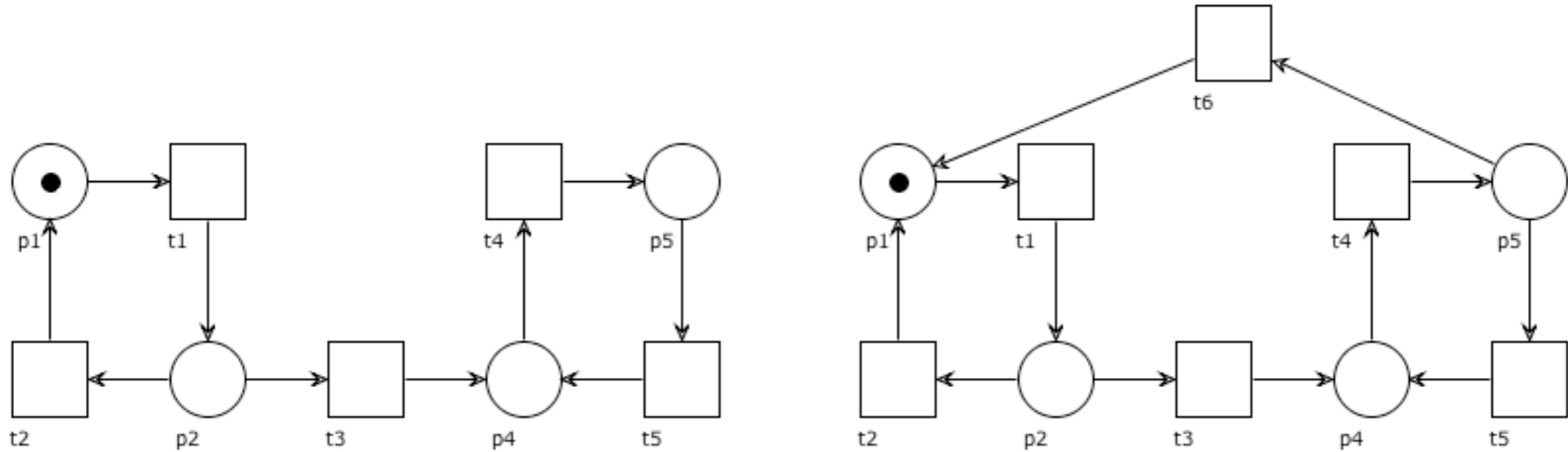
# Non-live vs Dead

$$\text{NonLive}(t, N) \equiv \exists M \in [M_0], \quad \text{Dead}(t, N' = (P, T, F, M))$$

$$\begin{aligned} \text{NonLive}(t, N) &\equiv \exists M \in [M_0], \quad \forall M' \in [M], \quad M' \not\stackrel{t}{\rightarrow} \\ \text{Dead}(t, N') &\equiv \forall M' \in [M], \quad M' \not\stackrel{t}{\rightarrow} \end{aligned}$$

a system is not live  
iff it has a non-live transition  
iff it has a transition that **can become** dead

# Liveness: example



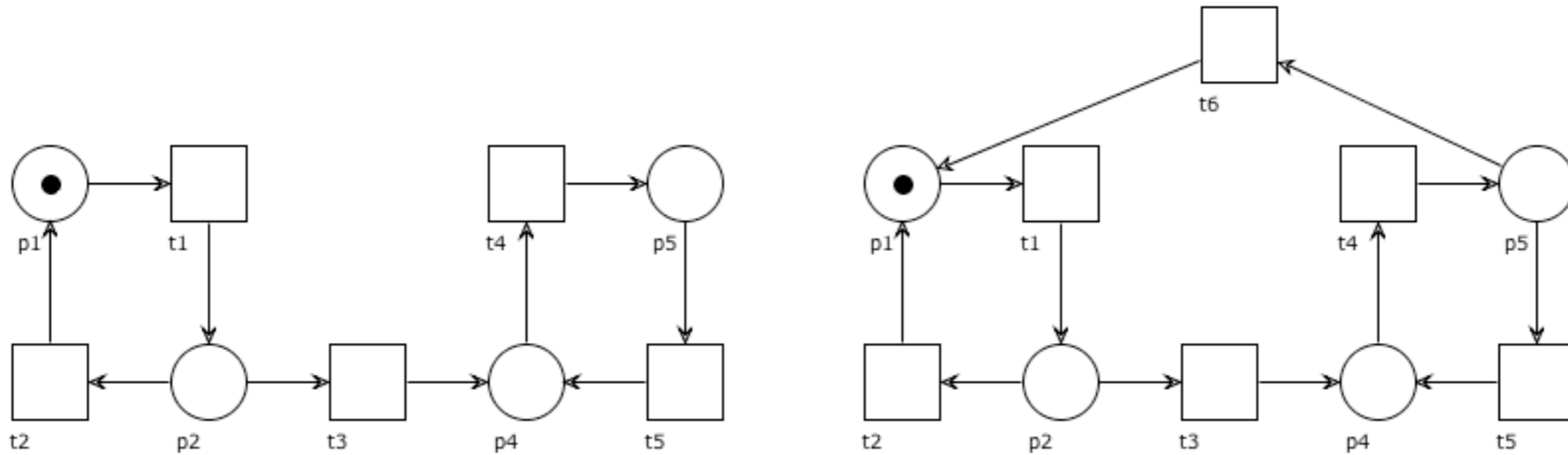
Which transitions are live?

Which are not?

Which are dead?

Is the net live?

# Liveness: example



<p><b>t4, t5</b></p> <p><b>t1, t2, t3</b></p> <p><b>none</b></p> <p><b>No</b></p>	<p>Which transitions are live?</p> <p>Which are not?</p> <p>Which are dead?</p> <p>Is the net live?</p>	<p><b>all</b></p> <p><b>none</b></p> <p><b>none</b></p> <p><b>Yes</b></p>
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# Liveness on the occurrence graph

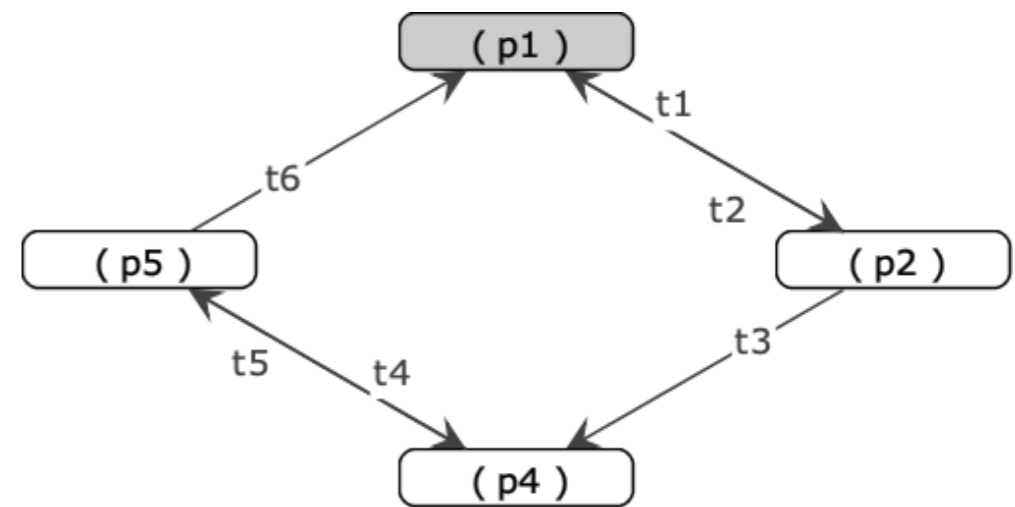
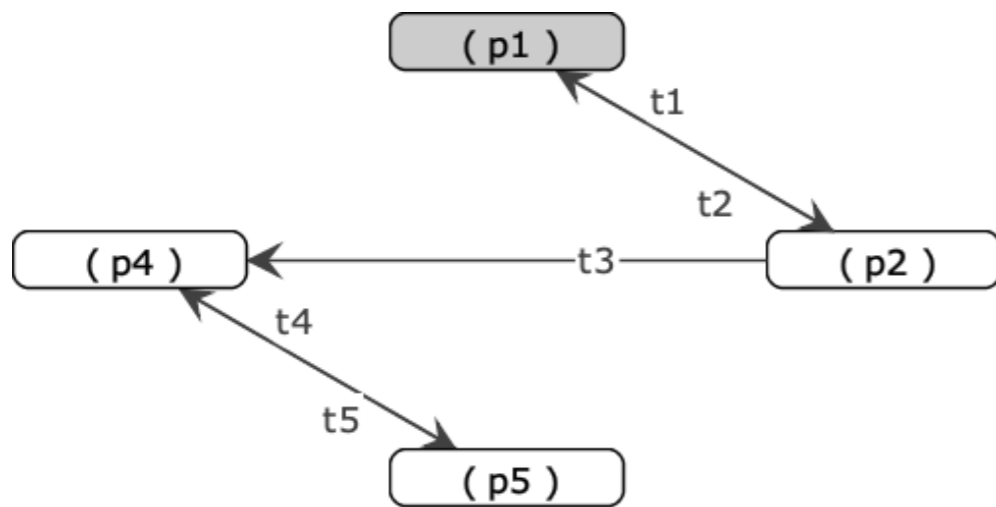
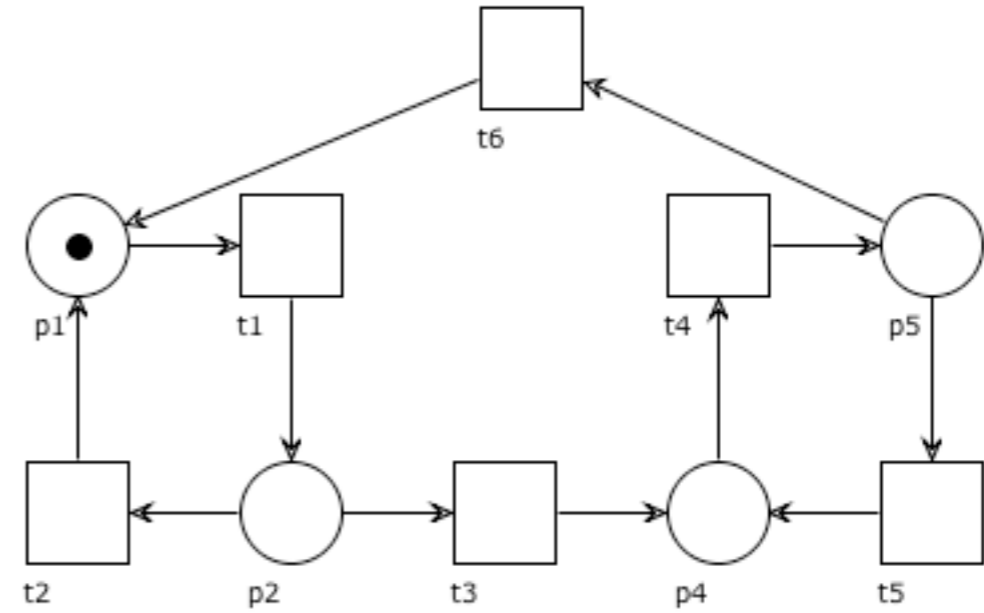
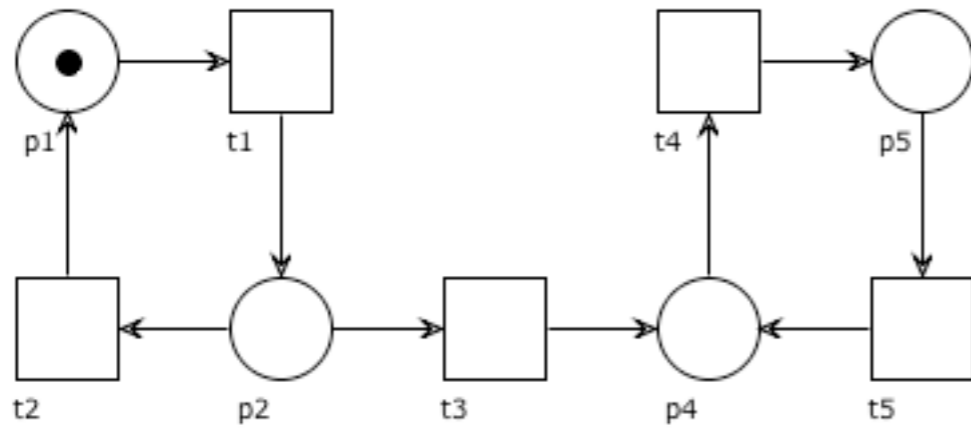
A transition  $t$  is live  
iff

From any node of the occurrence graph we can reach a  
node with an outgoing arc labelled by  $t$

A transition  $t$  is dead (at  $M_0$ )  
iff

There is no  $t$ -labelled arc in the occurrence graph

# Liveness: example



# Marked place

Given a marking  $M$

We say that a place  $p$  is **marked** (at  $M$ )  
if  $M(p) > 0$

(i.e., there is a token in  $p$  in the marking  $M$ )

We say that  $p$  is **unmarked**  
if  $M(p) = 0$

(i.e., there is no token in  $p$  in the marking  $M$ )

# Place-liveness, intuitively

A place  $p$  is **live** if  
every time it becomes unmarked  
there is still the possibility to be marked in the future  
(or if it always stays marked)

