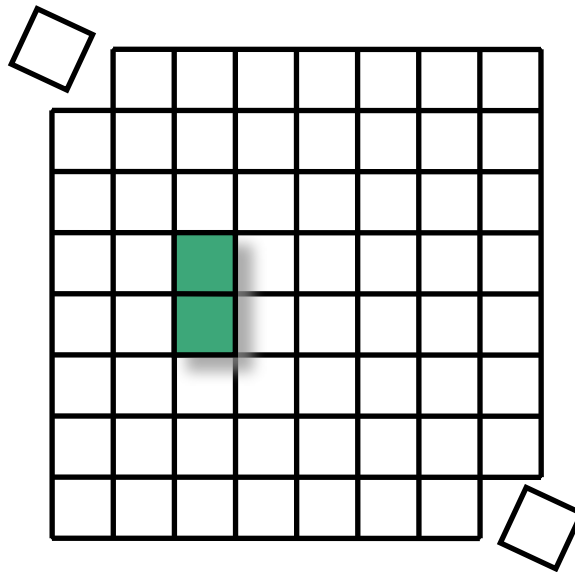


Object

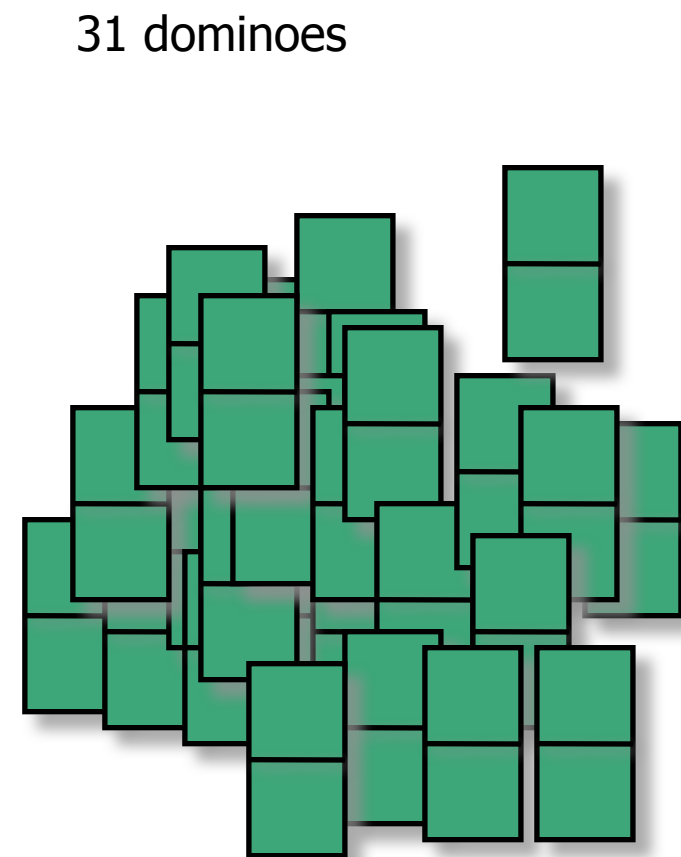
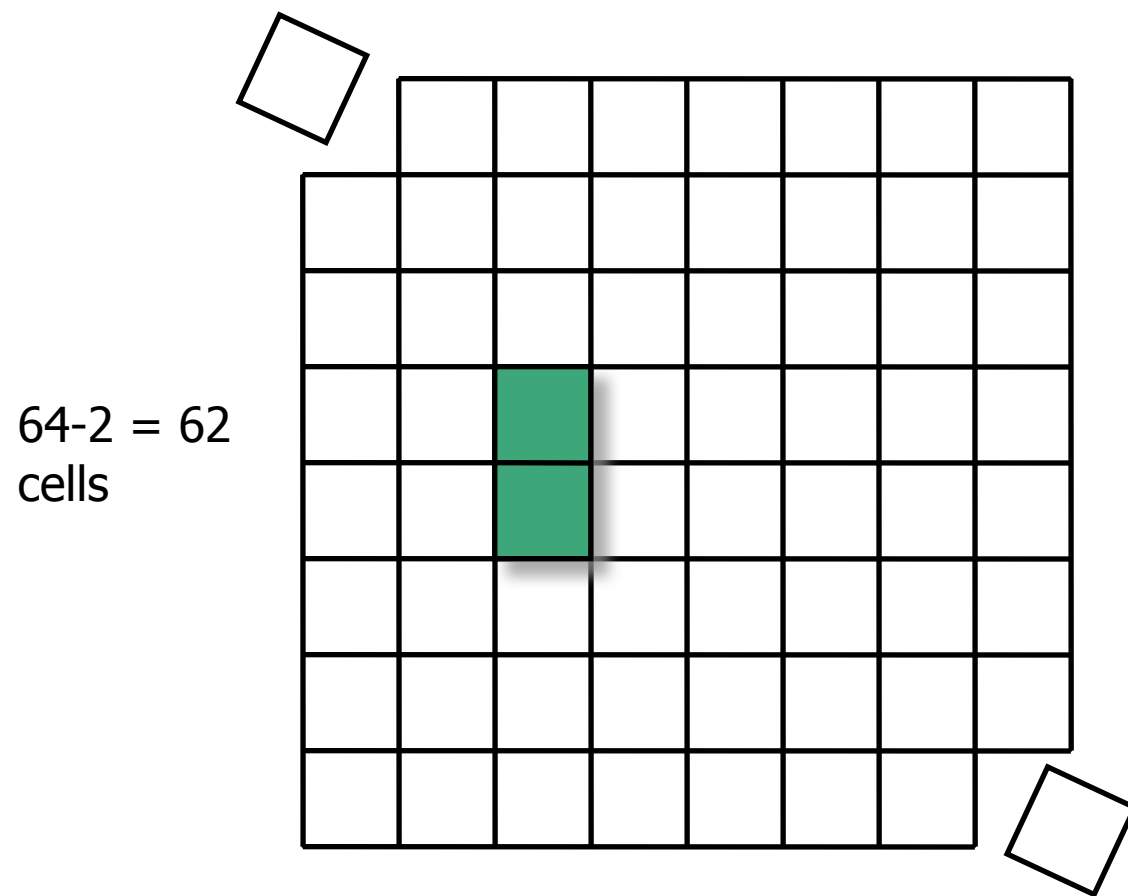


We introduce two relevant kinds of invariants for
Petri nets

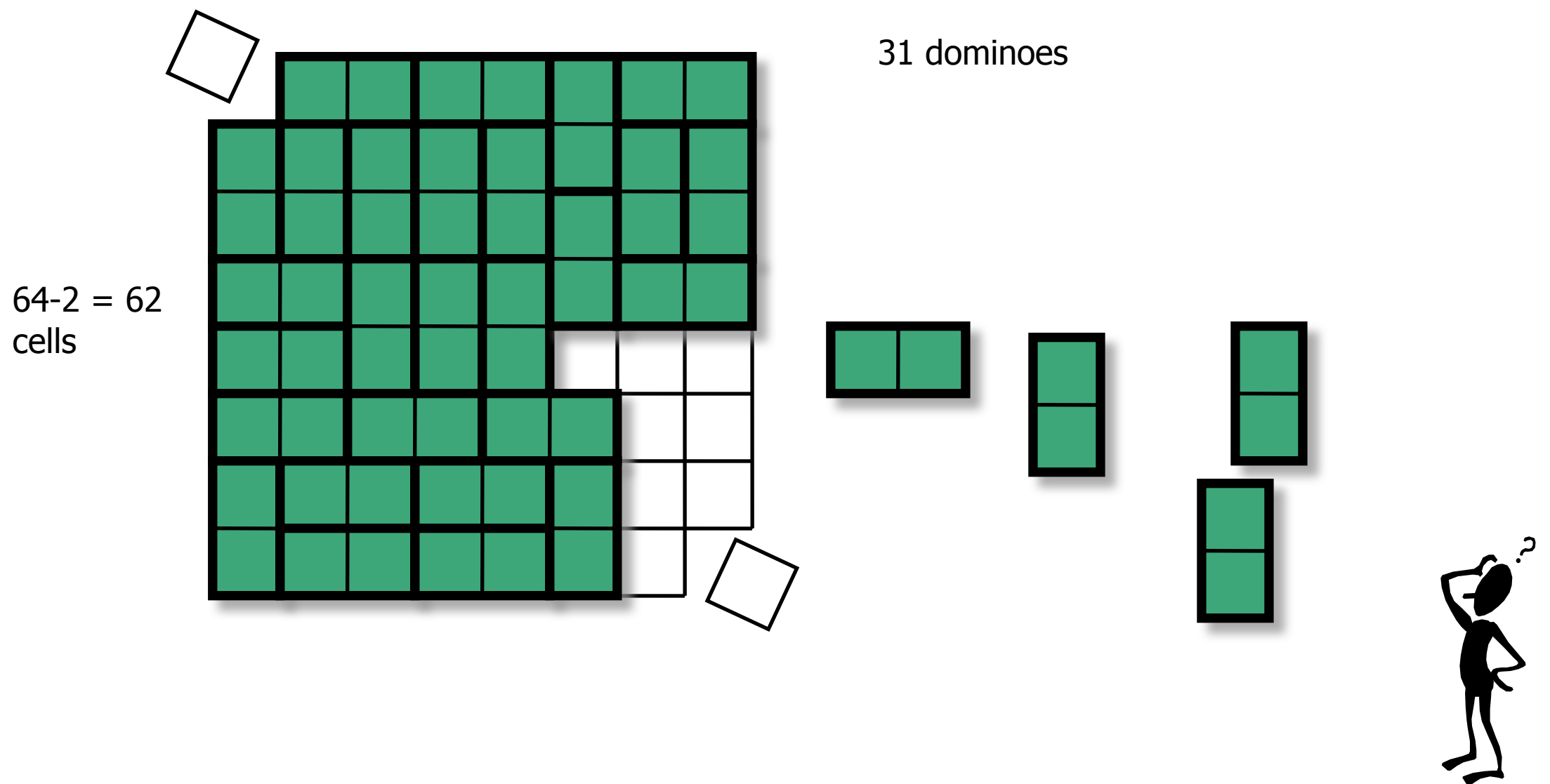
Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

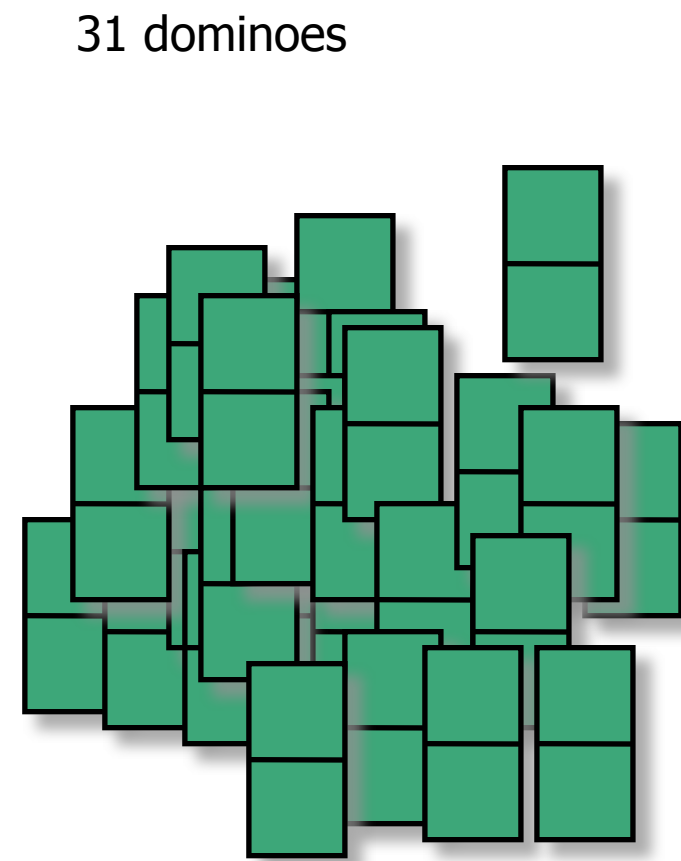
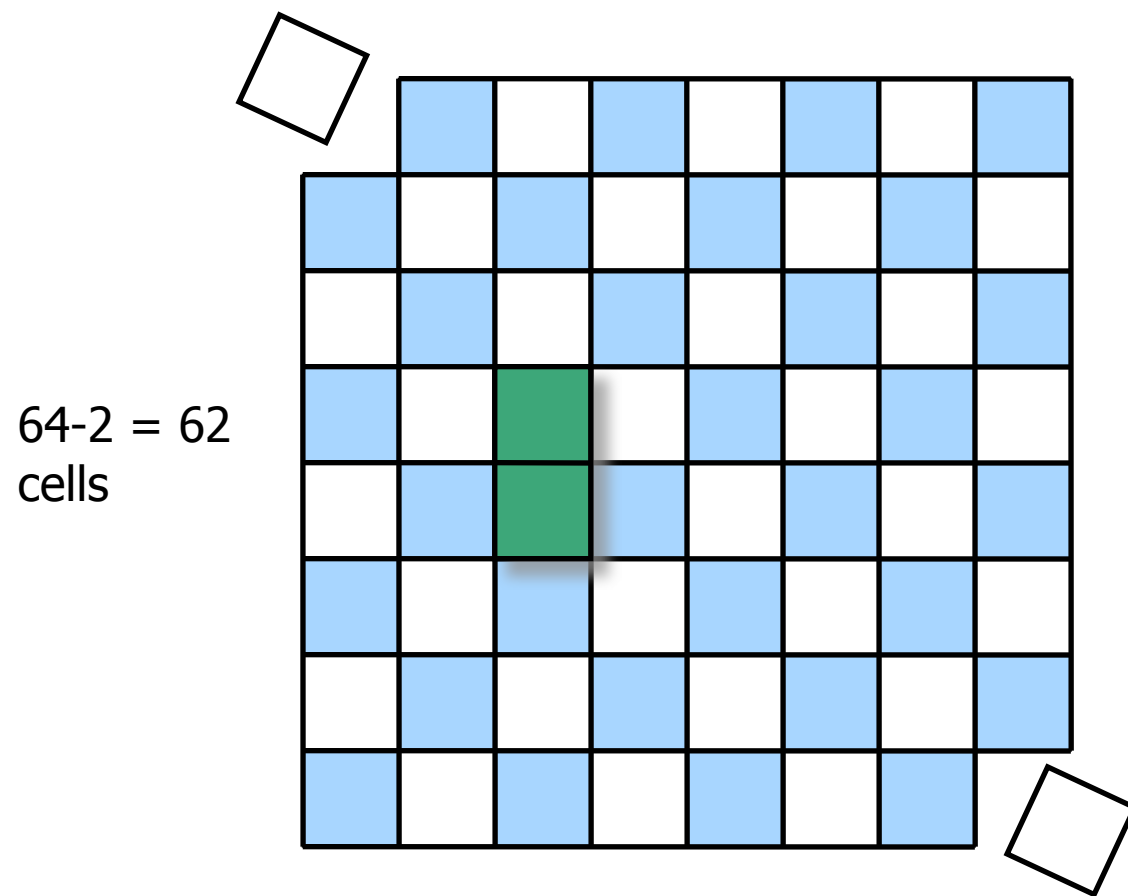
Puzzle time: tiling a chessboard with dominoes