A Petri net is **place-live** if all of its places are live



# Live place

**Definition:** Let  $(P, T, F, M_0)$  be a net system.

A place  $p \in P$  is **live** if  $\forall M \in [M_0 \rangle. \exists M' \in [M \rangle. M'(p) > 0$

# Place-liveness, formally

$$(P, T, F, M_0)$$

$$\forall p \in P. \quad \forall M \in [M_0]. \quad \exists M' \in [M]. \quad M'(p) > 0$$

# Dead nodes, intuitively

Given a marking  $M$

A transition  $t$  is **dead** at  $M$

if  $t$  will never be enabled in the future

(i.e.,  $t$  is not enabled in any marking reachable from  $M$ )

A place  $p$  is **dead** at  $M$

if  $p$  will never be marked in the future

(i.e.,  $p$  is unmarked in any marking reachable from  $M$ )

# Dead nodes

**Definition:** Let  $(P, T, F)$  be a net

A transition  $t \in T$  is **dead** at  $M$  if  $\forall M' \in [M \rangle. M' \not\xrightarrow{t}$

A place  $p \in P$  is **dead** at  $M$  if  $\forall M' \in [M \rangle. M'(p) = 0$

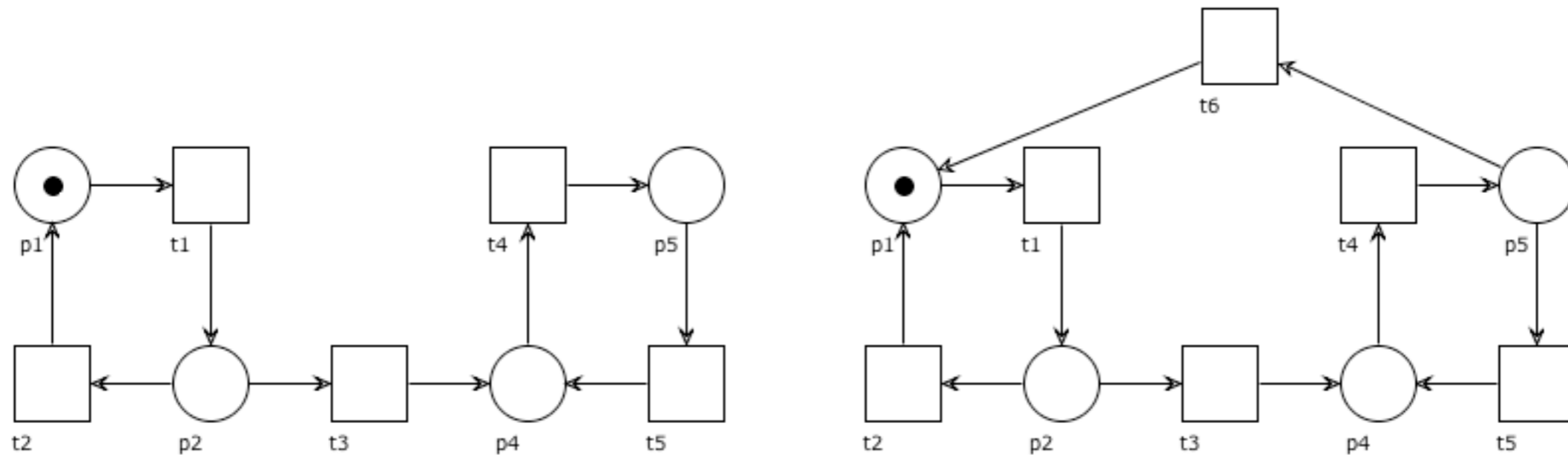
# Non-live vs Dead

If a transition is dead at some reachable marking  $M$   
then it is non-live

If a place is dead at some reachable marking  $M$   
then it is non-live

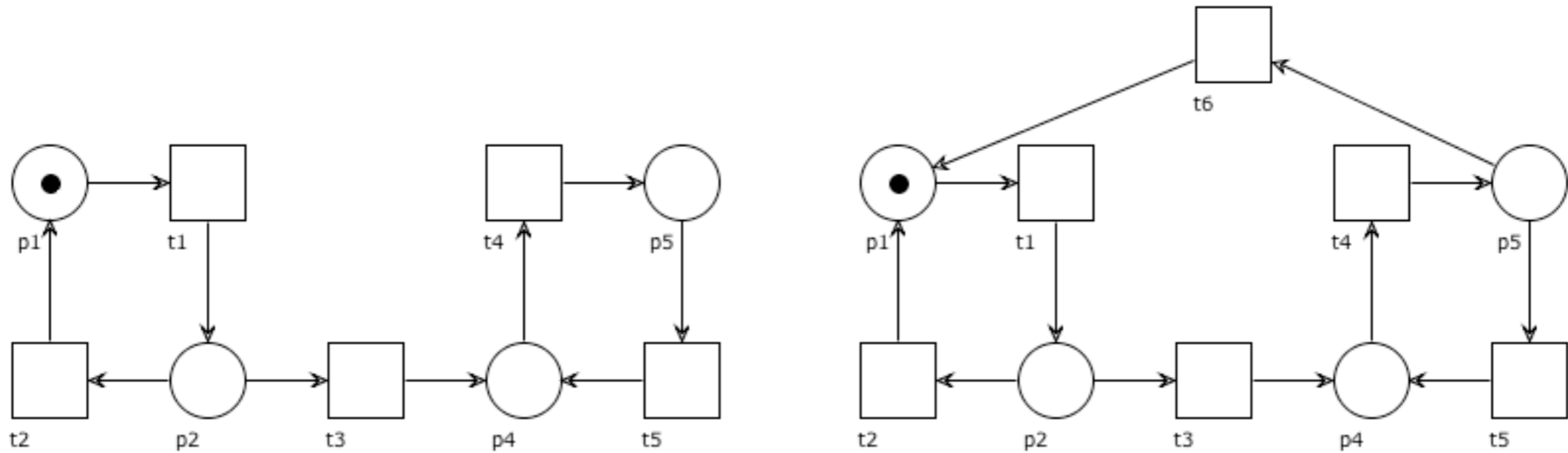
being non-live implies possibly becoming dead  
(but not necessarily in the current marking)

# Place Liveness: example



Which places are live?  
Which are not?  
Which are dead?  
Is the net place live?

# Place Liveness: example



p4, p5  
 p1, p2  
 none  
 No

Which places are live?  
 Which are not?  
 Which are dead?  
 Is the net place live?

all  
 none  
 none  
 Yes

# Place liveness, a formal recap

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{PLive}(p, N) \equiv \forall M \in [M_0], \exists M' \in [M], M'(p) > 0$$

$$\text{NonPLive}(p, N) \equiv \neg \text{PLive}(p, N)$$

$$\text{Dead}(p, N) \equiv \forall M \in [M_0], M(p) = 0$$

$$\text{NonDead}(p, N) \equiv \neg \text{Dead}(p, N)$$

$$\text{PLive}(N) \equiv \forall p \in P, \text{PLive}(p, N)$$

$$\text{NonPLive}(N) \equiv \neg \text{PLive}(N)$$



# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{PLive}(p, N) \equiv \forall M \in [M_0], \exists M' \in [M], M'(p) > 0$$

$$\text{NonPLive}(p, N) \equiv \neg \text{PLive}(p, N)$$

write the explicit formula for NonPLive(p,N)

# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{PLive}(p, N) \equiv \forall M \in [M_0\rangle, \exists M' \in [M\rangle, M'(p) > 0$$

$$\text{NonPLive}(p, N) \equiv \neg \text{PLive}(p, N)$$

$$\equiv \exists M \in [M_0\rangle, \forall M' \in [M\rangle, M'(p) = 0$$

write the explicit formula for NonPLive(p,N)

# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{Dead}(p, N) \equiv \forall M \in [M_0], \quad M(p) = 0$$

$$\text{NonDead}(p, N) \equiv \neg \text{Dead}(p, N)$$

write the explicit formula for NonDead(p,N)

# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{Dead}(p, N) \equiv \forall M \in [M_0\rangle, \quad M(p) = 0$$

$$\text{NonDead}(p, N) \equiv \neg \text{Dead}(p, N)$$

$$\equiv \exists M \in [M_0\rangle, \quad M(p) > 0$$

write the explicit formula for NonDead(p,N)

# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{PLive}(p, N) \equiv \forall M \in [M_0], \exists M' \in [M], M'(p) > 0$$

$$\text{PLive}(N) \equiv \forall p \in P, \text{PLive}(p, N)$$

$$\text{NonPLive}(N) \equiv \neg \text{PLive}(N)$$

write the explicit formula for NonPLive(N)

# Question time

$$N = (P, T, F, M_0) \quad p \in P$$

$$\text{PLive}(p, N) \equiv \forall M \in [M_0], \exists M' \in [M], M'(p) > 0$$

$$\text{PLive}(N) \equiv \forall p \in P, \text{PLive}(p, N)$$

$$\text{NonPLive}(N) \equiv \neg \text{PLive}(N)$$

$$\equiv \exists p \in P, \exists M \in [M_0], \forall M' \in [M], M'(p) = 0$$

write the explicit formula for NonPLive(N)

# Some obvious facts

a system is not live iff it has a transition that can become dead at some reachable marking

a system is not place-live iff it has a place that can become dead at some reachable marking

If a place / transition is dead at  $M$ , then it remains dead at any marking reachable from  $M$   
(the set of dead nodes can only increase during a run)

# True or false?

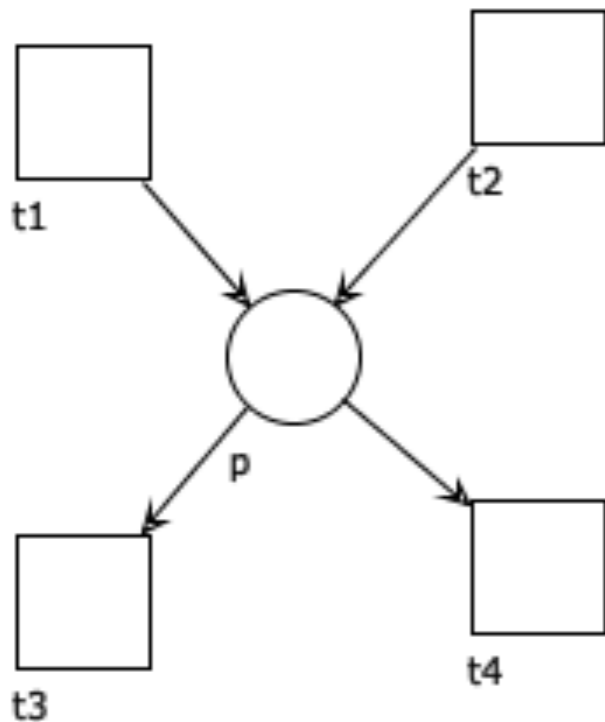
Every transition in the pre- or post-set of a dead place  
is also dead



# True or false?

Every transition in the pre- or post-set of a dead place is also dead

True



suppose  $p$  is dead  
if  $p$  remains empty then  $t_3$  and  $t_4$  cannot fire  
so they are dead

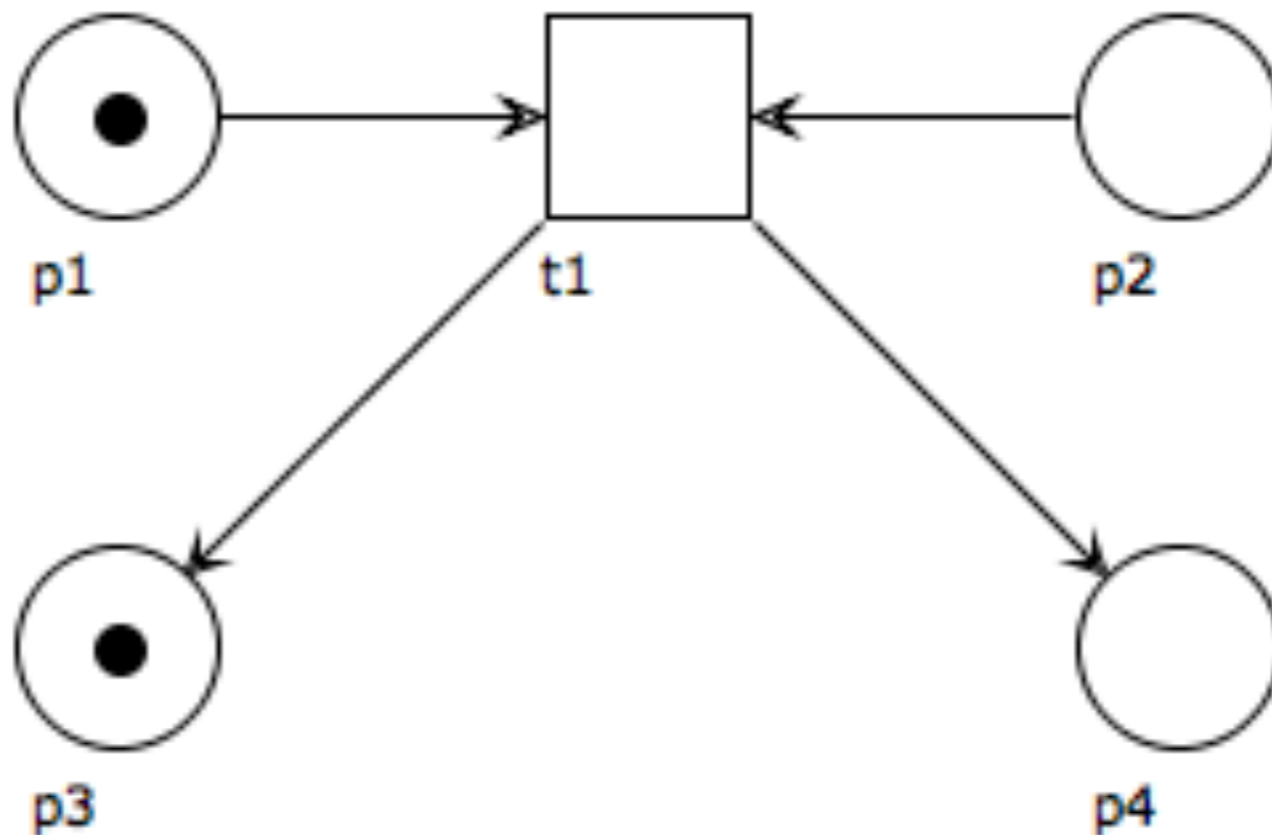
suppose  $p$  is dead  
if  $t_1$  or  $t_2$  could fire then some token would arrive in  $p$   
since no token can arrive in  $p$   
it means that  $t_1$  and  $t_2$  will never fire