Puzzle time: tiling a chessboard with dominoes



Puzzle time: tiling a chessboard with dominoes

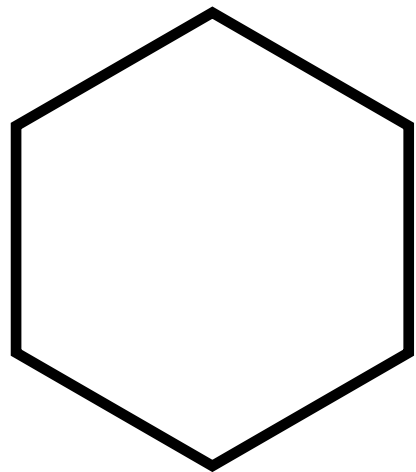


Invariant

An invariant of a dynamic system is an assertion that holds at every reachable state

Example

You have a polygon



You can rotate it

You can move it

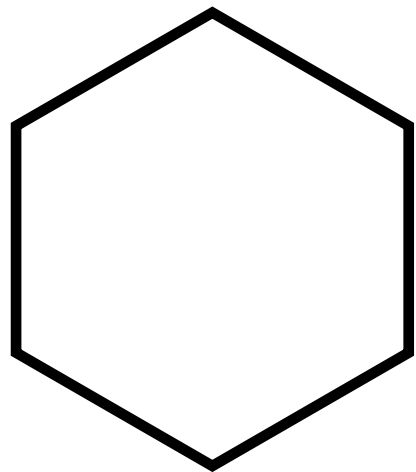
You can change its dimension

You can mirror it

Which invariants? **perimeter**

Example

You have a polygon



You can rotate it

You can move it

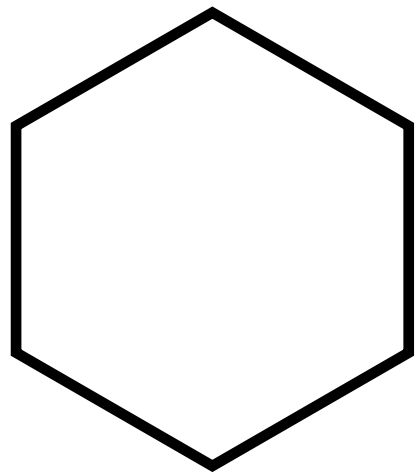
You can change its dimension

You can mirror it

Which invariants? **area**

Example

You have a polygon



You can rotate it

You can move it

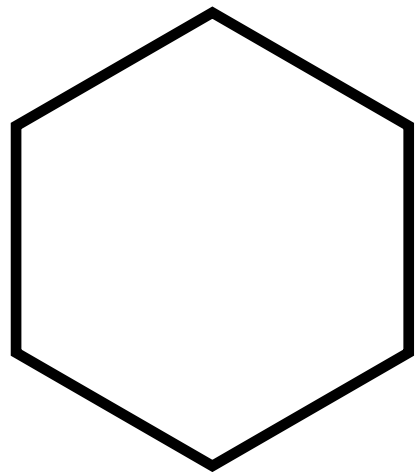
You can change its dimension

You can mirror it

Which invariants? **number of vertices**

Example

You have a polygon



You can rotate it

You can move it

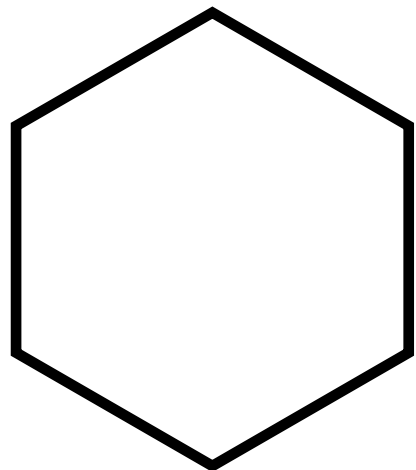
You can change its dimension

You can mirror it

Which invariants? **number of sides**

Example

You have a polygon



You can rotate it

You can move it

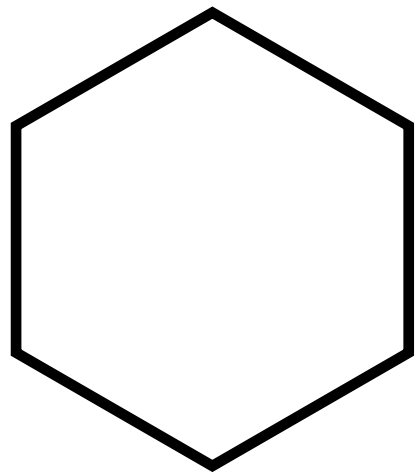
You can change its dimension

You can mirror it

Which invariants? **vertex degrees**

Example

You have a polygon



You can rotate it

You can move it

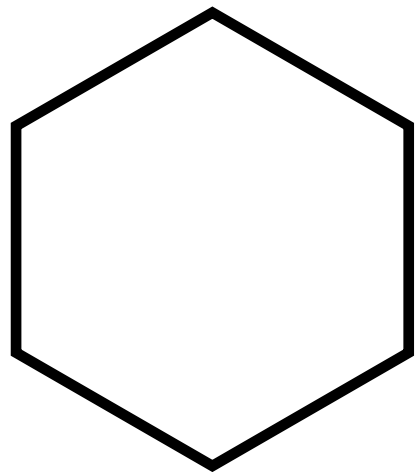
You can change its dimension

You can mirror it

Which invariants? **convexity**

Example

You have a polygon



You can rotate it

You can move it

You can change its dimension

You can mirror it

Which invariants? **color**

Puzzle: from MI to MU

You can compose words using symbols **M**, **I**, **U**

Given the initial word **MI**, you can apply the following transformations, in any order, as many times as you like:

1. Add a **U** to the end of any string ending in **I** (e.g., **MI** to **MIU**).
2. Double the string after the **M** (e.g., **MIU** to **MIUIU**).
3. Replace any **III** with a **U** (e.g., **MUIIU** to **MUUU**).
4. Remove any **UU** (e.g., **MUUU** to **MU**).

MI $\xrightarrow{2}$ **MII** $\xrightarrow{2}$ **MIII** $\xrightarrow{1}$ **MIIIU** $\xrightarrow{3}$ **MIUU** $\xrightarrow{4}$ **MI**

Puzzle: from **MI** to **MU**

You can compose words using symbols **M**, **I**, **U**

Given the initial word **MI**, you can apply the following transformations, in any order, as many times as you like:

1. Add a **U** to the end of any string ending in **I** (e.g., **MI** to **MIU**).
2. Double the string after the **M** (e.g., **MIU** to **MIUIU**).
3. Replace any **III** with a **U** (e.g., **MUIIIU** to **MUUU**).
4. Remove any **UU** (e.g., **MUUU** to **MU**).

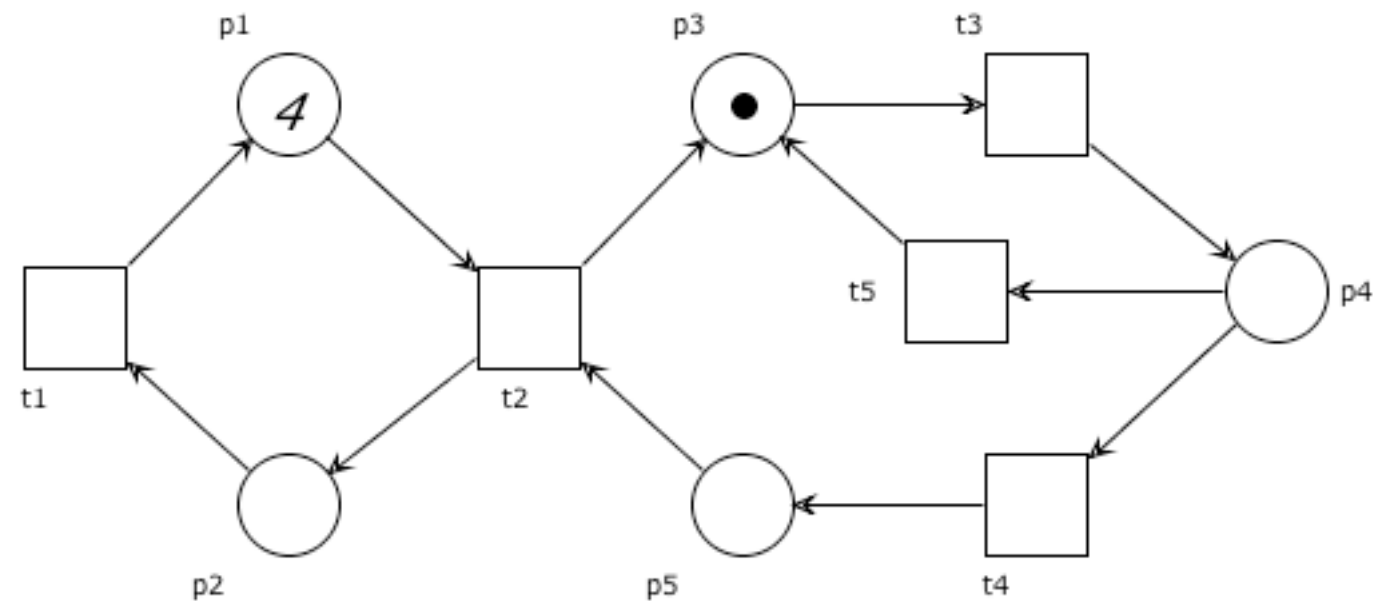
Can you transform **MI** to **MU**?
(*Hint*: count the **I**s modulo 3)

Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any
currently enabled
transition



Which invariants?

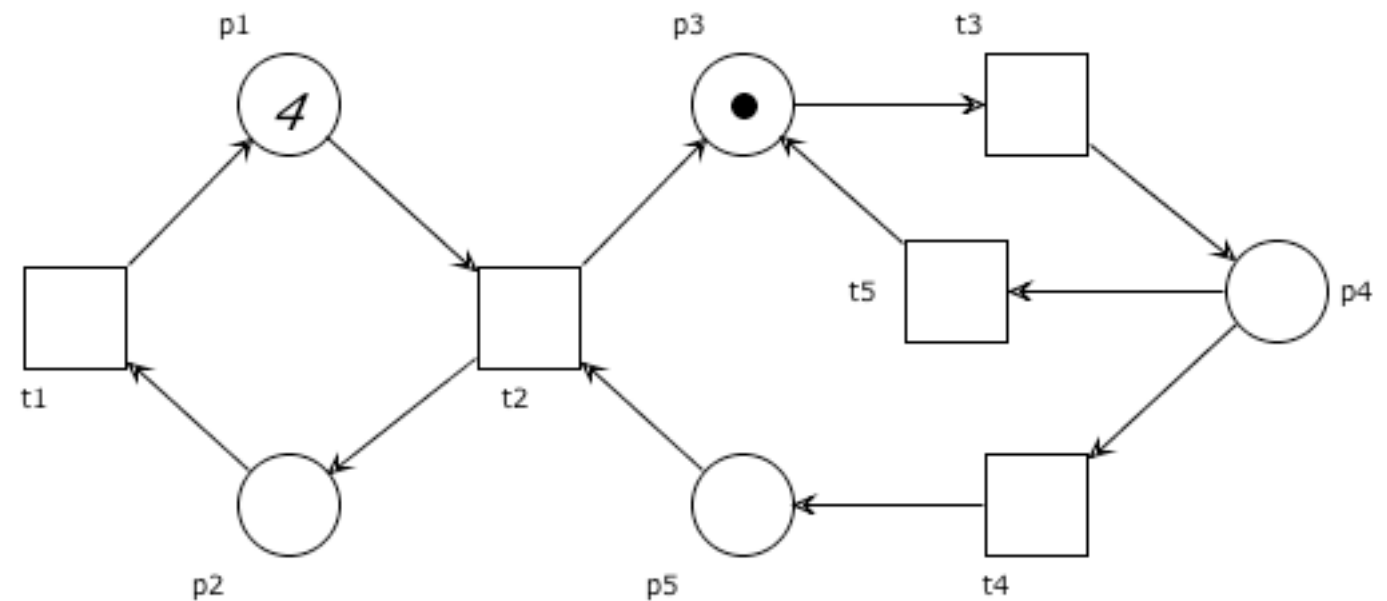
color

Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any
currently enabled
transition



Which invariants?

P, T, F

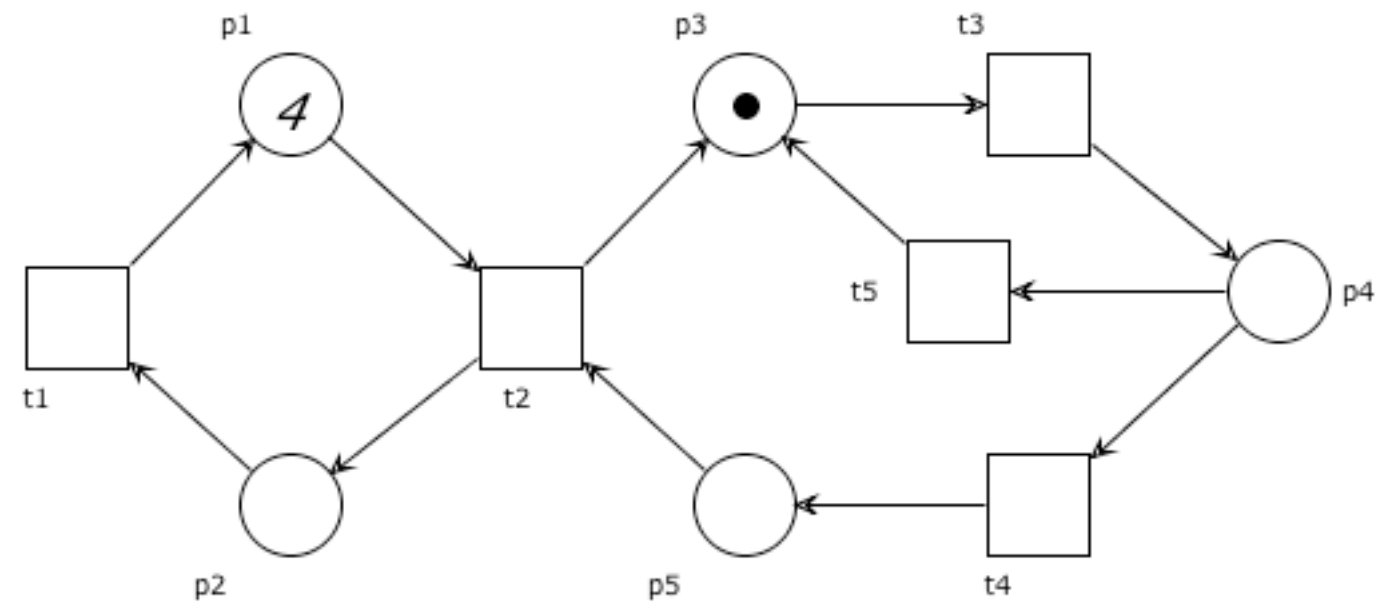
Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any currently enabled transition

Which invariants?