# True or false?

Every place in the pre- or post-set of a dead transition  
is also dead

# True or false?

Every place in the pre- or post-set of a dead transition is also dead



False

$t_1$  is dead but  $p_1$  and  $p_3$  are not dead

# Liveness implies place-liveness

**Proposition:** Live systems are also place-live

Assume the net is live

Take any  $p \in P$  and  $M \in [M_0\rangle$

We want to find  $M' \in [M_0\rangle$  s.t.  $M'(p) > 0$

Take any  $t \in \bullet p \cup p \bullet$

By liveness: there are  $M'', M''' \in [M_0\rangle$  s.t.  $M'' \xrightarrow{t} M'''$

Then  $M''(p) > 0$  or  $M'''(p) > 0$

# Place liveness on the occurrence graph

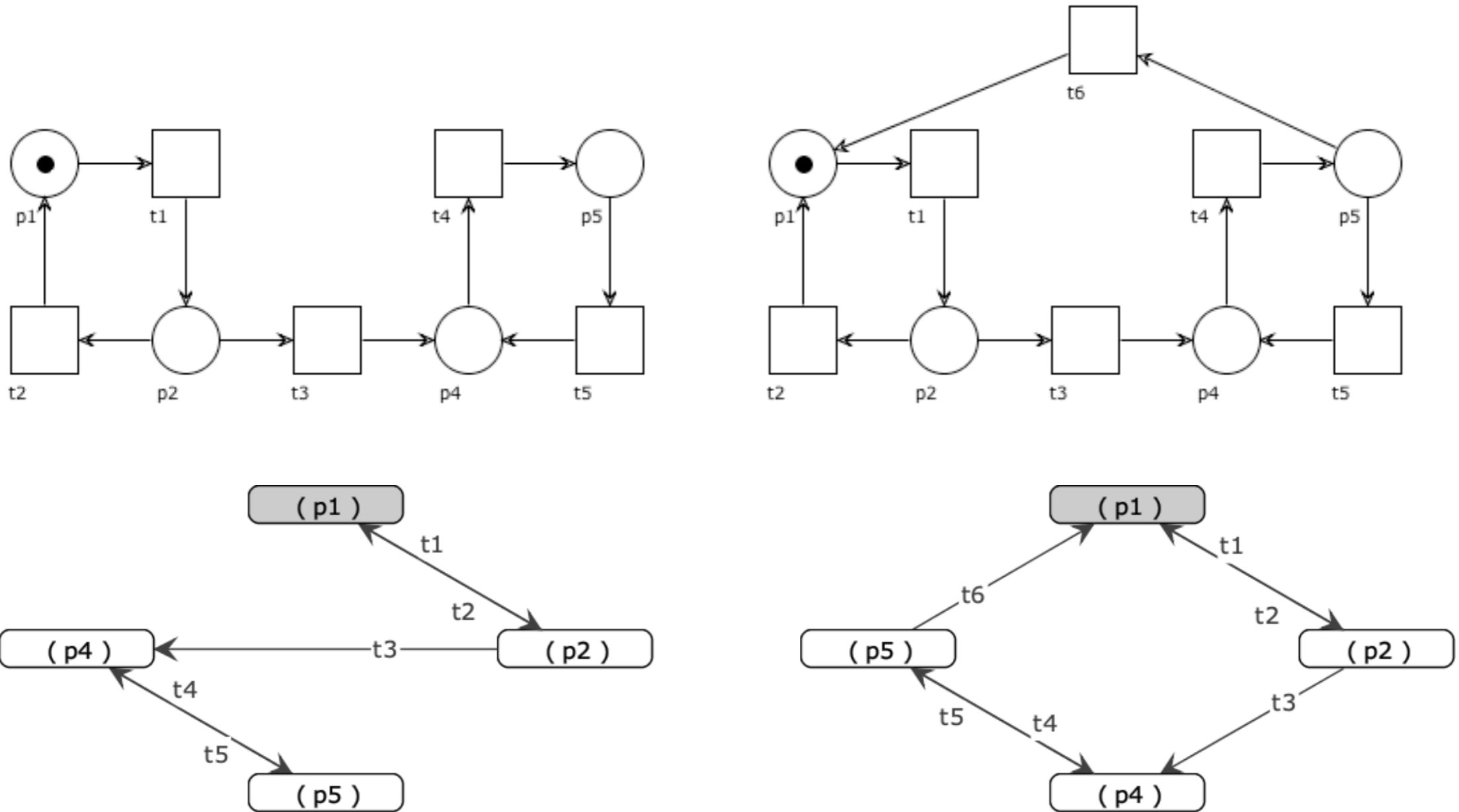
A place  $p$  is live  
iff

From any node of the occurrence graph we can reach a  
node with a token in  $p$

A place  $p$  is dead (at  $M_0$ )  
iff

All the nodes of the occurrence graph have no token in  $p$

# Place Liveness: example



# Exercise

Draw a net that  
is place-live but not live  
(if you can)

# Deadlock-freedom

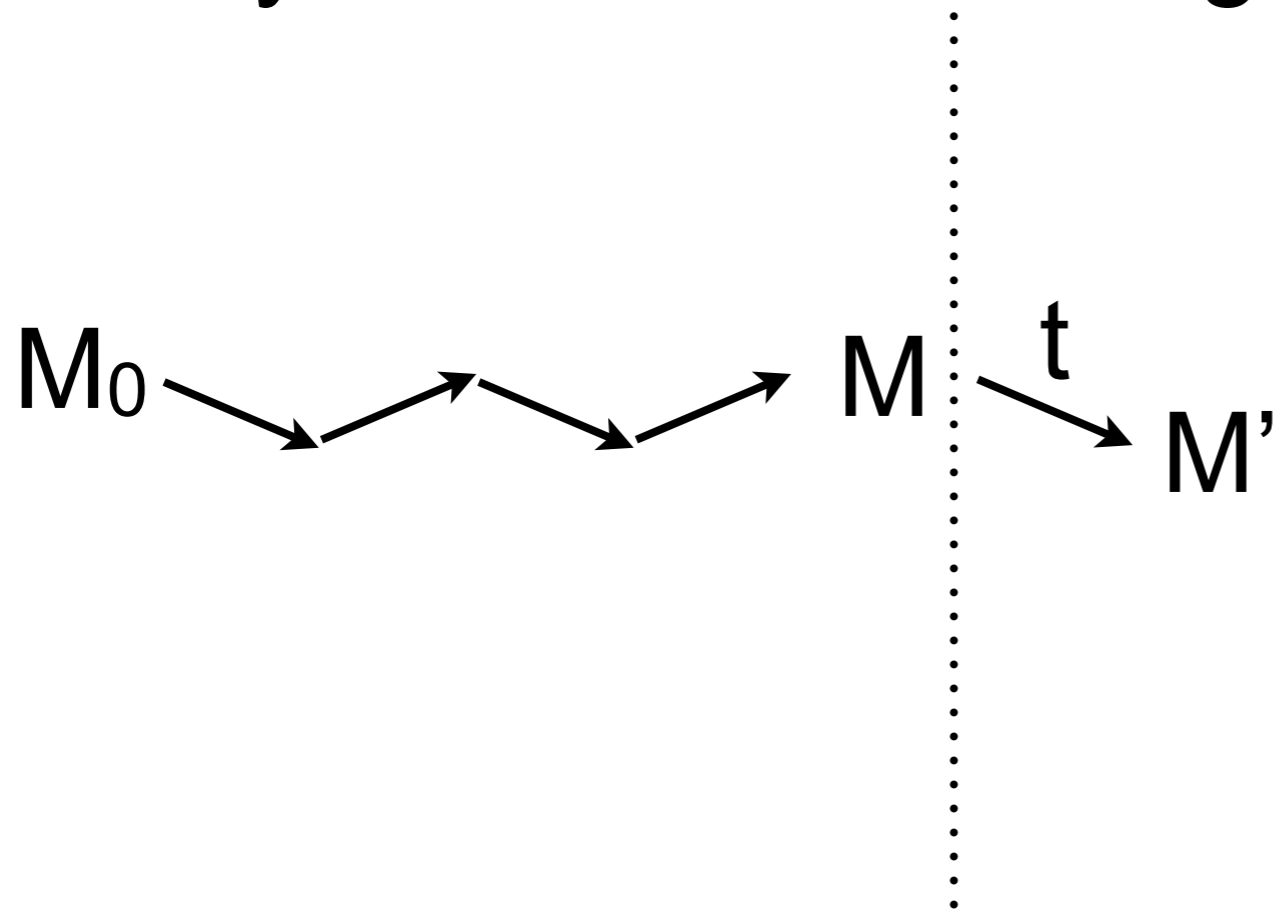
A Petri net is **deadlock free**, if every reachable marking enables some transition

In other words, we are guaranteed that at any point in time of the computation, some transition can be fired



# Deadlock-freedom illustrated

For any reachable marking  $M$ ...



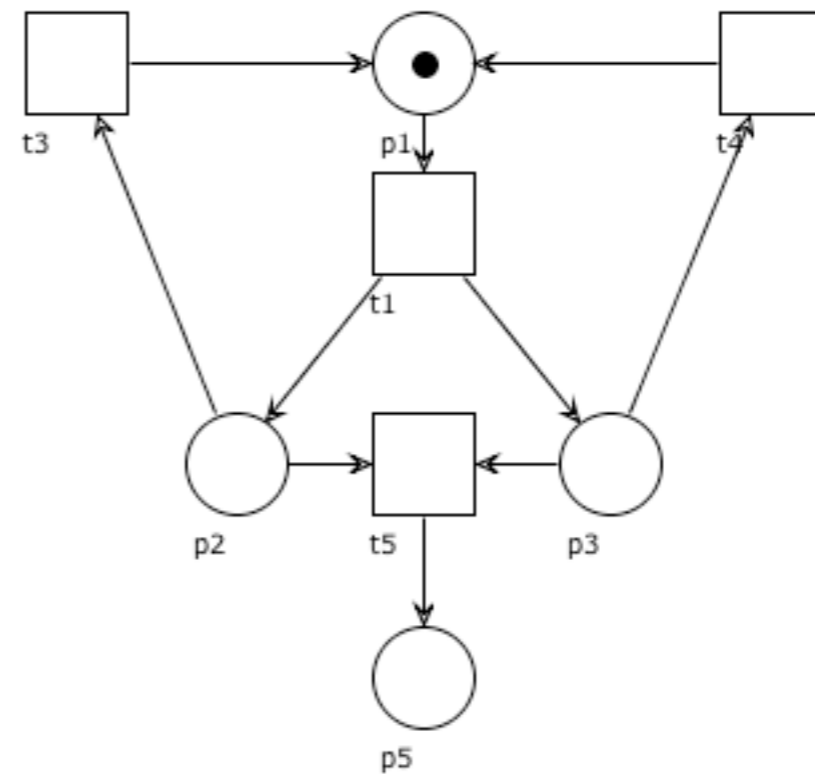
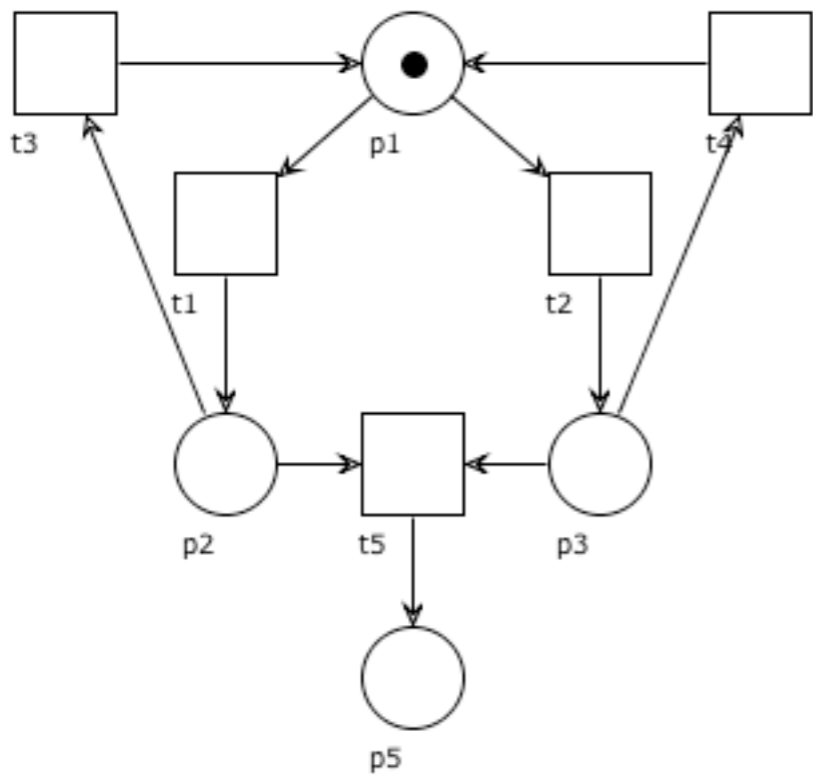
...can we fire some transition?

# Deadlock freedom, formally

$(P, T, F, M_0)$

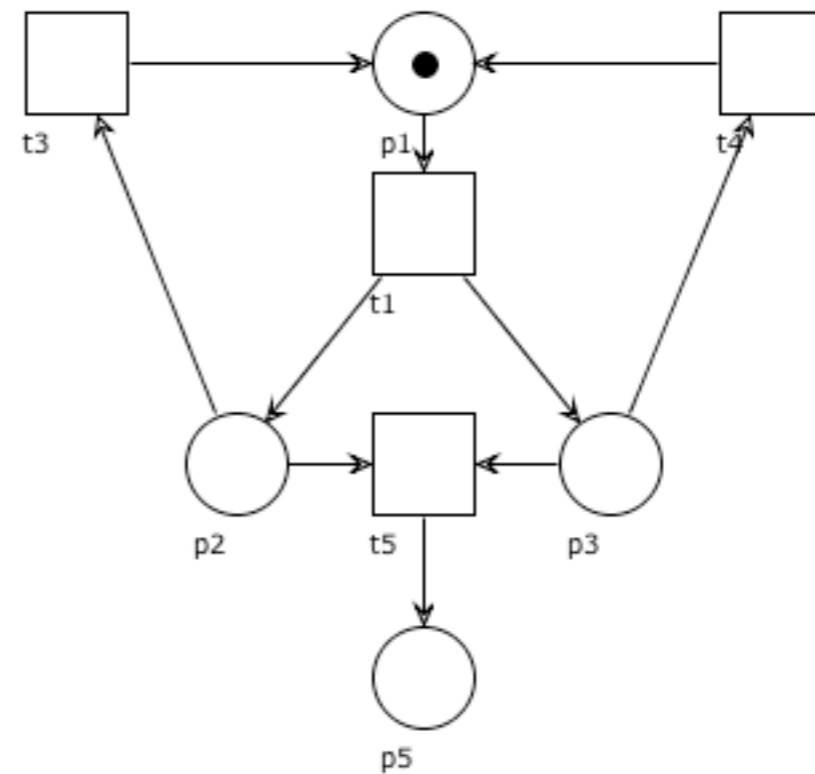
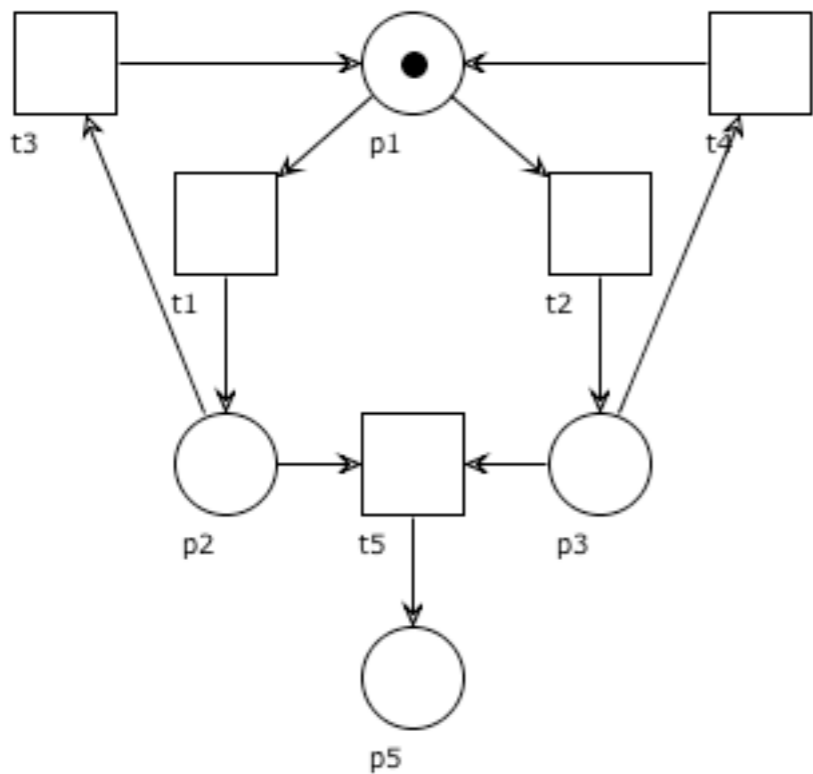
$\forall M \in [M_0 \rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$

# Deadlock-freedom: example



Is the net deadlock-free?

# Deadlock-freedom: example



**Yes** Is the net deadlock-free? **No**

$$[p_1] = \{ p_1, p_2, p_3 \}$$

$$[p_1] = \{ p_1, p_2 + p_3, p_5, \dots \}$$

$$p_1 \rightarrow \quad p_2 \rightarrow \quad p_3 \rightarrow$$

$$p_5 \nrightarrow$$

# Deadlock freedom on the occurrence graph

A net is deadlock free  
iff

Every node of the occurrence graph has an outgoing arc

# Question time

Does liveness imply deadlock-freedom?

(Can you exhibit a live Petri net that is not deadlock-free?)

Does deadlock-freedom imply liveness?

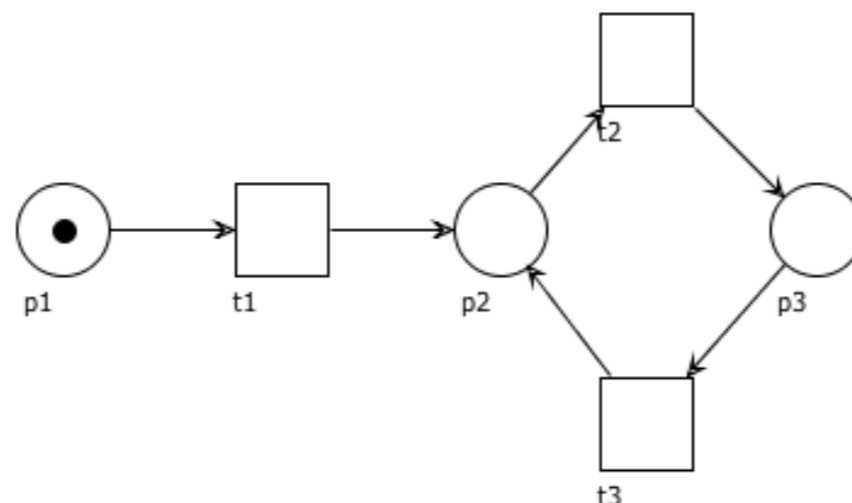
(Can you exhibit a deadlock-free net that is not live?)

# Question time

Does liveness imply deadlock-freedom? **YES**  
(Can you exhibit a live Petri net that is not deadlock-free?)

**NO**

Does deadlock-freedom imply liveness? **NO**  
(Can you exhibit a deadlock-free net that is not live?) **YES**



# Liveness implies deadlock freedom

**Lemma** If  $(P, T, F, M_0)$  is live, then it is deadlock-free

By contradiction, let  $M \in [M_0 \rangle$ , with  $M \not\rightarrow$

Let  $t \in T$  ( $T$  cannot be empty).

By liveness,  $\exists M' \in [M \rangle$  with  $M' \xrightarrow{t}$ .

Since  $M \not\rightarrow$ , we have  $[M \rangle = \{M\}$ .

Therefore  $M = M' \xrightarrow{t}$ , which is absurd.



# Digression: for next exercises

Contraposition

$$P \Rightarrow Q \quad \equiv \quad (\neg Q) \Rightarrow \neg P$$

# Exercises

Prove each of the following properties  
or give some counterexamples

If a system is not place-live, then it is not live

If a system is not live, then it is not place-live

If a system is place-live, then it is deadlock-free

If a system is deadlock-free, then it is place-live

# $k$ -Boundedness

Let  $k$  be a natural number

A place  $p$  is  **$k$ -bounded** if no reachable marking has more than  $k$  tokens in place  $p$

A net is  **$k$ -bounded** if all of its places are  $k$ -bounded

In other words, if a net is  $k$ -bounded, then  $k$  is a capacity constraint that can be imposed over places without any risk of causing “overflow”

# Safe nets

A place  $p$  is **safe** if it is 1-bounded

A net is **safe** if all of its places are safe

In other words, if the net is safe, then we know that, in any reachable marking, each place contains one token at most

# Boundedness

A place  $p$  is **bounded** if it is  $k$ -bounded for some natural number  $k$

A net is **bounded** if all of its places are bounded

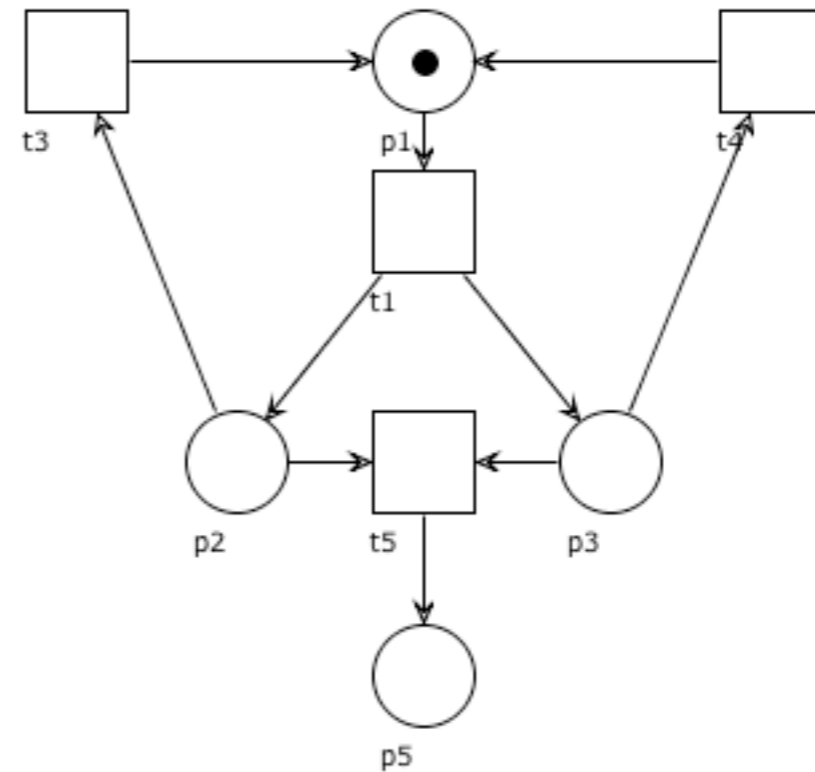
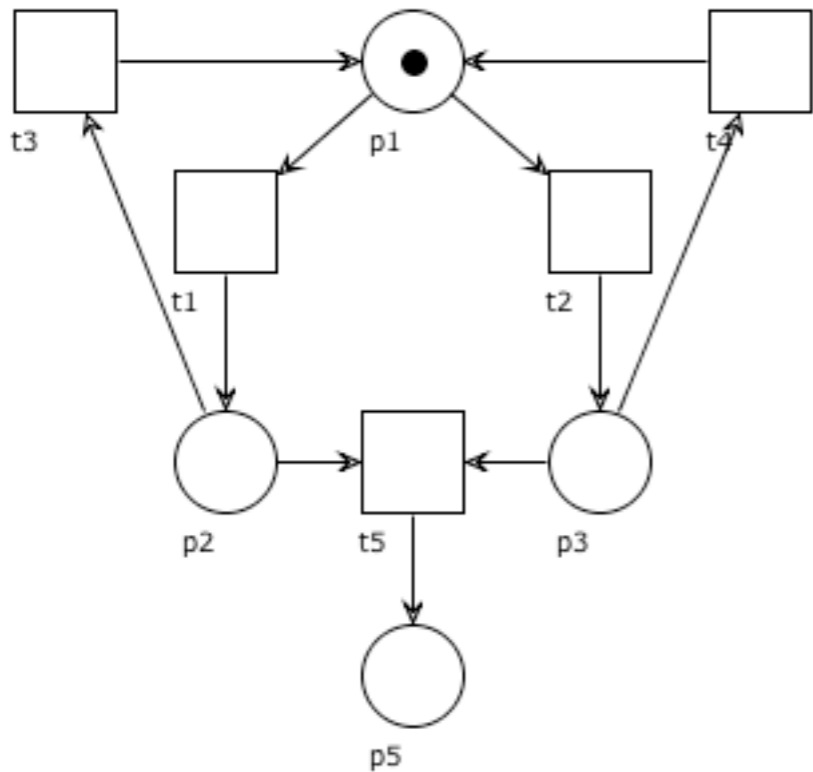
A net is **unbounded** if it is not bounded (i.e., there is at least one place in which any number of tokens can appear)