Any property that holds for any reachable marking

Recall:

Liveness, formally

$$(P, T, F, M_0)$$

$$\forall t \in T, \quad \forall M \in [M_0 \rangle, \quad \exists M' \in [M \rangle, \quad M' \xrightarrow{t}$$

Liveness as invariant

Lemma

If (P, T, F, M_0) is live and $M \in [M_0 \rangle$, then (P, T, F, M) is live.

Let $t \in T$ and $M' \in [M \rangle$.

Since $M \in [M_0 \rangle$, then $M' \in [M_0 \rangle$.

Since (P, T, F, M_0) is live, $\exists M'' \in [M' \rangle$ with $M'' \xrightarrow{t}$.

Therefore (P, T, F, M) is live.

Recall: Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [M_0 \rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

Deadlock freedom as invariant

Lemma: If (P, T, F, M_0) is deadlock-free and $M \in [M_0 \rangle$, then (P, T, F, M) is deadlock-free.

Let $M' \in [M \rangle$.

Since $M \in [M_0 \rangle$, then $M' \in [M_0 \rangle$.

Since (P, T, F, M_0) is deadlock-free, $\exists t \in T$ with $M' \xrightarrow{t}$.

Therefore (P, T, F, M) is deadlock-free.

Exercise

Give the formal definition of Boundedness

Then prove that Boundedness is an invariant

Or give a counter-example

Exercise

Give the formal definition of Cyclicity

Then prove that Cyclicity is an invariant

Or give a counter-example

Structural invariants

In the case of Petri nets, it is possible to compute certain vectors of **rational** numbers^(*) (directly from the structure of the net) (independently from the initial marking) which induce nice invariants, called

S-invariants

T-invariants

(*) it is not necessary to consider real-valued solutions, because incidence matrices only have integer entries

Why invariants?

Can be calculated efficiently
(polynomial time for a basis)

Independent of initial marking

Structural property with behavioural consequences

However, the main reason is didactical!
You only truly understand a model if you think
about it in terms of invariants!

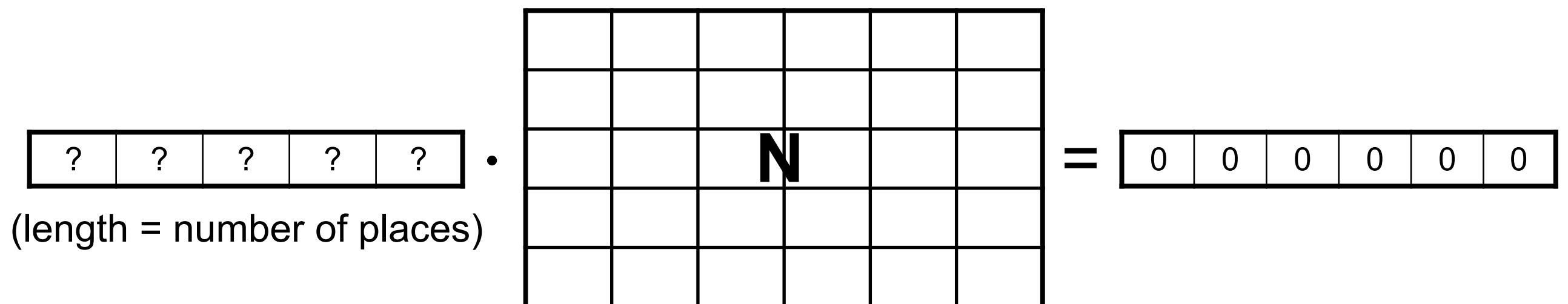


S-invariants

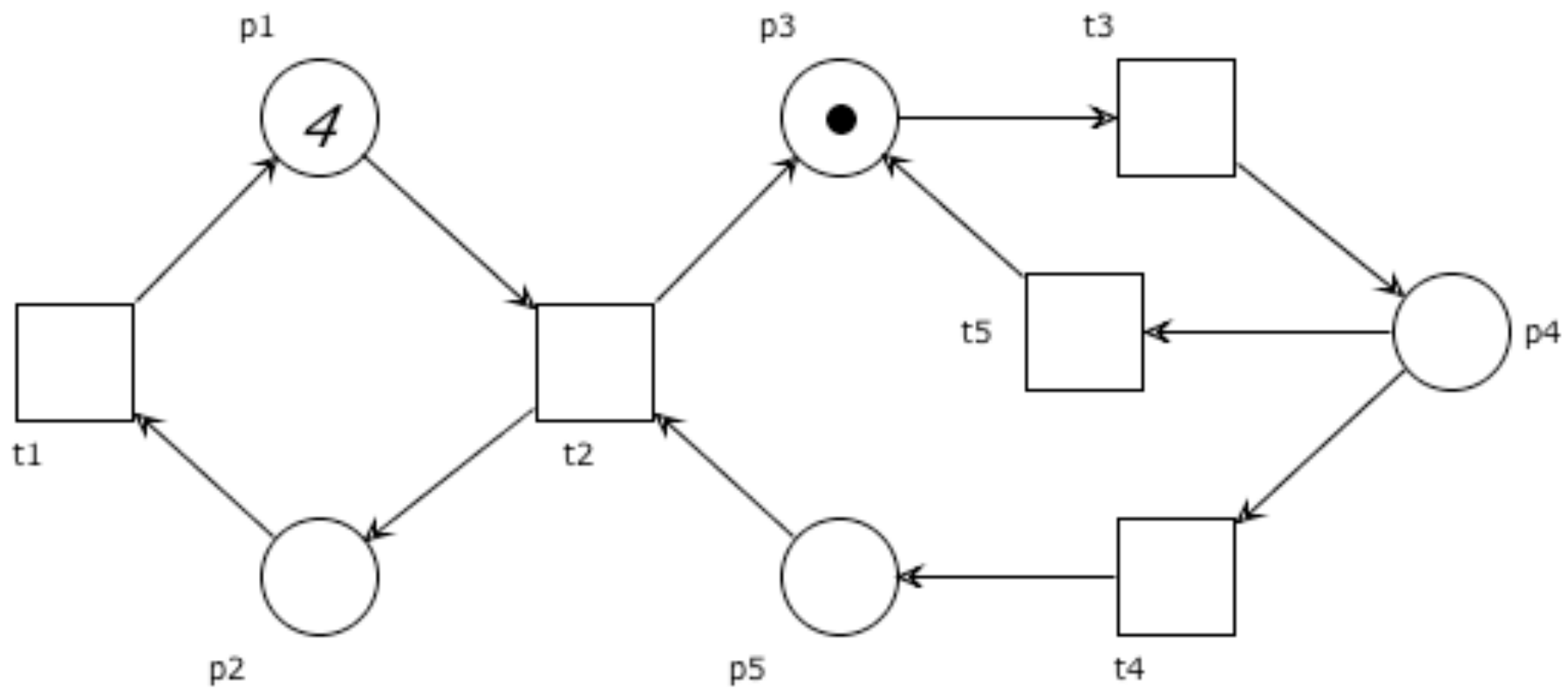
S-invariant (aka place-invariant)

Definition: An **S-invariant** of a net $N=(P,T,F)$ is a rational-valued solution x of the equation

$$x \cdot N = 0$$



Example



Find some/all S-invariants for the net above

Fundamental property of S -invariants

Proposition: Let \mathbf{I} be an invariant of N .

For any $M \in [M_0 \rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

$$\begin{array}{|c|c|c|c|c|} \hline & & \mathbf{I} & & \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline \\ \hline M \\ \hline \\ \hline \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & \mathbf{I} & & \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \\ \hline \\ \hline M_0 \\ \hline \\ \hline \\ \hline \end{array}$$

Fundamental property of S -invariants

Proposition: Let \mathbf{I} be an invariant of N .

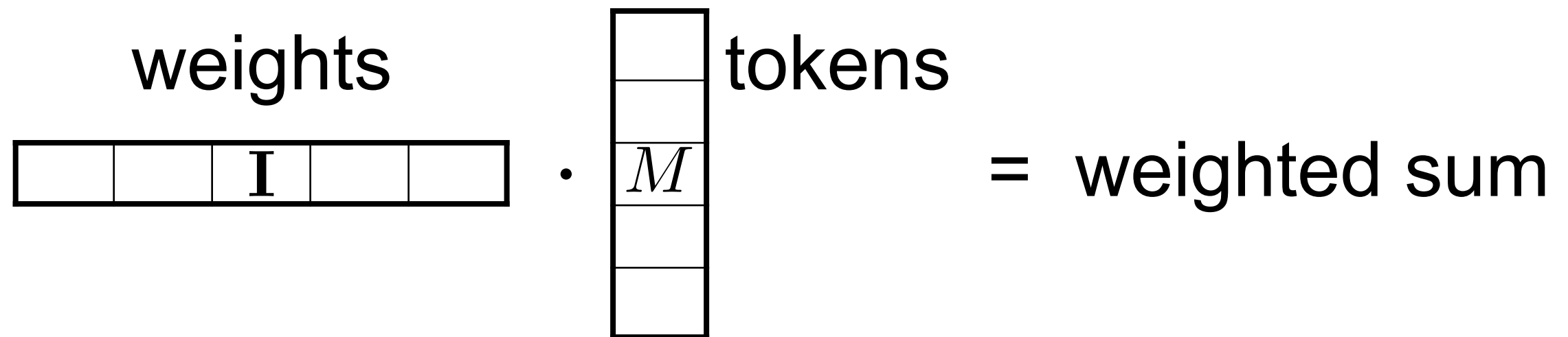
For any $M \in [M_0 \rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since $M \in [M_0 \rangle$, there is σ s.t. $M_0 \xrightarrow{\sigma} M$

By the marking equation: $M = M_0 + \mathbf{N} \cdot \vec{\sigma}$

$$\begin{aligned} \text{Therefore: } \mathbf{I} \cdot M &= \mathbf{I} \cdot (M_0 + \mathbf{N} \cdot \vec{\sigma}) \\ &= \mathbf{I} \cdot M_0 + \mathbf{I} \cdot \mathbf{N} \cdot \vec{\sigma} \\ &= \mathbf{I} \cdot M_0 + \mathbf{0} \cdot \vec{\sigma} \\ &= \mathbf{I} \cdot M_0 \end{aligned}$$

Place-invariant, intuitively



Place-invariant, intuitively

A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

For example, you can imagine that tokens are coins, places are the different kinds of available coins, the S-invariant assigns a value to each coin: the value of a marking is the sum of the values of the tokens/coins in it and it is not changed by firings

Place-invariant, intuitively

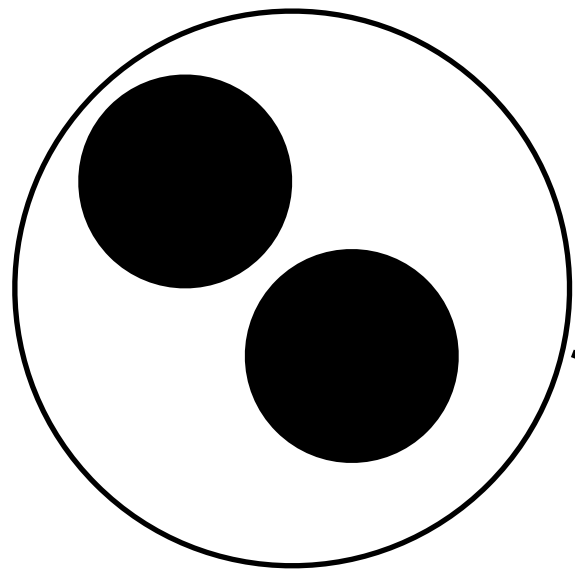
A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

For example, you can imagine that tokens are molecules, places are different kinds of molecules, the S-invariant assigns the number of atoms needed to form each molecule:
the overall number of atoms is not changed by firings