# Boundedness, formally

$$(P, T, F, M_0)$$

$$\exists k \in \mathbb{N}, \quad \forall M \in [M_0 \rangle, \quad \forall p \in P, \quad M(p) \leq k$$

# Boundedness: example



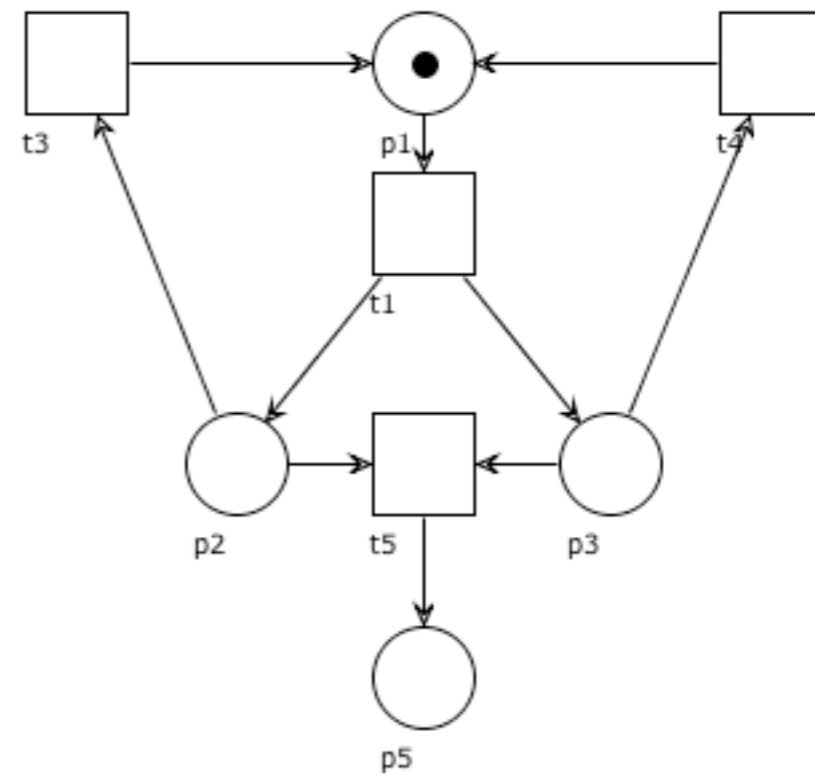
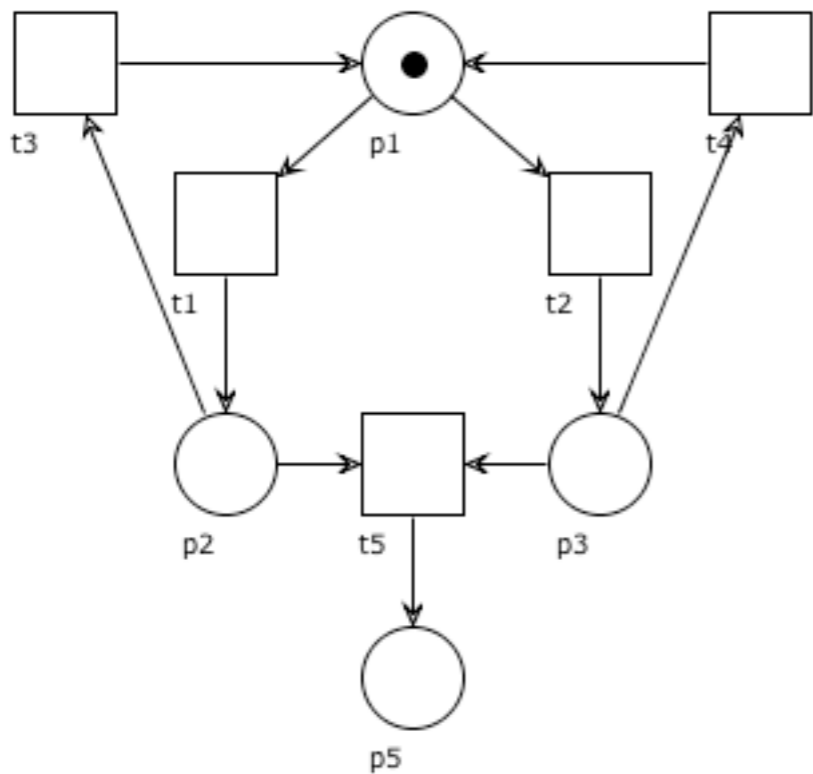
Which places are bounded?

Is the net bounded?

Which places are safe?

Is the net safe?

# Boundedness: example



all  
Yes  
all  
Yes

Which places are bounded?

Is the net bounded?

Which places are safe?

Is the net safe?

none

No

none

No



# Boundedness and the reachability graph

A system is bounded  
iff  
its reachability graph is finite

# Boundedness implies finiteness

**Theorem:** If a system is **bounded**  
then its reachability graph is finite

**Proof:** if the system is bounded there exists  $k$   
such that each place contains at most  $k$  tokens.

If there are  $n$  places it means that there are at  
most  $(k+1)^n$  reachable markings.

Hence the occurrence graph has a finite number  
of nodes

# Finiteness implies boundedness

**Theorem:** A system is **bounded**  
if its reachability graph is finite

**Proof:** for each node  $M$  we take  $k_M$  be the  
maximum number of tokens in the same place.

Then we let  $k$  be the largest among all  $k_M$   
 $k = \max \{k_M \mid M \text{ is a node of the graph}\}$   
( $k$  exists because the reachability graph is finite)

Clearly the system is  $k$ -bounded and thus bounded

# Cyclicity (aka Reversibility)

A marking  $M$  is a **home marking** if it can be reached from every reachable marking

A net is **cyclic** (or **reversible**) if its initial marking is a home marking

# Orthogonal properties

Liveness, boundedness and cyclicity are independent of each other

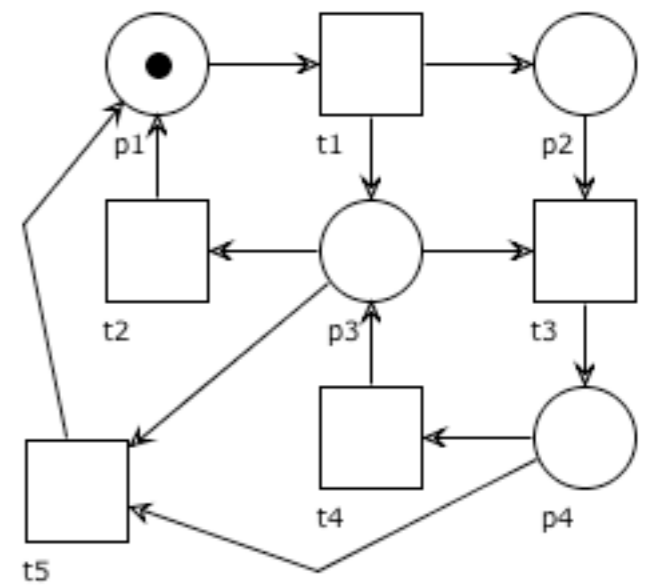
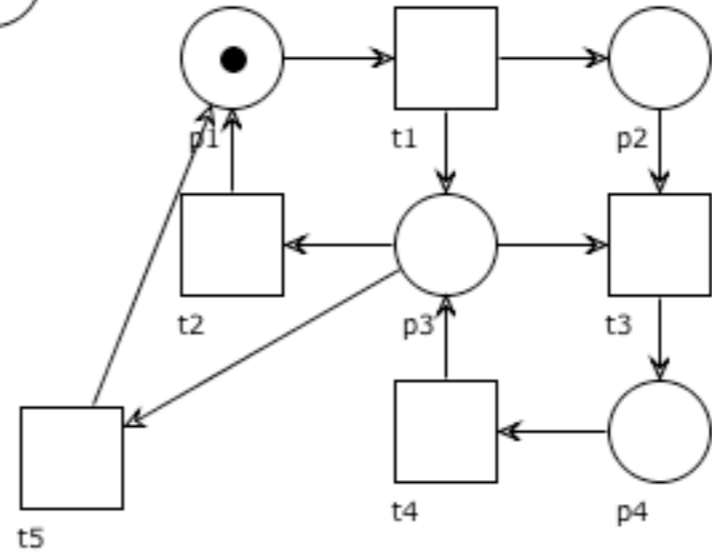
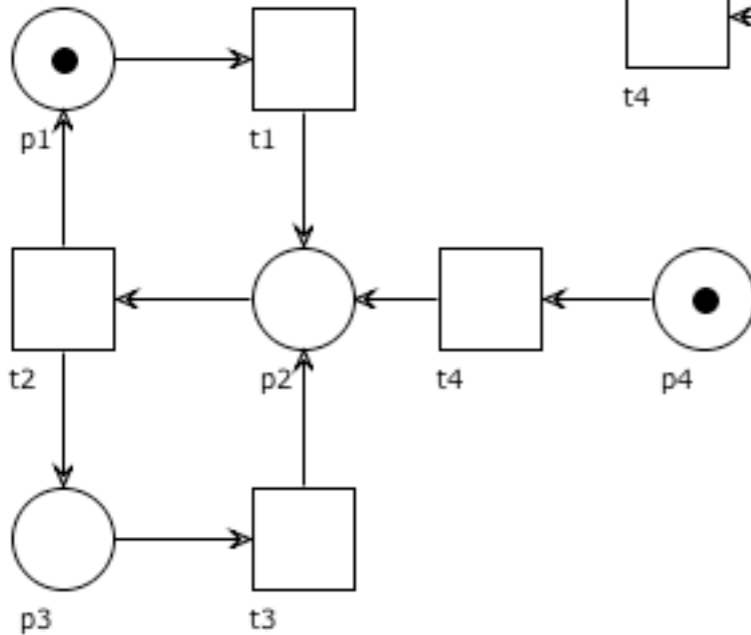
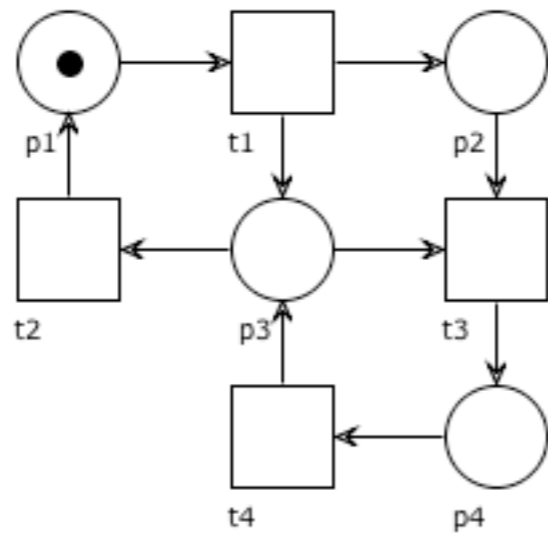
In other words, you can find nets that satisfy any arbitrary combination of the above three properties (and not the others)

# Exercises

Write the formal definition of the predicates  
 $\text{Home}(M,N)$  and  $\text{Cyclic}(N)$