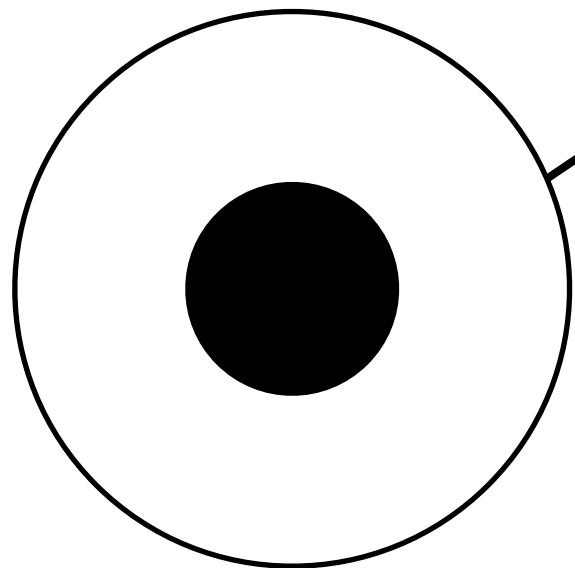
Intuition: bubbles

within tokens

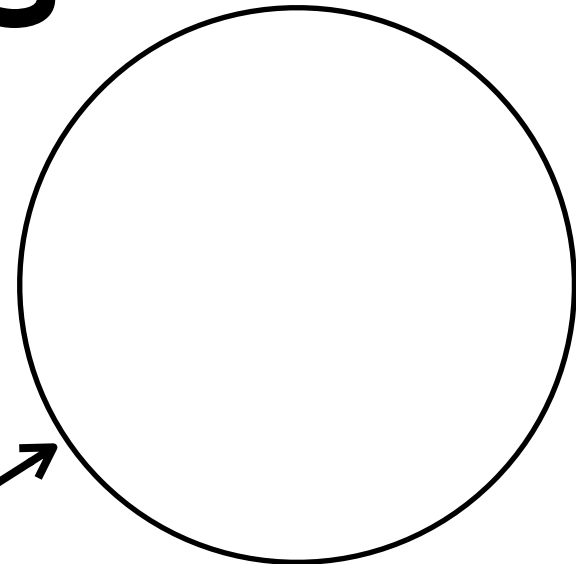
$$I(p_1)=2$$



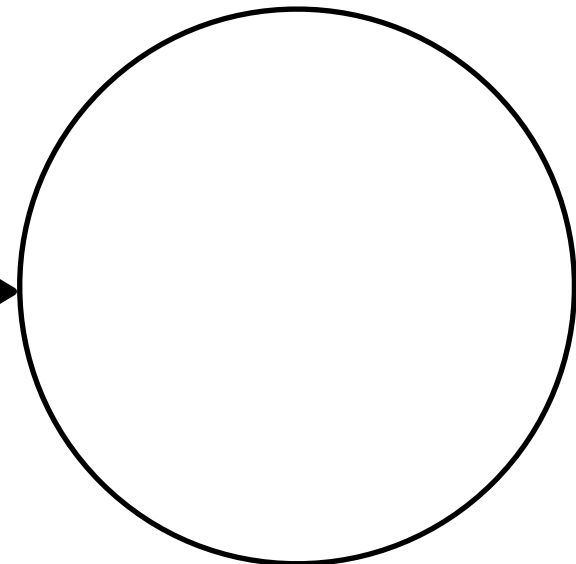
$$I(p_2)=3$$



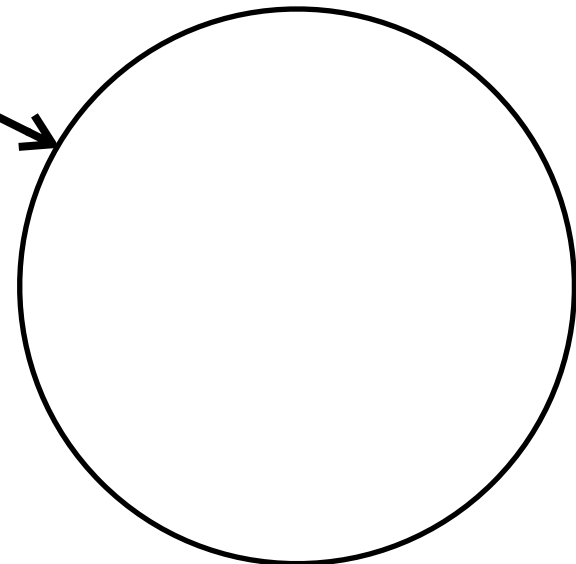
$$I(p_3)=0$$



$$I(p_4)=1$$



$$I(p_5)=4$$

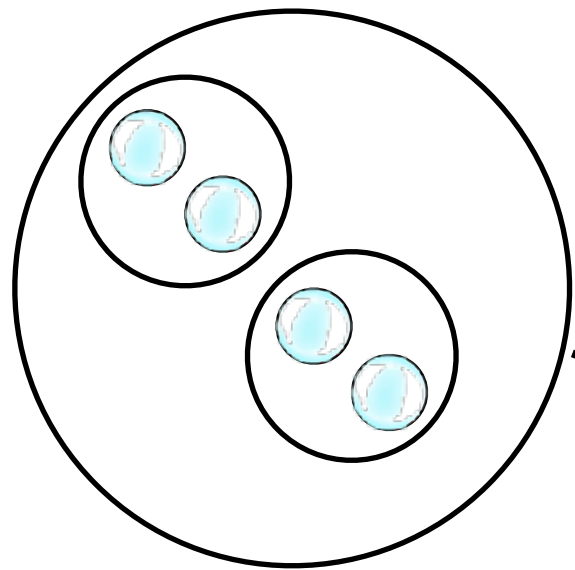


$$I = [2 \ 3 \ 0 \ 1 \ 4 \ \dots]$$

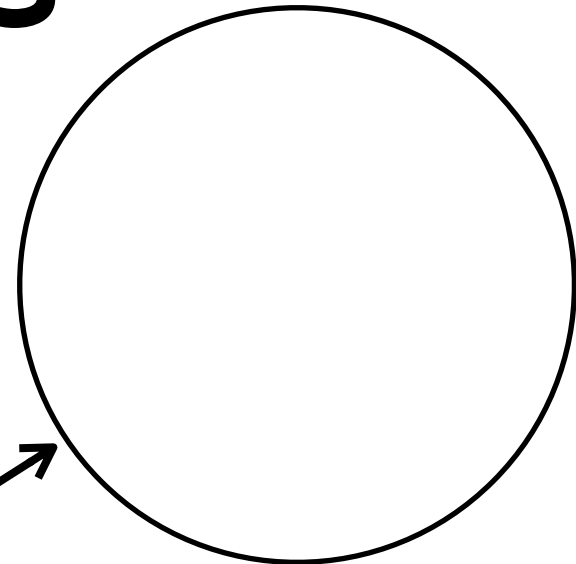
Intuition: bubbles

within tokens

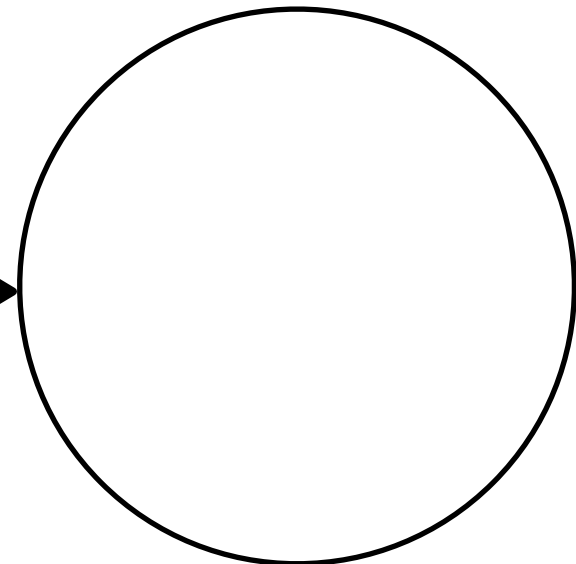
$$I(p_1)=2$$



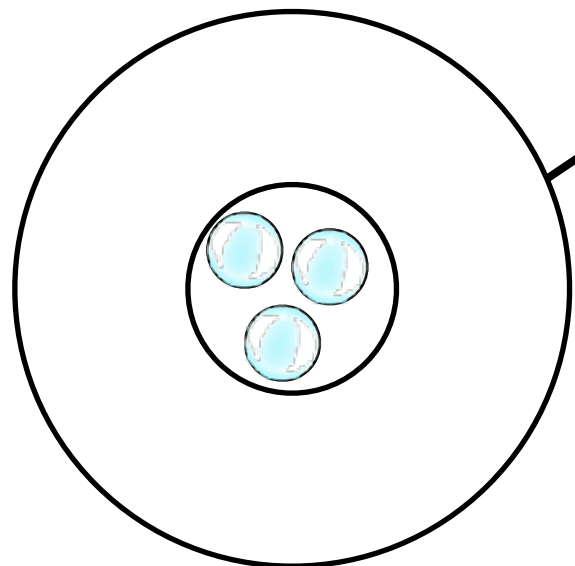
$$I(p_3)=0$$



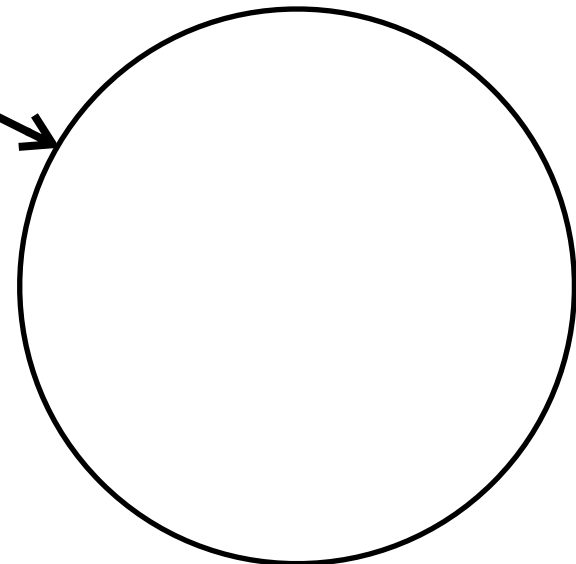
$$I(p_4)=1$$



$$I(p_2)=3$$



$$I(p_5)=4$$

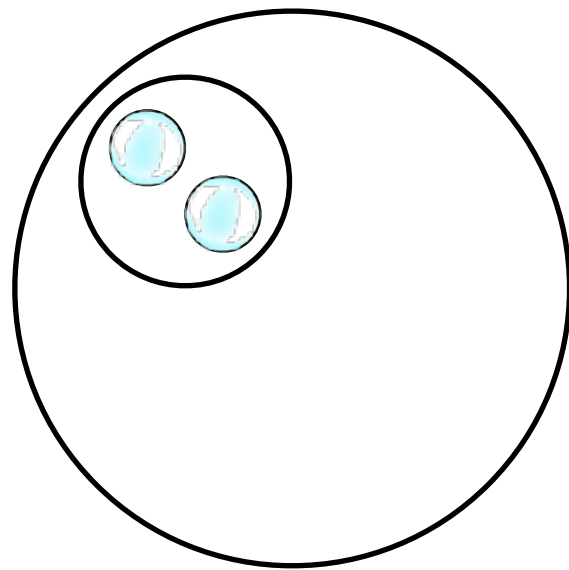


$$I = [2 \ 3 \ 0 \ 1 \ 4 \ \dots]$$

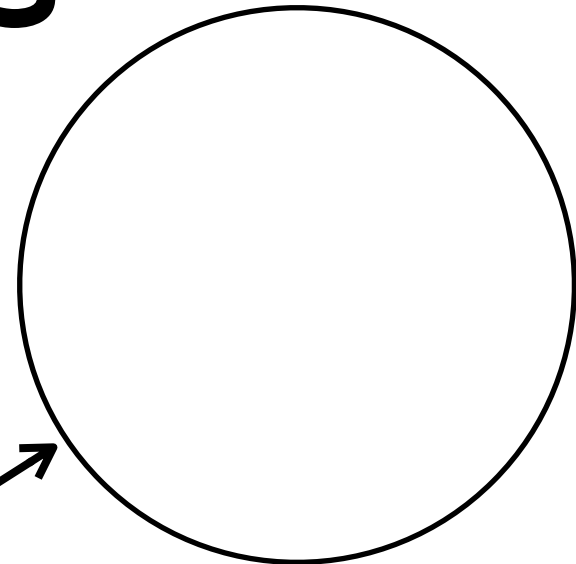
Intuition: bubbles

within tokens

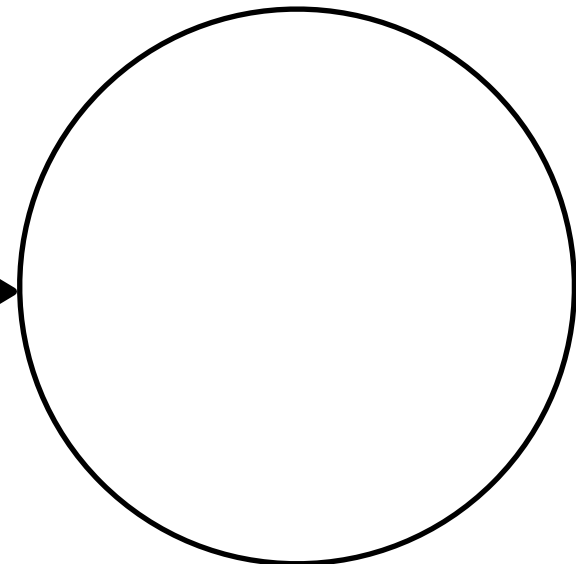
$$I(p_1)=2$$



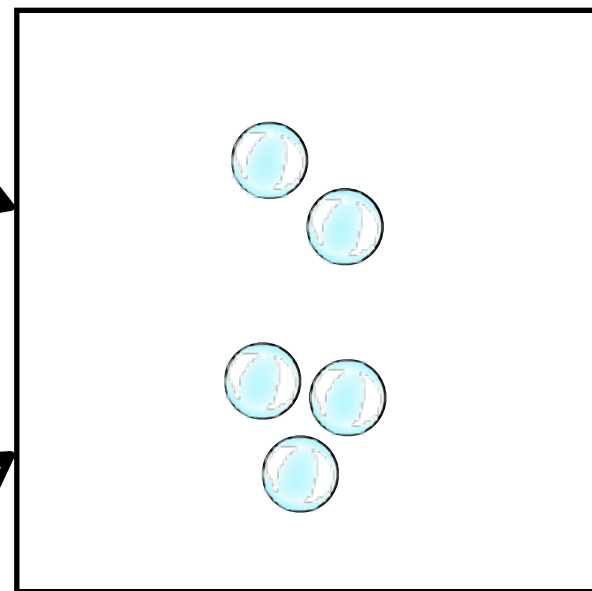
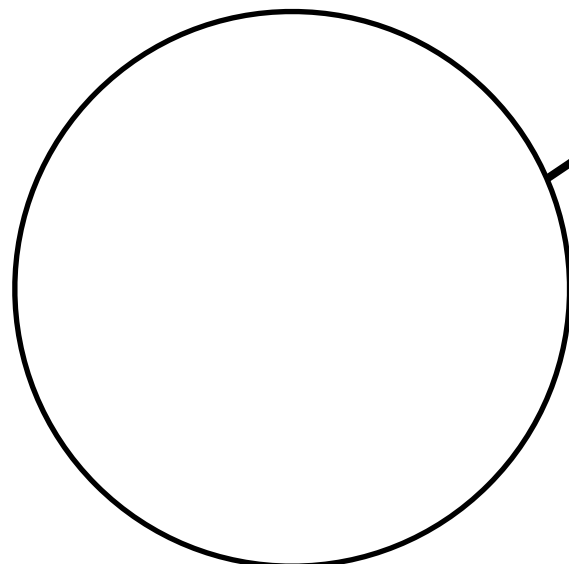
$$I(p_3)=0$$