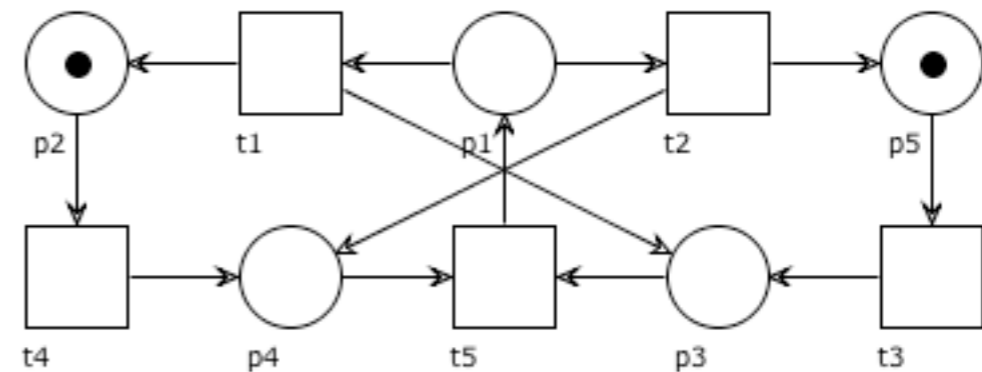
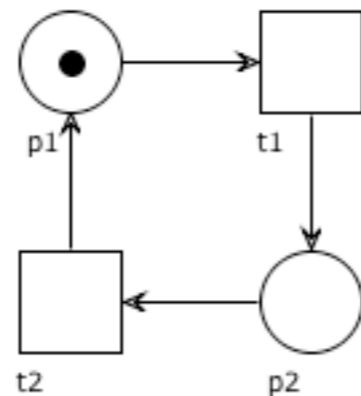
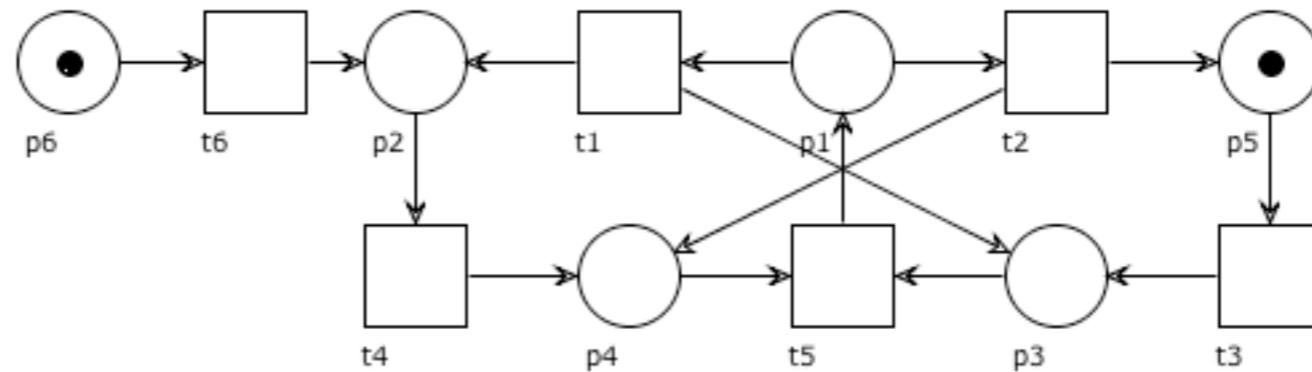
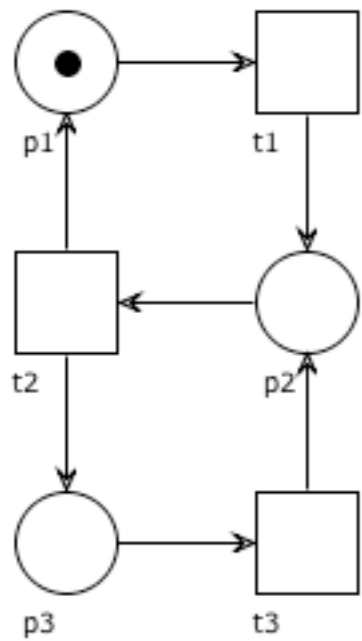
# Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic



# Exercises

For each of the following nets, say if they are live, deadlock-free, bounded, safe, cyclic





# Petri nets: structural properties

# Structural properties

All the properties we have seen so far are  
**behavioural (or dynamic)**

(i.e. they depend on the initial marking and firing rules)

It is sometimes interesting to connect them to  
**structural** properties

(i.e. the shape of the graph representing the net)

This way we can give **structural characterization** of  
behavioural properties for a class of nets  
(computationally less expensive to check)

# A matter of terminology

To better reflect the above distinction, it is frequent:

to use the term **net system** for denoting a Petri net  
**with** a given initial marking  
(we study behavioural properties of systems)

to use the term **net** for denoting a Petri net  
**without** specifying any initial marking  
(we study structural properties of nets)

# Paths and circuits

$$(P, T, F)$$

A **path** is a non-empty sequence of nodes  $x_1x_2\dots x_k$  such that

$$(x_i, x_{i+1}) \in F \quad \text{for every } 1 \leq i \leq k$$

and we say it leads from  $x_1$  to  $x_k$

A path  $x_1x_2\dots x_k$  is called a **circuit** if

$$\text{all its nodes are distinct and } (x_k, x_1) \in F$$

since there is no node  $x$  with  $(x, x) \in F$ , any circuit has at least two nodes

# Paths and circuits

$$(P, T, F)$$

An **undirected path** is a non-empty sequence of nodes  $x_1x_2\dots x_k$  s.t.

$$(x_i, x_{i+1}) \in (F \cup F^{-1}) \quad \text{for every } 1 \leq i \leq k$$

(denotes the inverse of a binary relation)

$$F^{-1} = \{ (y, x) \mid (x, y) \in F \}$$

(a path where we disregard the orientation of arcs)

# Connectedness

A net  $(P, T, F)$  is **weakly connected** if there is an undirected path between any two distinct nodes

A net  $(P, T, F)$  is **strongly connected** if there is a path between any two distinct nodes

# Connectedness, again

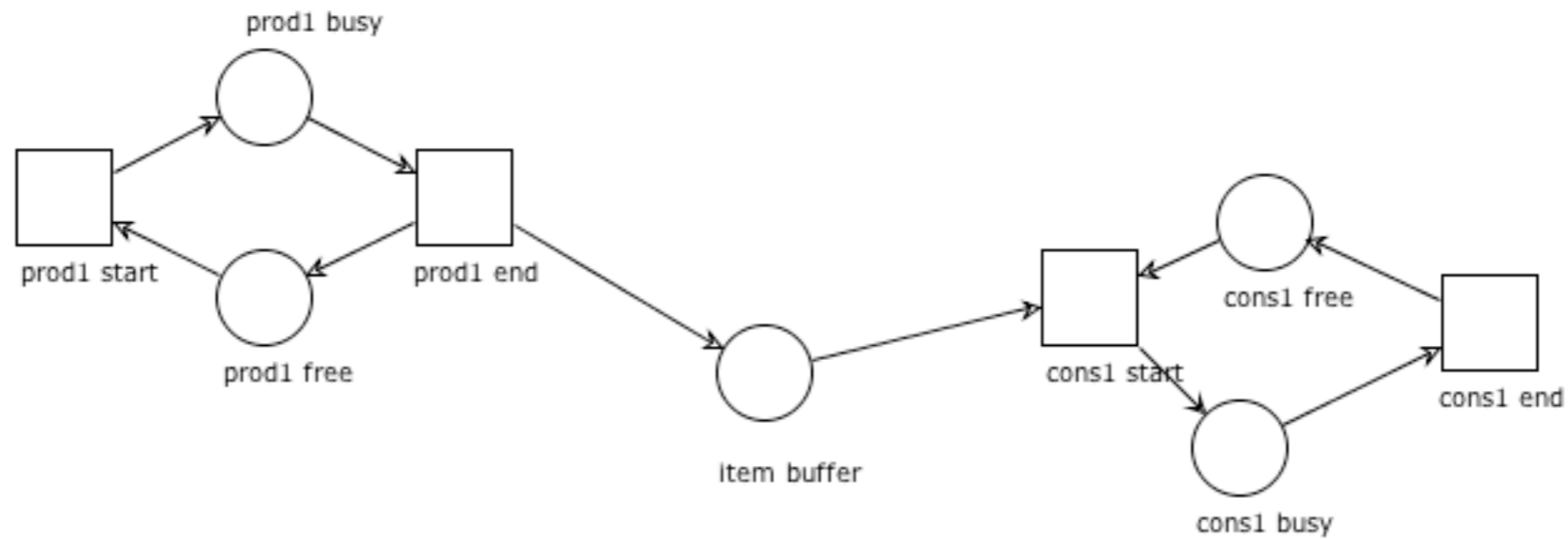
A net  $(P,T,F)$  is **weakly connected**  
iff

it cannot be splitted in separated components

A weakly connected net is **strongly connected**  
iff

for every arc  $(x,y)$  there is a path from  $y$  to  $x$

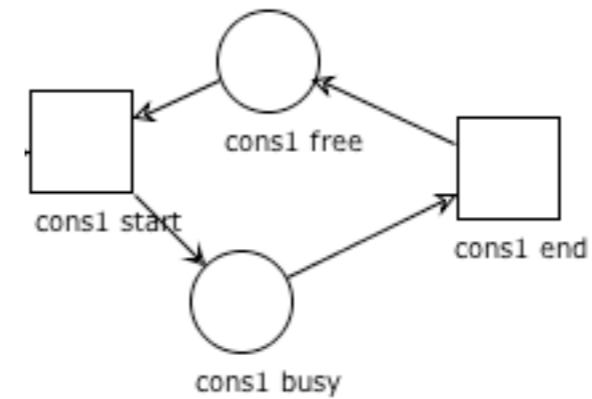
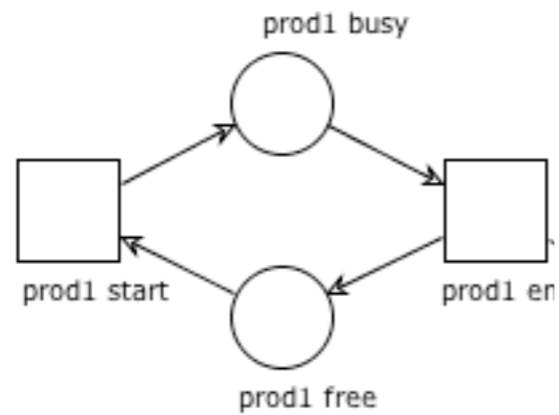
# Examples