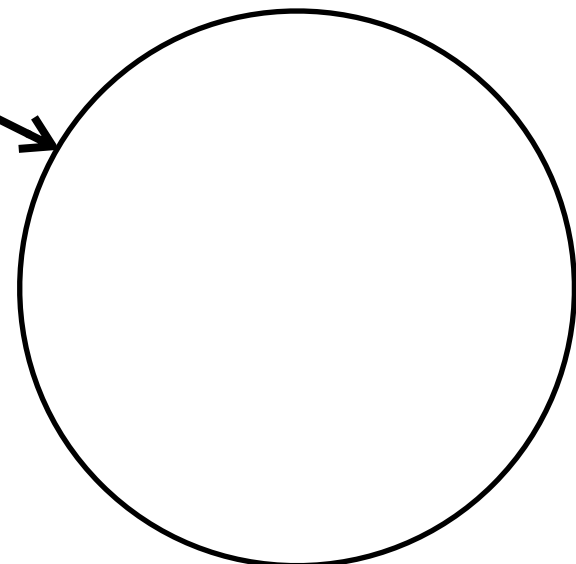
$$I(p_4)=1$$



$$I(p_2)=3$$



$$I(p_5)=4$$



$$I = [2 \ 3 \ 0 \ 1 \ 4 \ \dots]$$

Intuition: bubbles

within tokens

$$I(p_1)=2$$

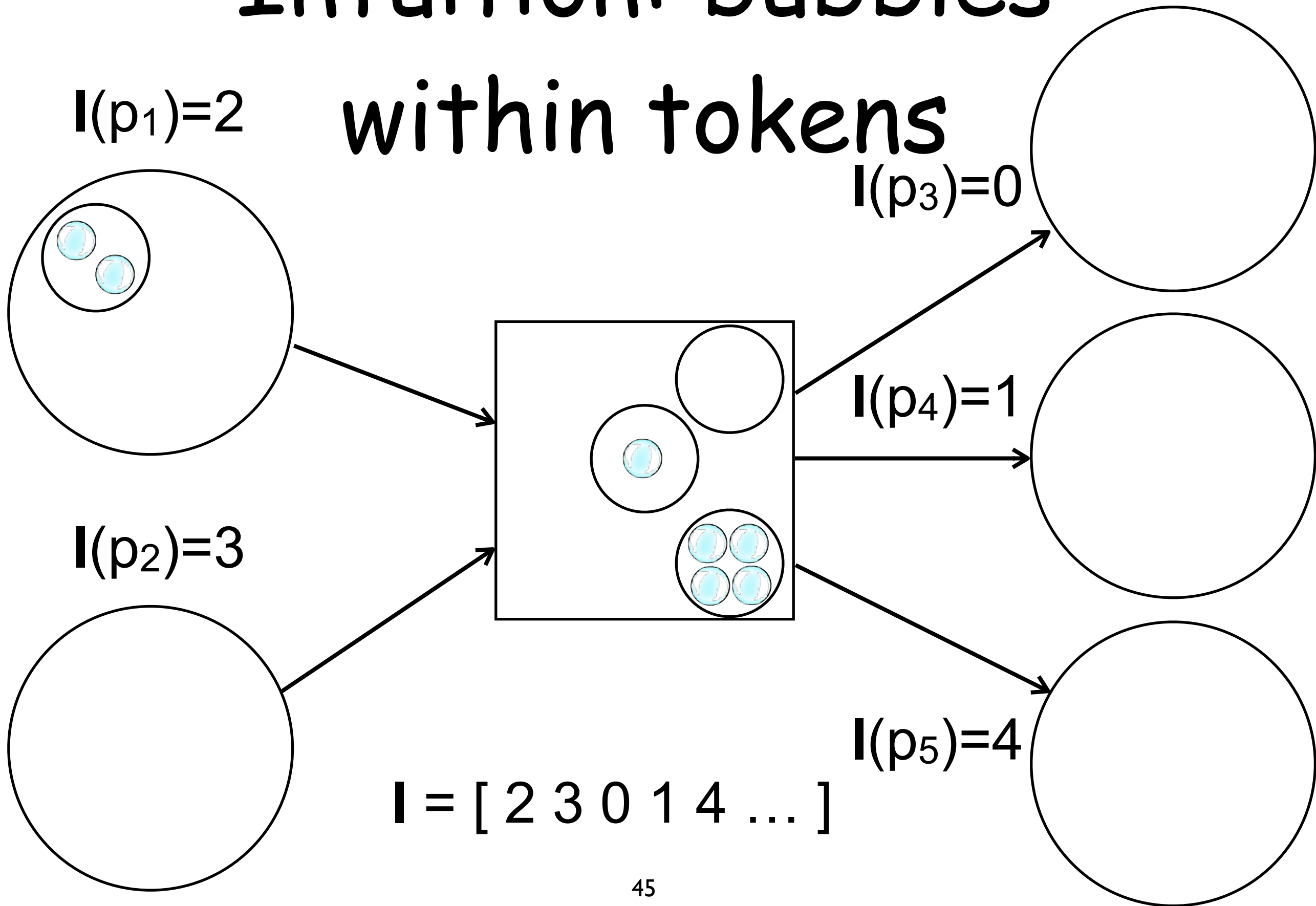
$$I(p_3)=0$$

$$I(p_2)=3$$

$$I(p_4)=1$$

$$I(p_5)=4$$

$$I = [2 \ 3 \ 0 \ 1 \ 4 \ \dots]$$



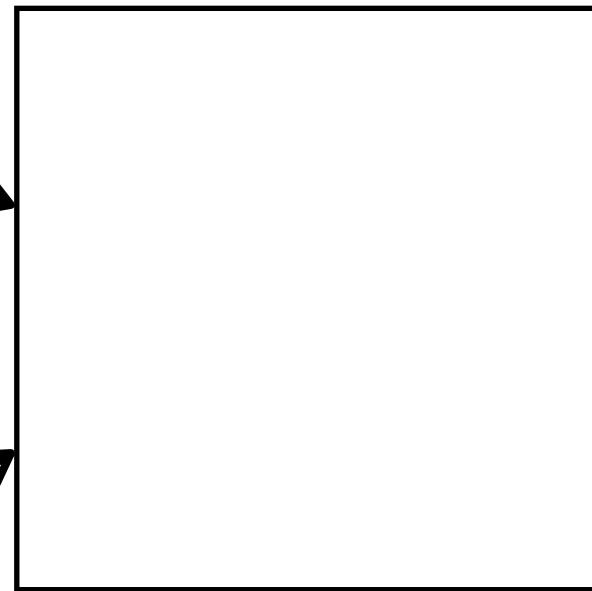
Intuition: tokens

as coins

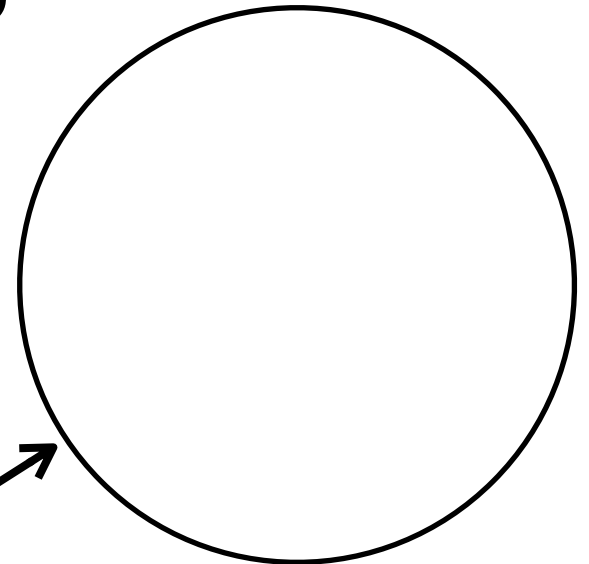
$I(p_1) = 10\text{¢}$



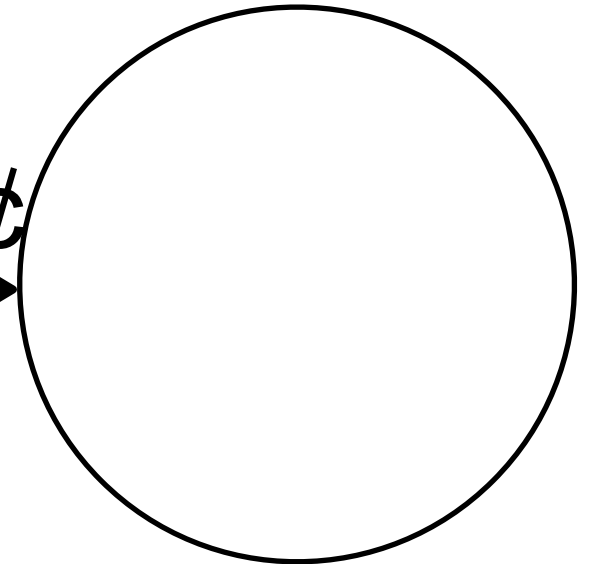
$I(p_2) = 50\text{¢}$



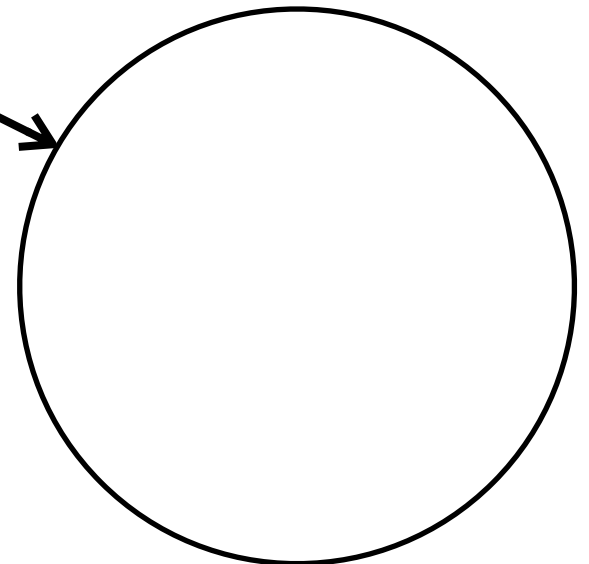
$I(p_3) = 20\text{¢}$



$I(p_4) = 20\text{¢}$



$I(p_5) = 20\text{¢}$



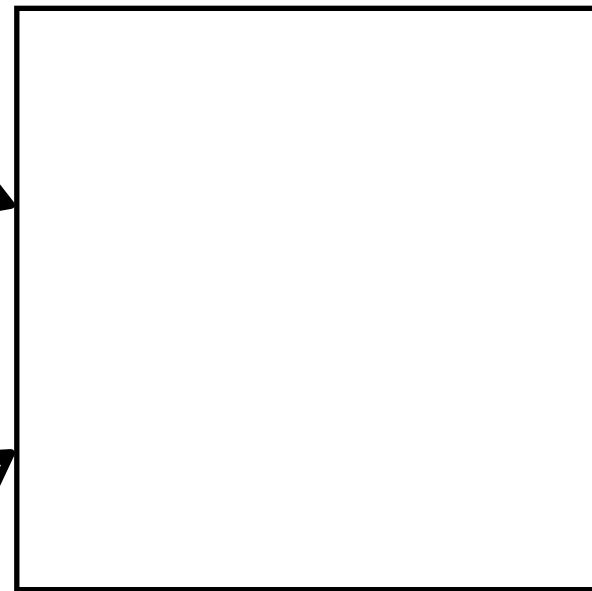
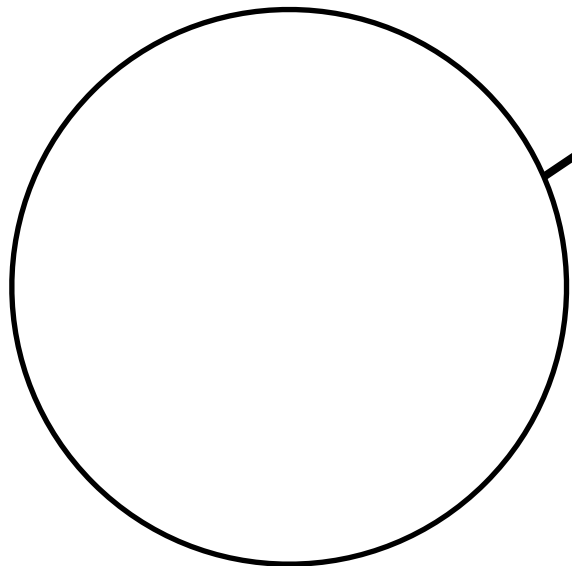
Intuition: tokens

as coins

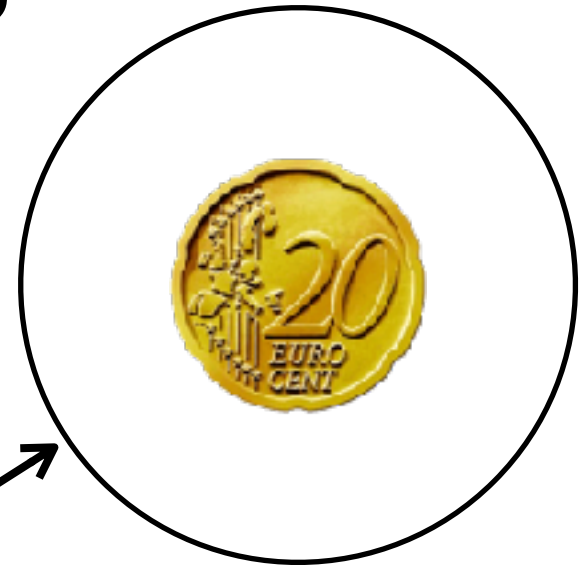
$I(p_1) = 10\text{¢}$



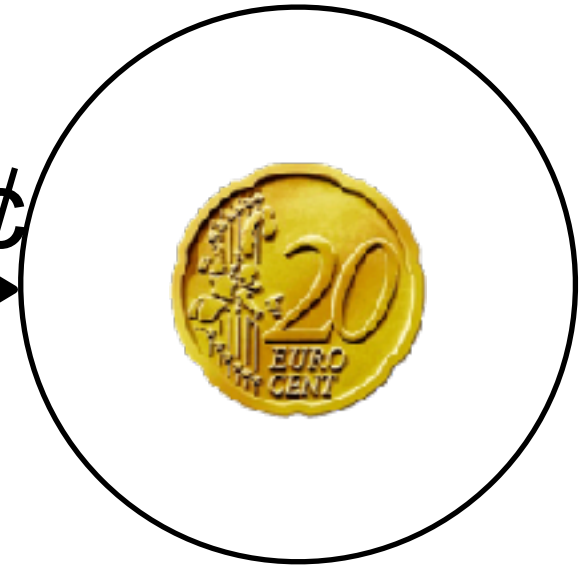
$I(p_2) = 50\text{¢}$



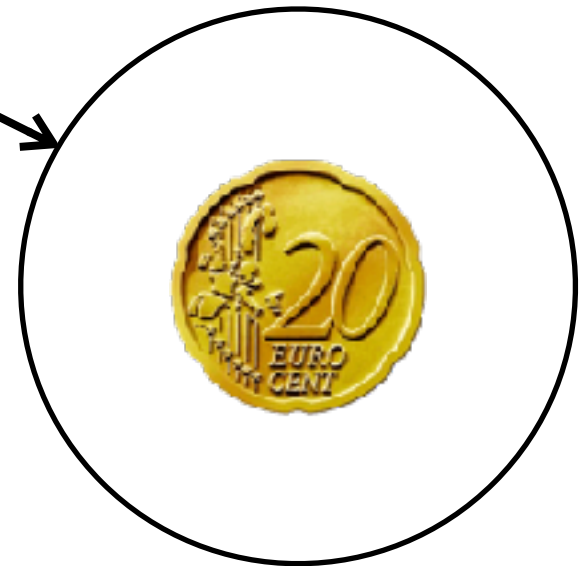
$I(p_3) = 20\text{¢}$



$I(p_4) = 20\text{¢}$



$I(p_5) = 20\text{¢}$



Linear combination

Proposition:

Any linear combination of S-invariants is an S-invariant

Take any two S-Invariants \mathbf{I}_1 and \mathbf{I}_2 and any two values k_1, k_2 . We want to prove that $k_1 \mathbf{I}_1 + k_2 \mathbf{I}_2$ is an S-invariant.

$$\begin{aligned}(k_1 \mathbf{I}_1 + k_2 \mathbf{I}_2) \cdot \mathbf{N} &= k_1 \mathbf{I}_1 \cdot \mathbf{N} + k_2 \mathbf{I}_2 \cdot \mathbf{N} \\ &= k_1 \mathbf{0} + k_2 \mathbf{0} \\ &= \mathbf{0}\end{aligned}$$

Alternative definition of S -invariant

Proposition:

A mapping $\mathbf{I} : P \rightarrow \mathbb{Q}$ is an S -invariant of N iff for any $t \in T$:

$$\sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$

Exercise

Prove the proposition about the alternative characterization of S -invariants

Consequence of alternative definition

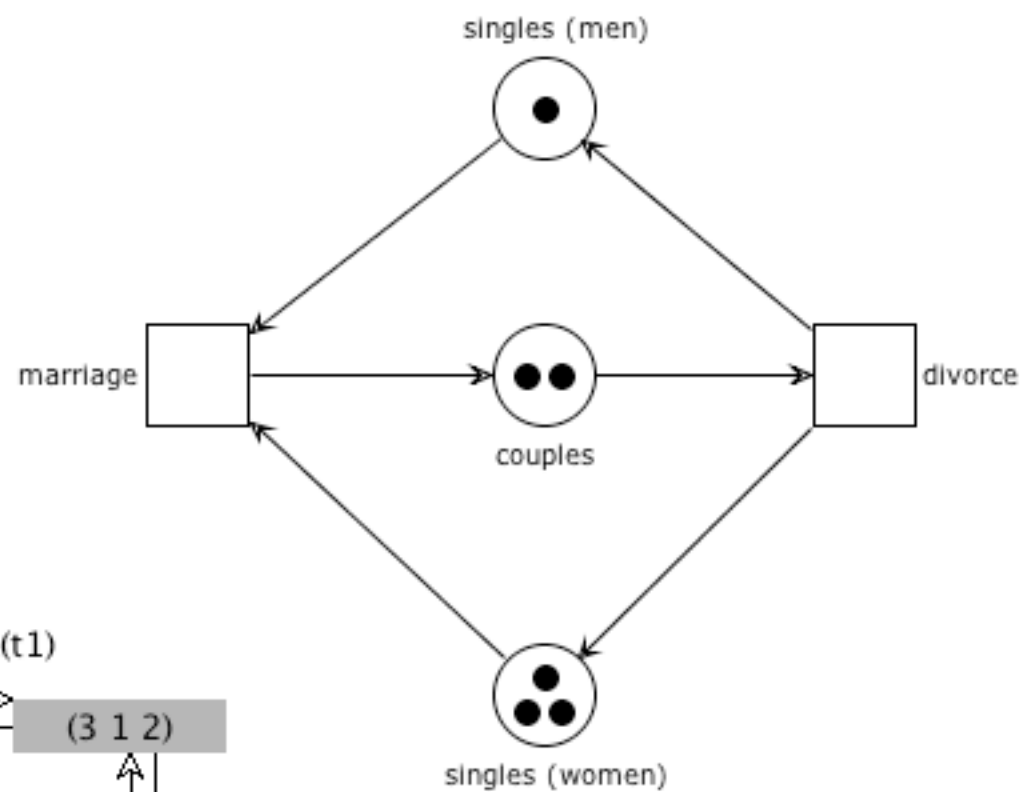
Very useful in proving S-invariance!

The check is possible without constructing
the incidence matrix

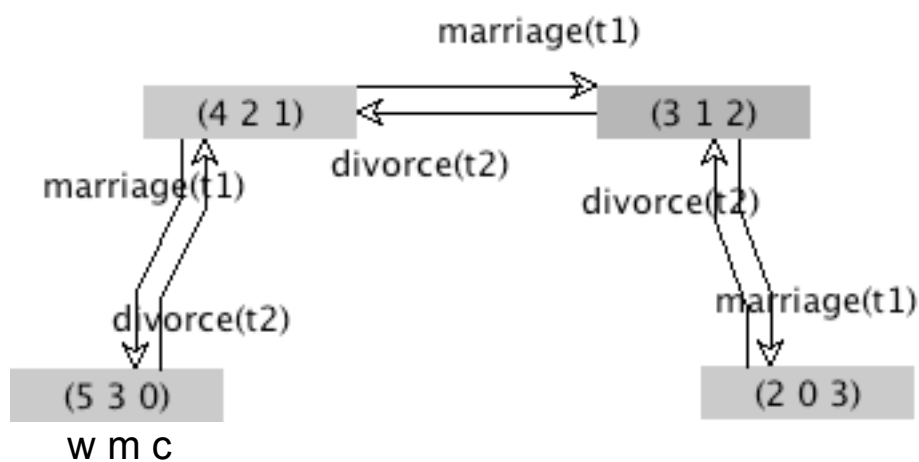
It can also help to build S-invariants
directly over the picture

Question time

Which of the following are S-invariants?

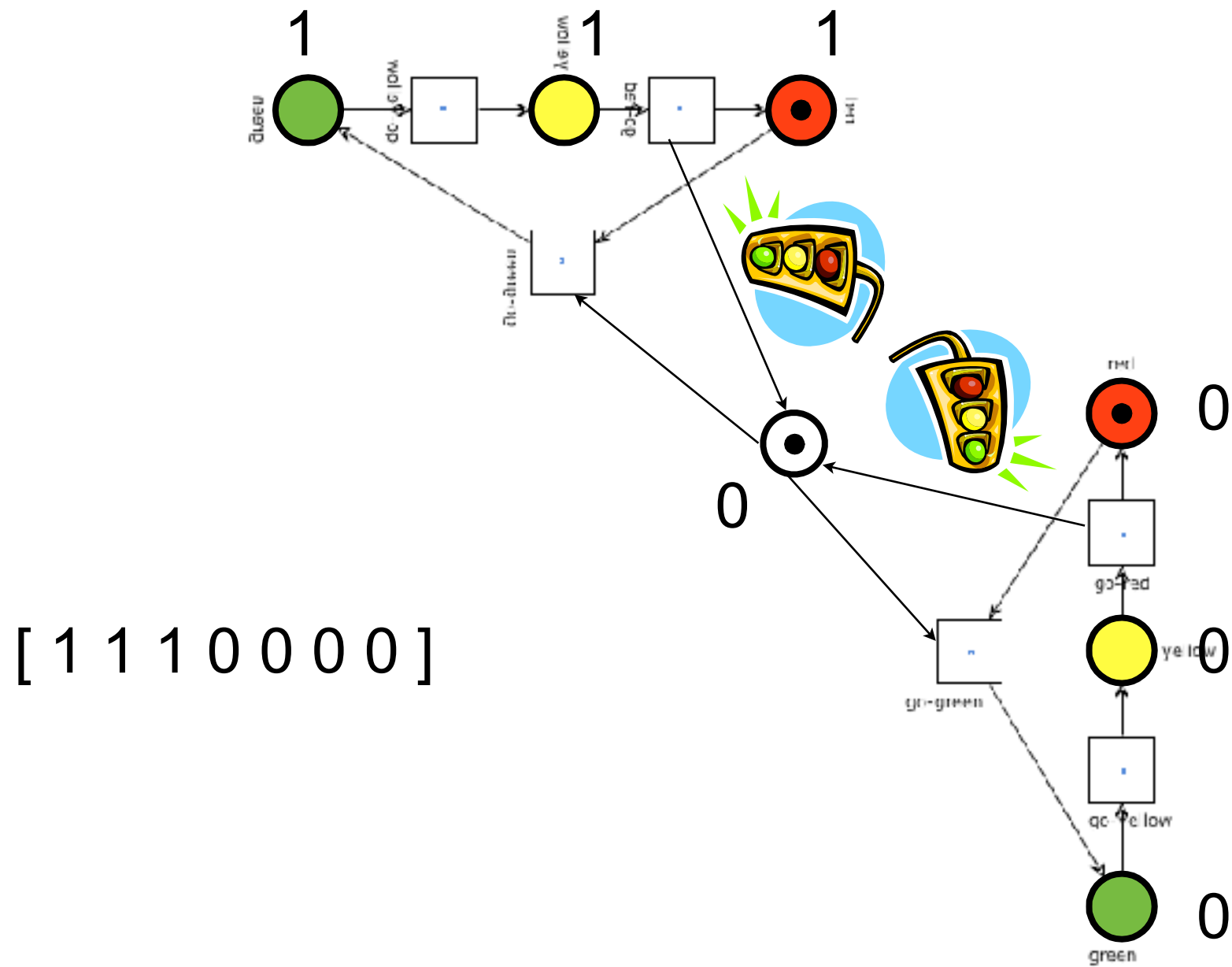


	m	w	c
$[\quad]$	1	1	-1
$[\quad]$	1	0	1
$[\quad]$	0	1	1
$[\quad]$	1	1	1
$[\quad]$	1	-1	0
$[\quad]$	1	1	2
$[\quad]$	1	2	2



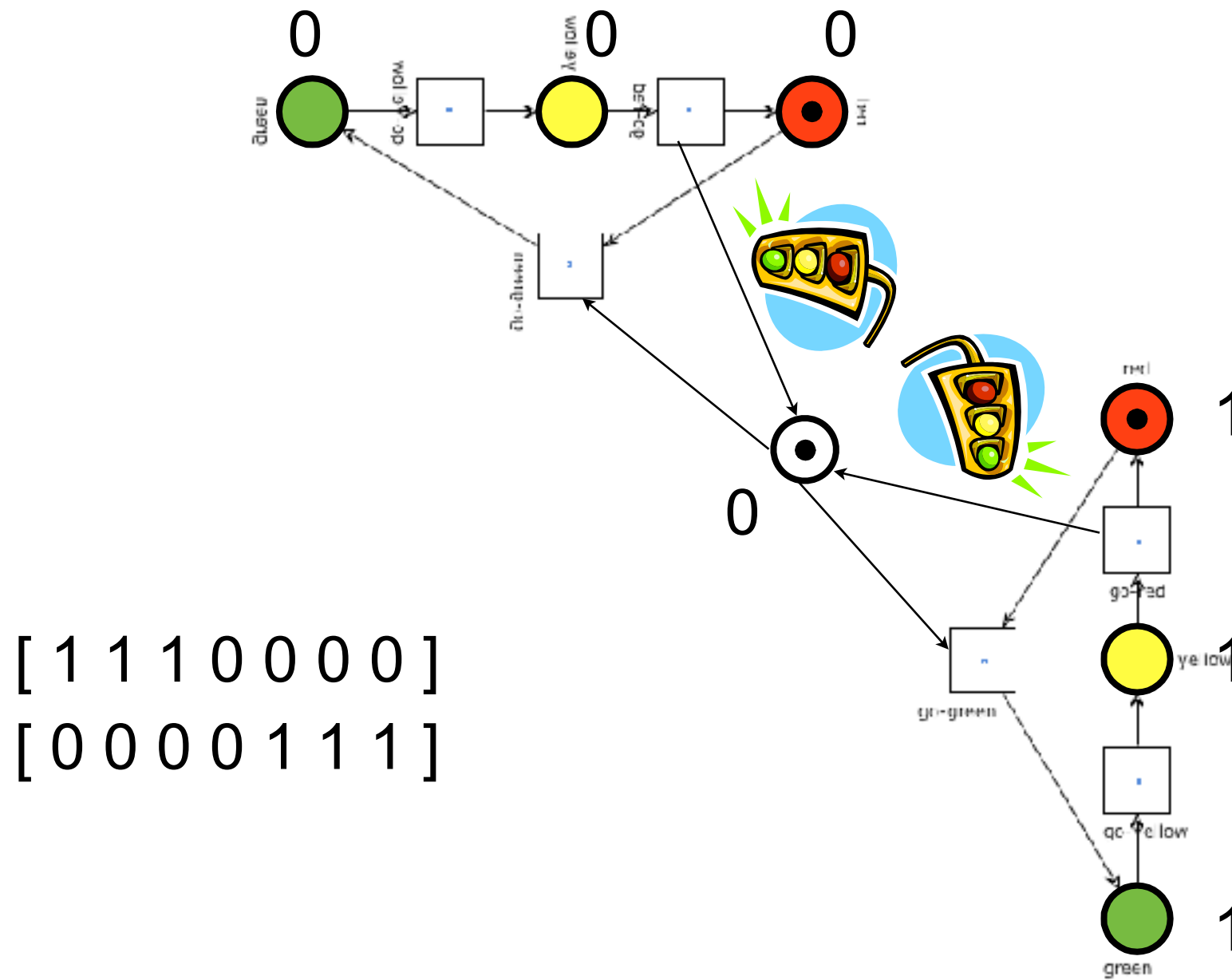
$$\forall t \in T, \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t \bullet} \mathbf{I}(p)$$