weakly connected  
not strongly connected

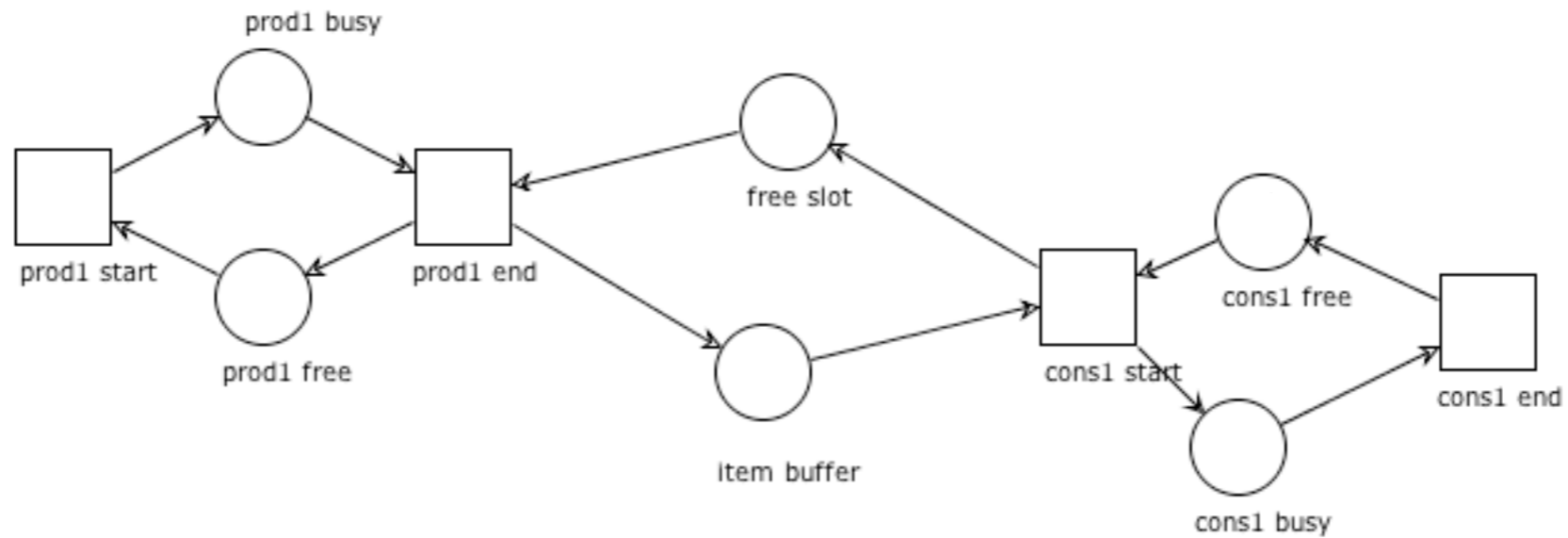


# Examples



not weakly connected  
not strongly connected

# Examples



weakly connected  
strongly connected

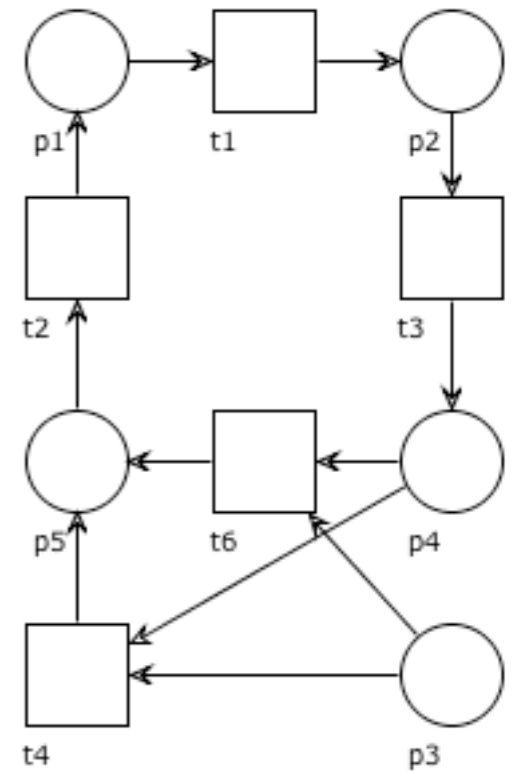
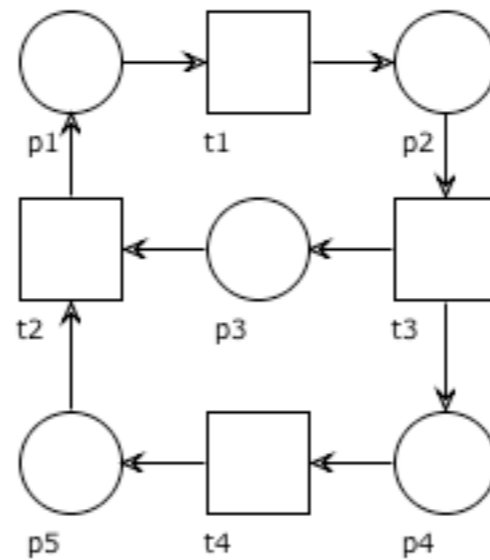
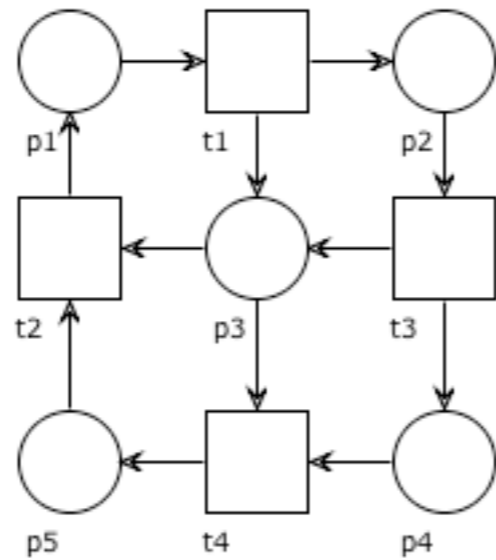
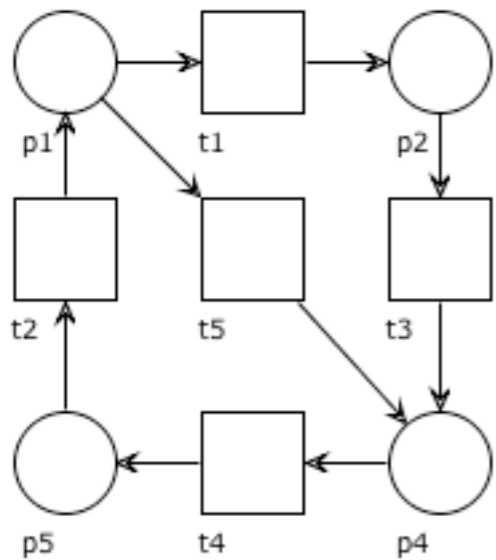
# A note

In the following we will consider (implicitly) weakly connected nets only

(if they are not, then we can study each of their subsystems separately)

# Question time

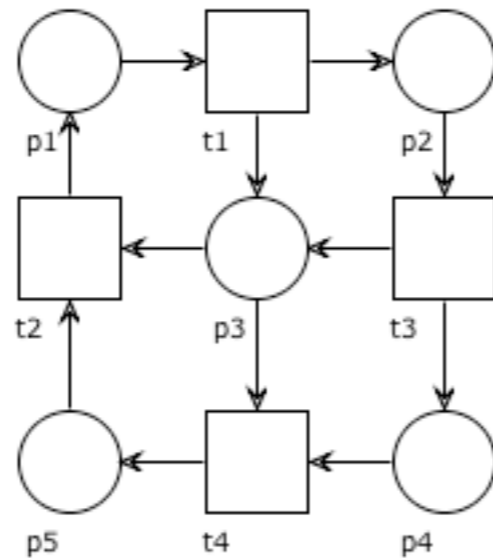
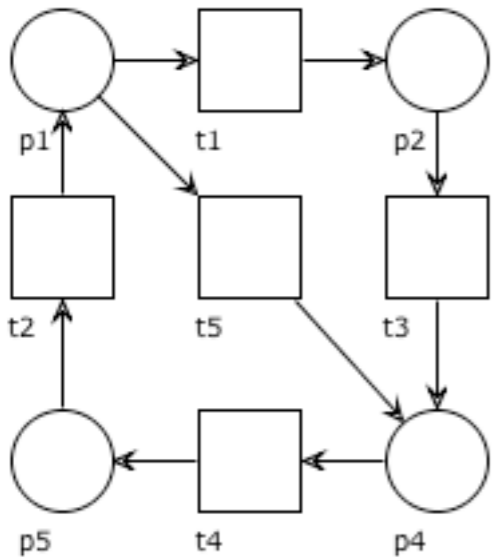
Is the net strongly connected?



# Question time

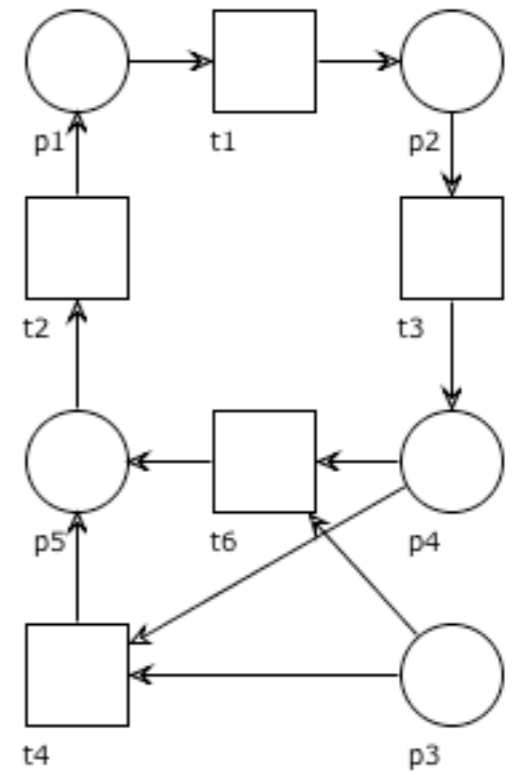
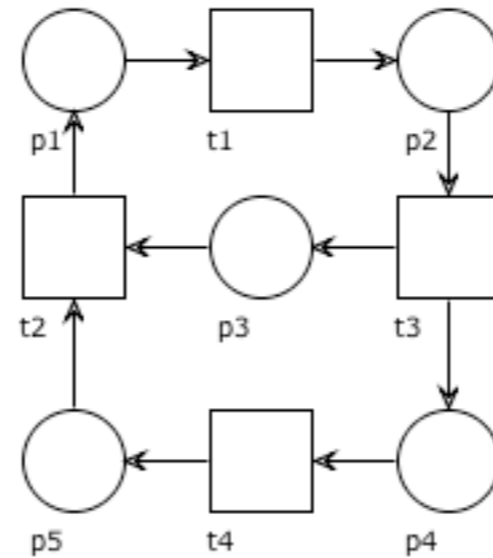
Is the net strongly connected?

YES



YES

YES



NO

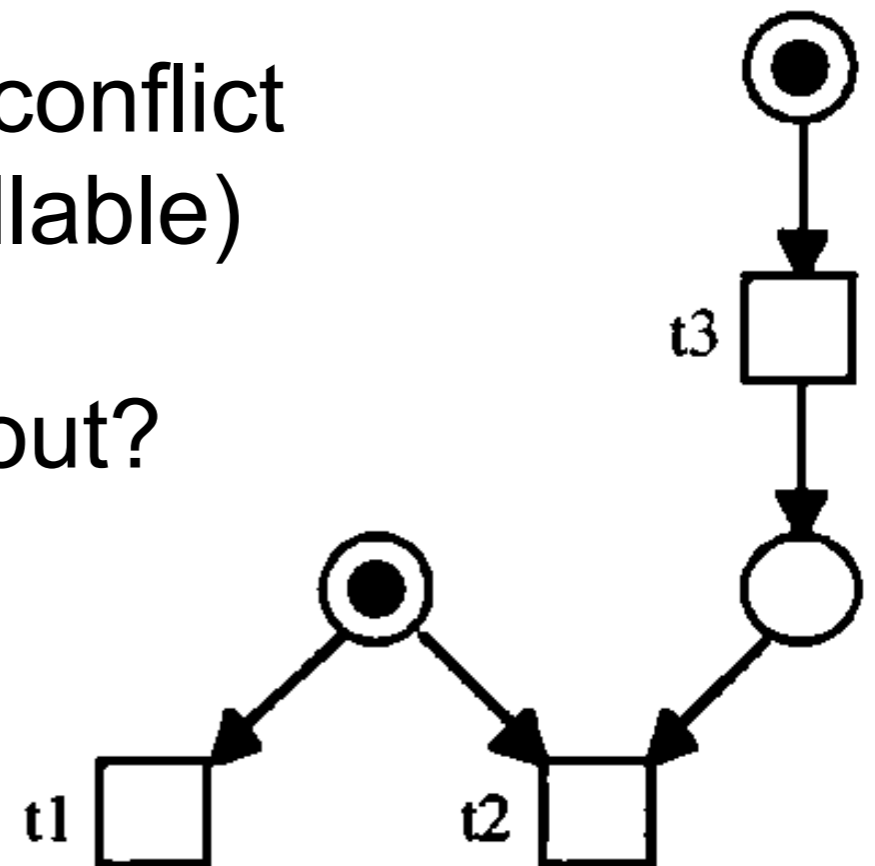
# Interference of conflicts and synch

Typical situation:

initially t1 and t2 are not in conflict

but when t3 fires they are in conflict  
(the firing of t3 is not controllable)

How to rule this situation out?



# S-systems / S-nets

A Petri net is called **S-system** if every transition has one input place and one output place  
(S comes from *Stellen*, the German word for place)

This way any synchronization is ruled out

The theory of S-systems is very simple

# T-systems / T-nets

A Petri net is called **T-system** if every place has one input transition and one output transition

This way all choices/conflicts are ruled out

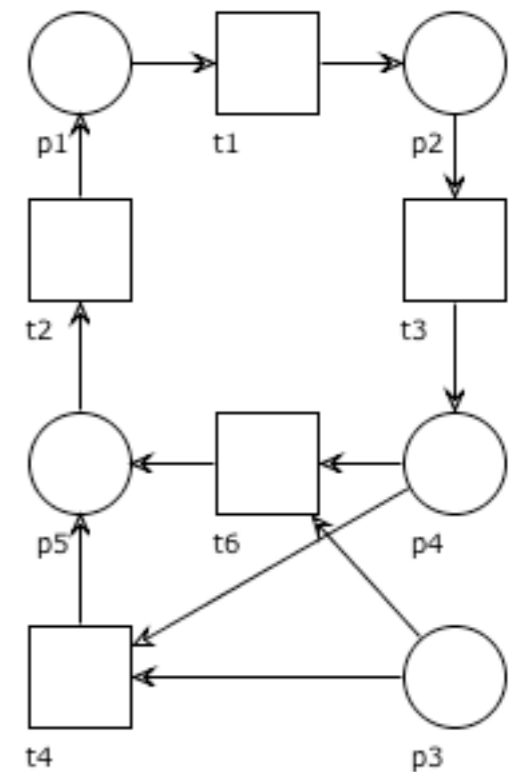
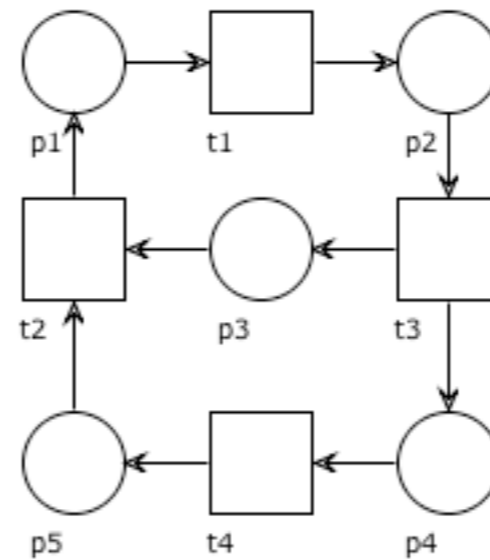
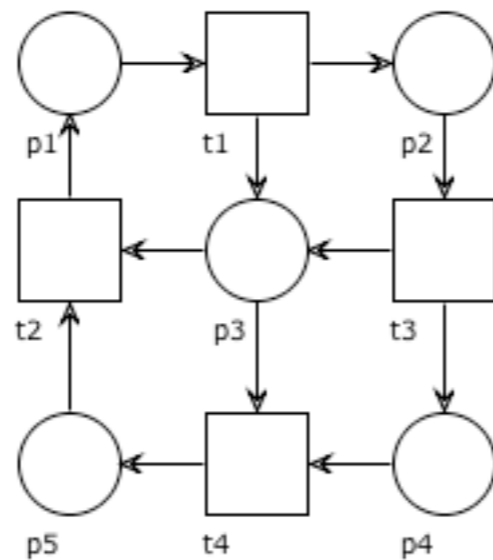
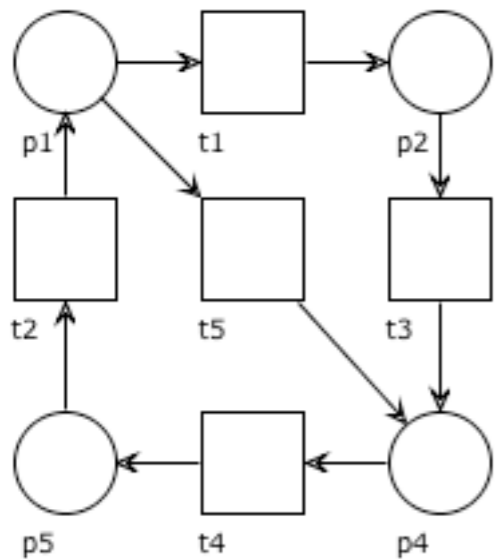
T-systems are concurrent but essentially deterministic