Traffic-lights example



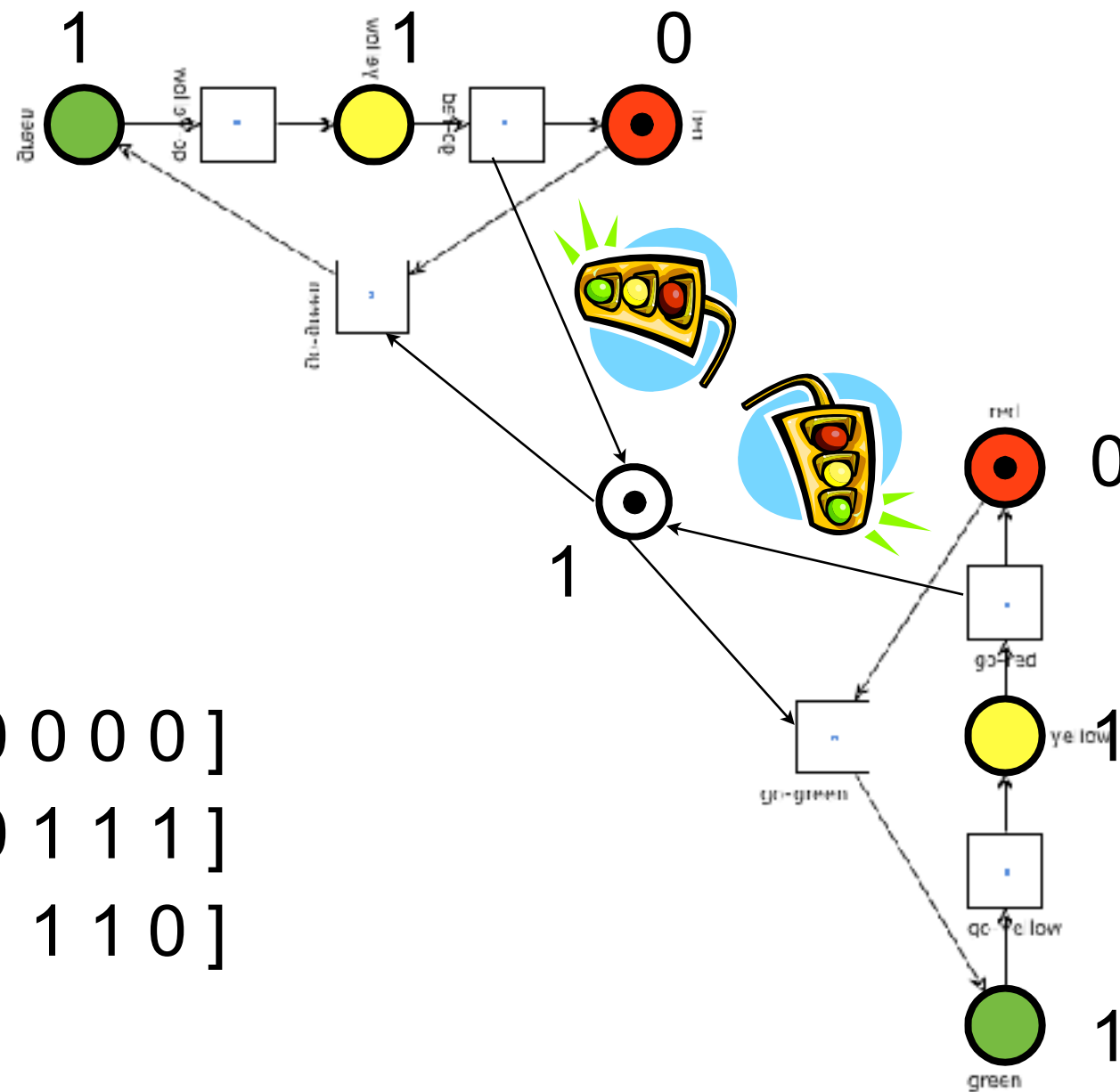
[1 1 1 0 0 0 0]

Traffic-lights example



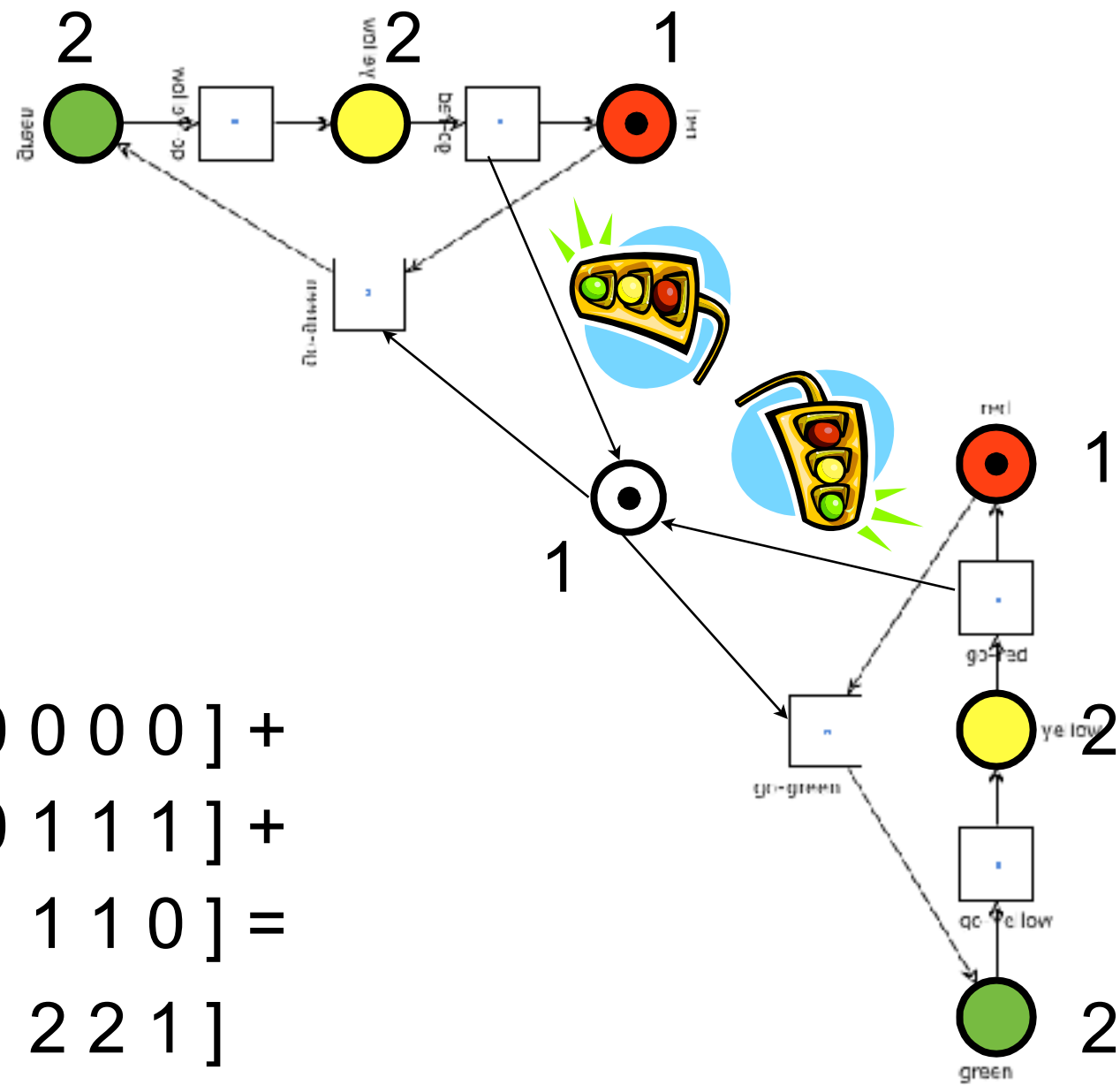
[1 1 1 0 0 0 0]
 [0 0 0 0 1 1 1]

Traffic-lights example



```
[ 1 1 1 0 0 0 0 ]
[ 0 0 0 0 1 1 1 ]
[ 1 1 0 1 1 1 0 ]
```

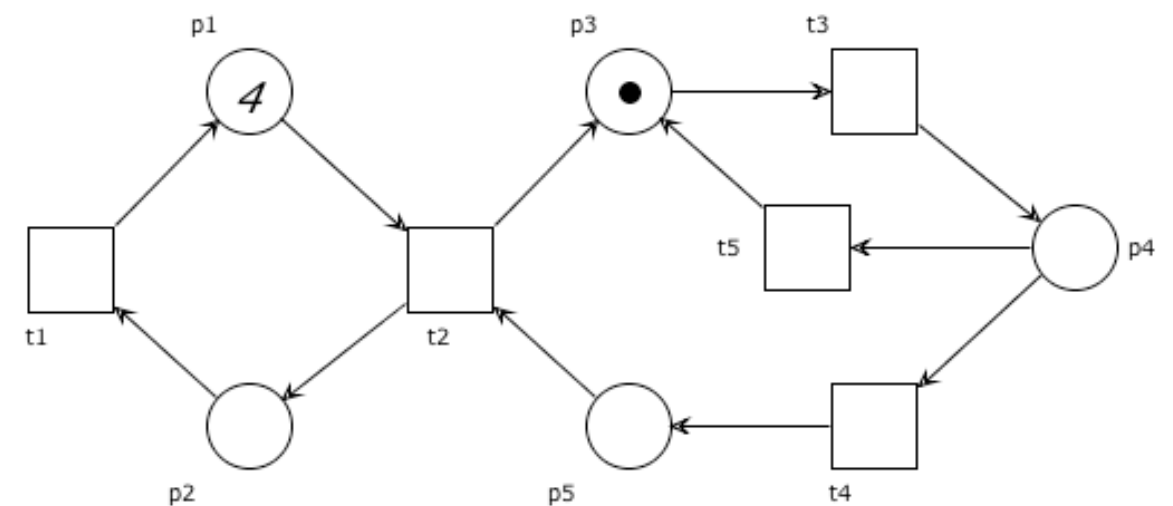
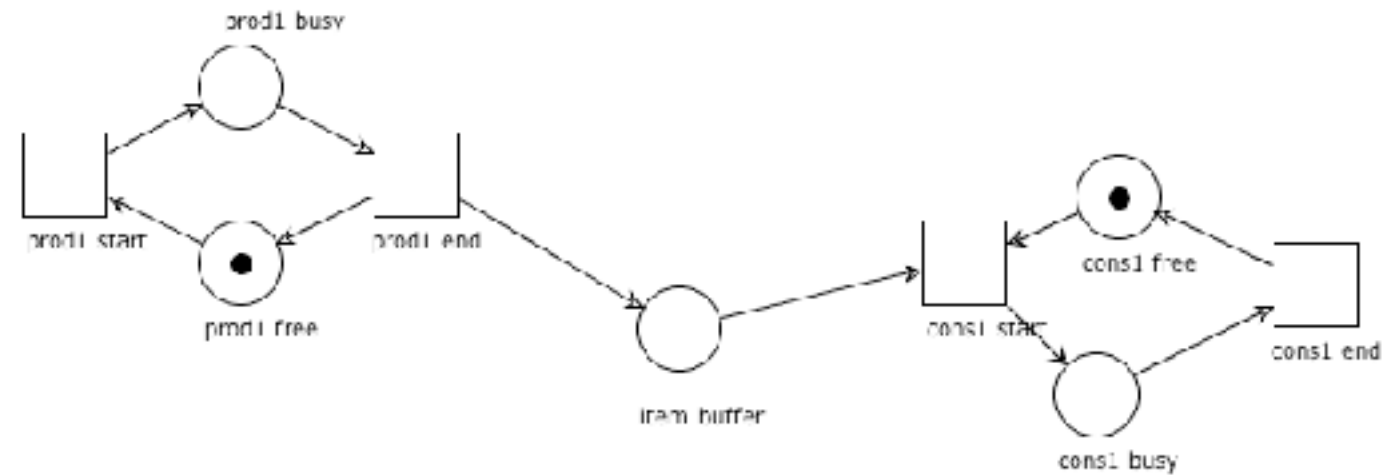
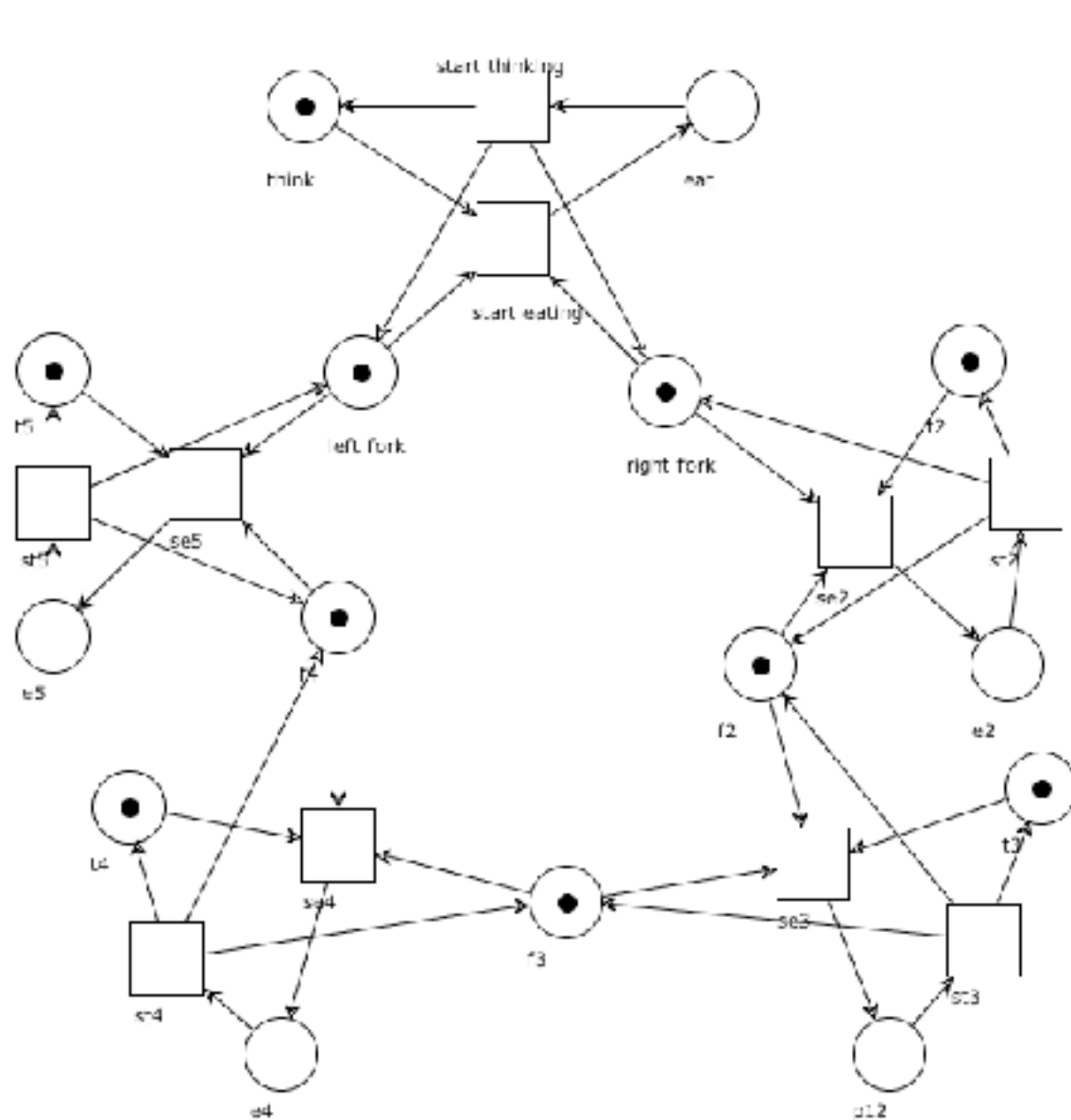
Traffic-lights example



$$\begin{aligned}
 & [1\ 1\ 1\ 0\ 0\ 0\ 0] + \\
 & [0\ 0\ 0\ 0\ 1\ 1\ 1] + \\
 & [1\ 1\ 0\ 1\ 1\ 1\ 0] = \\
 & [2\ 2\ 1\ 1\ 2\ 2\ 1]
 \end{aligned}$$

Exercises

Define two (linearly independent) S-invariants for each of the nets below



S-invariants and system properties

Semi-positive S-invariants

The S-invariant \mathbf{I} is **semi-positive** if $\mathbf{I} > \mathbf{0}$
(i.e. $\mathbf{I} \geq \mathbf{0}$ and $\mathbf{I} \neq \mathbf{0}$)

all entries are non-negative
and at least one is positive

The **support** of \mathbf{I} is: $\langle \mathbf{I} \rangle = \{ p \mid \mathbf{I}(p) > 0 \}$

The S-invariant \mathbf{I} is **positive** if $\mathbf{I} \succ \mathbf{0}$ all entries are positive
(i.e. $\mathbf{I}(p) > 0$ for any place $p \in P$)
(i.e. $\langle \mathbf{I} \rangle = P$)

A (semi-positive) S-invariant whose coefficients are all 0 and 1 is called **uniform**

Note

Notation: $\bullet S = \bigcup_{s \in S} \bullet s$

Every semi-positive invariant
satisfies the equation

transitions that produce tokens
in some places of the support

$$\bullet \langle \mathbf{I} \rangle = \langle \mathbf{I} \rangle \bullet$$

transitions that consume tokens
from some places of the support

pre-sets of support equal post-sets of support

(the result holds for both S-invariants and T-invariants)

A sufficient condition for boundedness

Theorem:

If (P, T, F, M_0) has a positive S-invariant then it is bounded

Let $M \in [M_0 \rangle$ and let \mathbf{I} be a positive S-invariant.