T-systems have been studied extensively since the early Seventies



# Question time

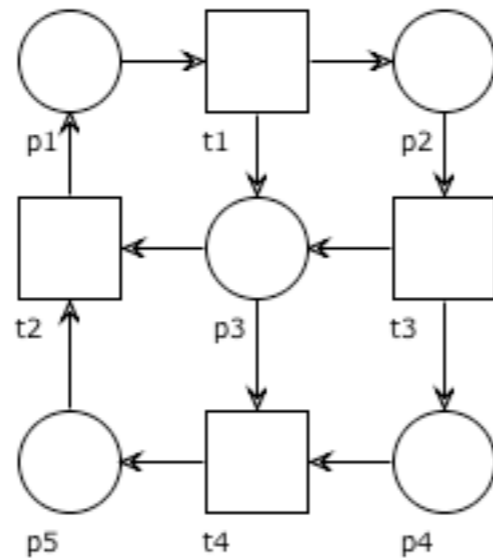
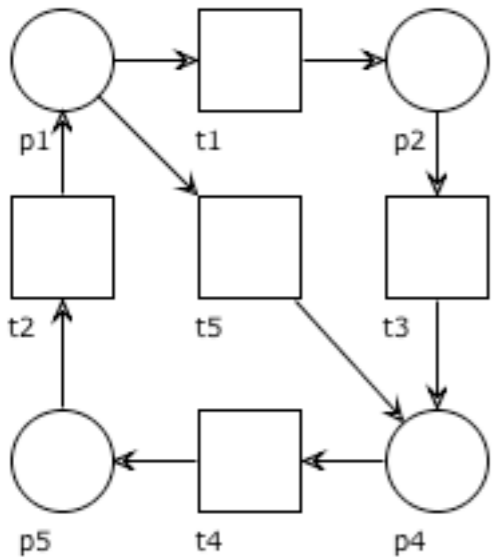
Is the net an S-net, a T-net?



# Question time

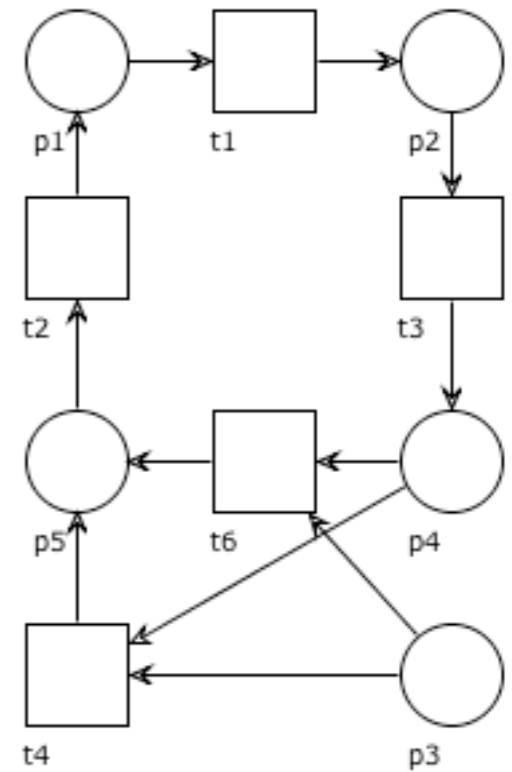
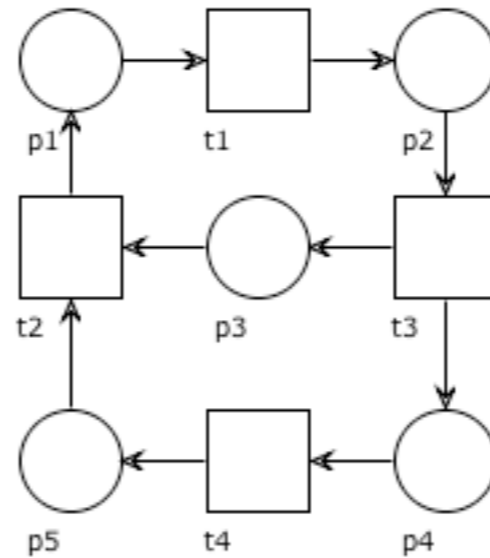
Is the net an S-net, a T-net?

S-net  
T-net



S-net  
T-net

S-net  
T-net



S-net  
T-net

# Free-choice nets

The aim is to avoid that a choice between transitions is influenced by the rest of the system

Easiest way:

keep places with more than one output transition apart from transitions with more than one input place

In other words, if  $(p,t)$  is an arc, then it means that  $t$  is the only output transition of  $p$  (no conflict)

OR

$p$  is the only input place of  $t$  (no synch)

# Free-choice systems / nets

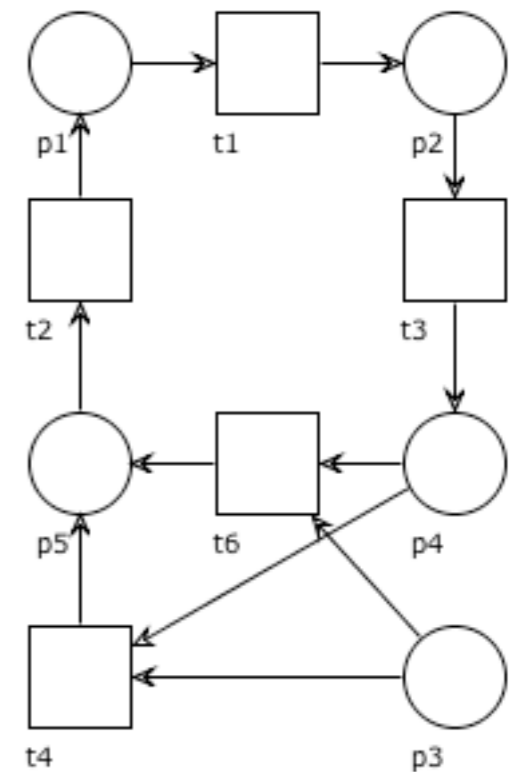
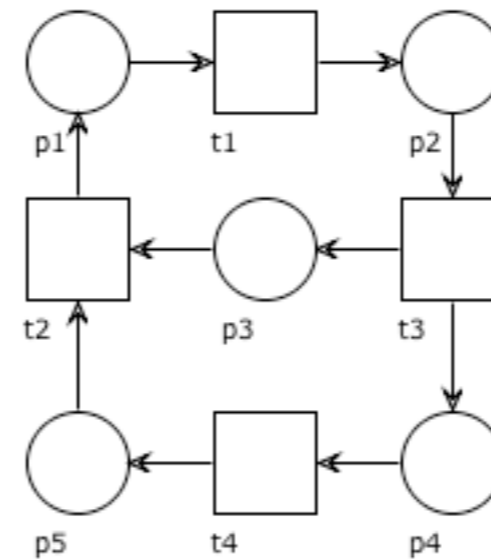
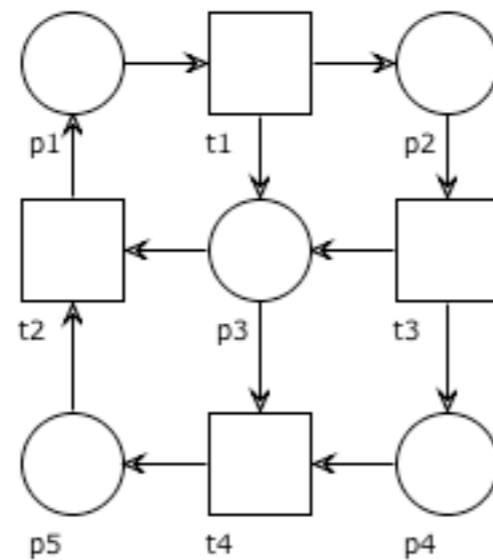
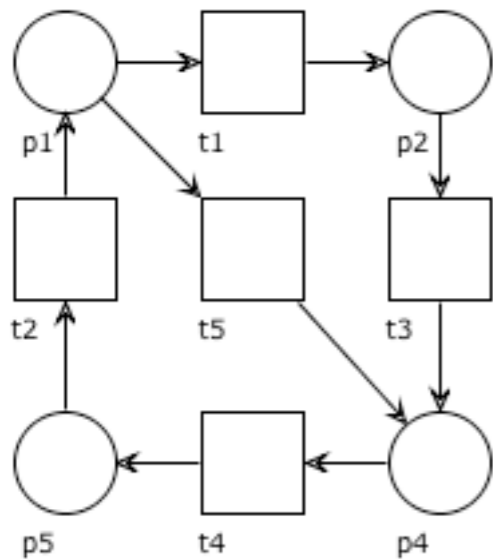
But we can study a slightly more general class of nets by requiring a weaker constraint

A Petri net is **free-choice** if  
for any pair of transitions  
their pre-sets are either disjoint or equal

or, equivalently, if  
for any pair of places  
their post-sets are either disjoint or equal

# Question time

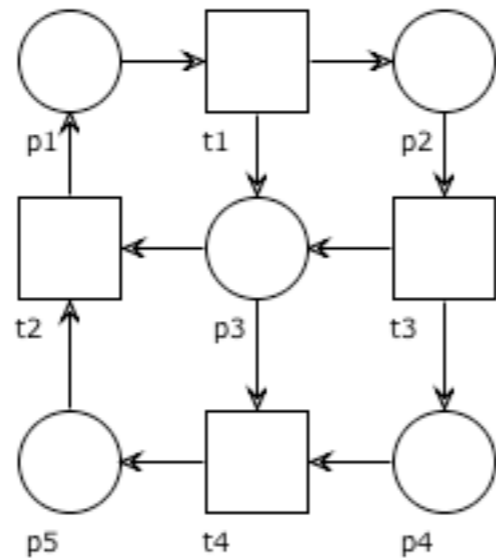
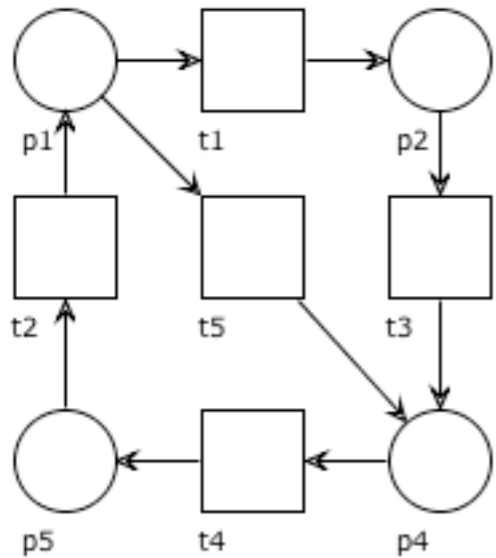
Is the net free-choice?



# Question time

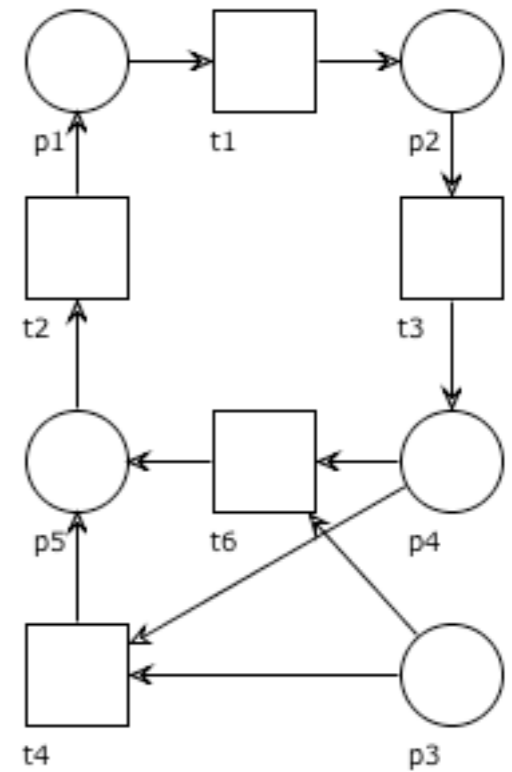
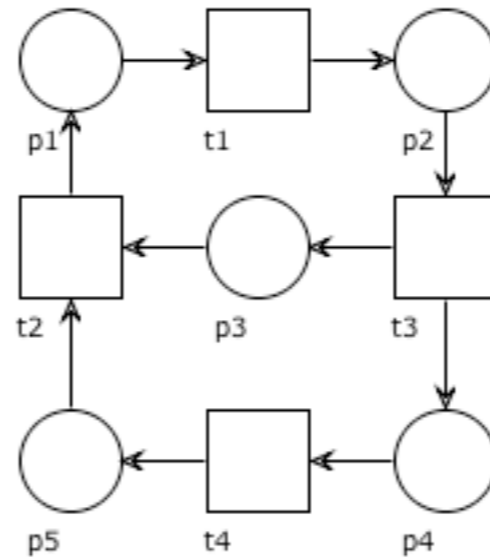
Is the net free-choice?

FC-net



FC-net

FC-net



FC-net

# Exercises

Prove that every S-net is free-choice

Prove that every T-net is free-choice

Show a net for each area of the Eulero-Venn diagram below