Let $p \in P$. Then $\mathbf{I}(p)M(p) \leq \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since \mathbf{I} is positive, we can divide by $\mathbf{I}(p)$:

$$M(p) \leq (\mathbf{I} \cdot M_0) / \mathbf{I}(p)$$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q)M(q)$$

Consequences of previous theorem

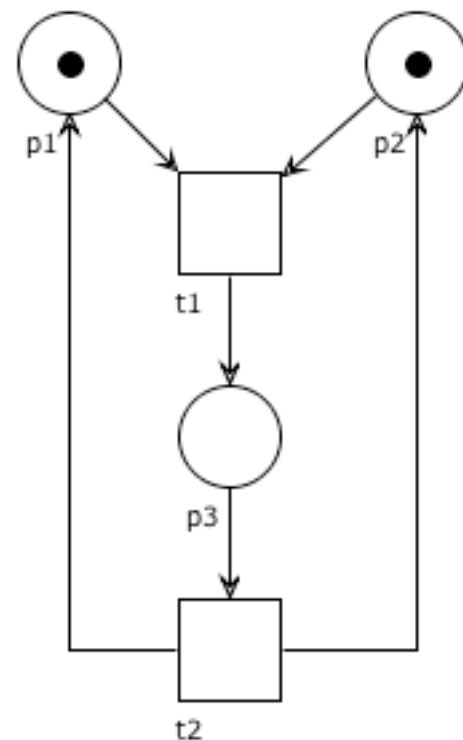
By exhibiting a positive S-invariant we can prove that the system is **bounded for any initial marking**

Note that all places in the support of a semi-positive S-invariant are **bounded for any initial marking**

$$M(p) \leq \frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p)} \quad \text{this value is independent from the reachable marking } M$$

Example

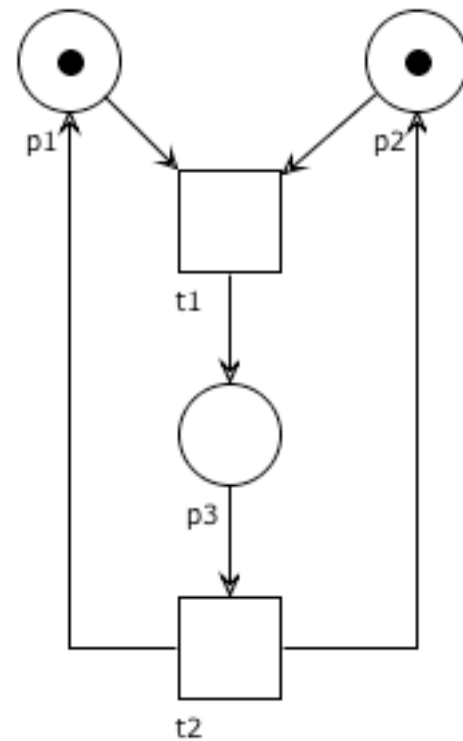
To prove that the system is bounded we can just exhibit a positive S-invariant



$$I = [1 \ 1 \ 2]$$

Example

How many tokens are at most in p_3 ?

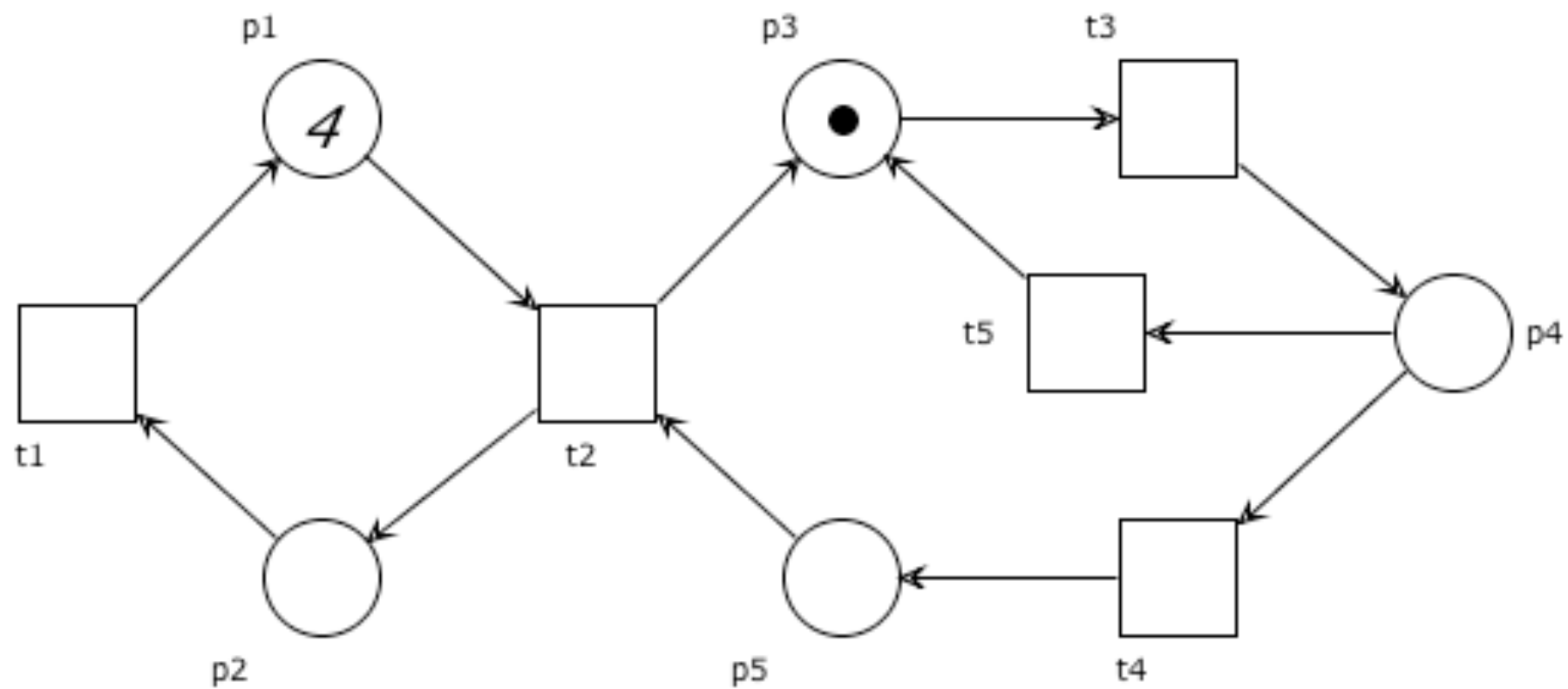


$$I = [1 \ 1 \ 2]$$

$$\frac{I \cdot M_0}{I(p_3)} = \frac{2}{2} = 1$$

Exercises

Find a positive S-invariant for the net below



A necessary condition for liveness

Theorem:

If (P, T, F, M_0) is live then for every semi-positive invariant \mathbf{I} :

$$\mathbf{I} \cdot M_0 > 0$$

Let $p \in \langle \mathbf{I} \rangle$ and take any $t \in \bullet p \cup p \bullet$.

By liveness, there are $M, M' \in [M_0 \rangle$ with $M \xrightarrow{t} M'$

Then, $M(p) > 0$ (if $t \in p \bullet$) or $M'(p) > 0$ (if $t \in \bullet p$)

If $M(p) > 0$, then $\mathbf{I} \cdot M \geq \mathbf{I}(p)M(p) > 0$

If $M'(p) > 0$, then $\mathbf{I} \cdot M' \geq \mathbf{I}(p)M'(p) > 0$

In any case, $\mathbf{I} \cdot M_0 = \mathbf{I} \cdot M = \mathbf{I} \cdot M' > 0$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q)M(q)$$

Consequence of previous theorem

If we find a semi-positive invariant such that

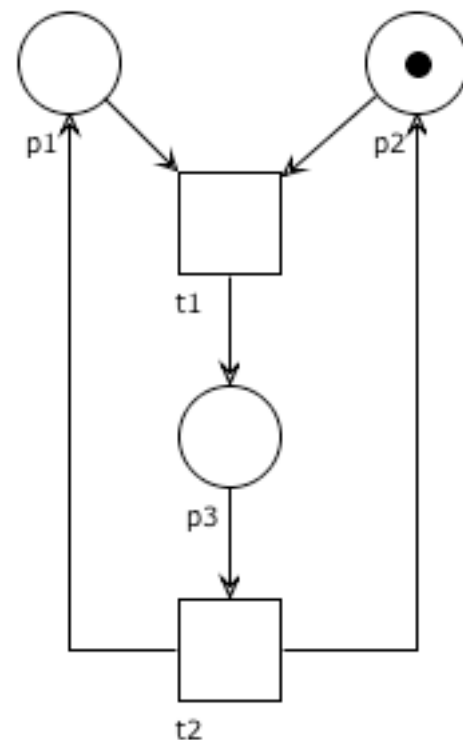
$$\mathbf{I} \cdot M_0 = 0$$

Then we can conclude that the system **is not live**

Example

the system is not live

It is immediate to check the counter-example



$$I = [1 \ 0 \ 1]$$

$$[1 \ 0 \ 1] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

I M₀

Markings that agree on all S -invariant

Definition: M and M' **agree on all S -invariants** if for every S -invariant I we have $I \cdot M = I \cdot M'$

Note: by properties of linear algebra, this corresponds to require that the equation on y
 $M + N \cdot y = M'$ has some rational-valued solution

Remark: In general, there can exist M and M' that agree on all S -invariants but such that none of them is reachable from the other

A necessary condition for reachability

Reachability is decidable, but computationally expensive
(EXPSpace-hard)

S-invariants provide a preliminary check that can be
computed efficiently

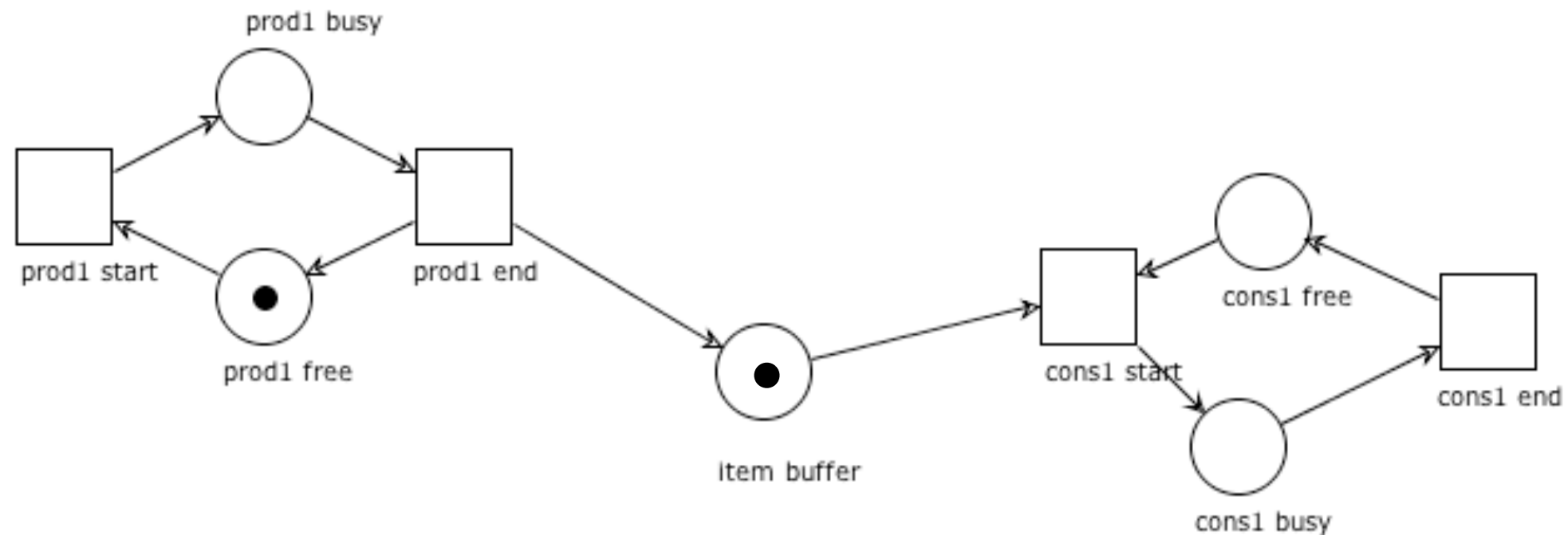
Let (P, T, F, M_0) be a system.

If there is an S-invariant \mathbf{I} s.t. $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$ then $M \notin [M_0 \rangle$

If the equation $\mathbf{N} \cdot \mathbf{y} = M - M_0$ has no rational-valued solution, then $M \notin [M_0 \rangle$

Example

Prove that the marking
 $M = \text{prod1free} + \text{cons1busy}$
is not reachable



$$I = [0 \ 0 \ 0 \ 1 \ 1]$$
$$I \cdot M_0 = 0 \neq 1 = I \cdot M$$

S-invariants: recap

Positive S-invariant \Rightarrow boundedness
Unboundedness \Rightarrow no positive S-invariant

Semi-positive S-invariant I and liveness $\Rightarrow I \cdot M_0 > 0$
Semi-positive S-invariant I and $I \cdot M_0 = 0 \Rightarrow$ non-live

S-invariant I and M reachable $\Rightarrow I \cdot M = I \cdot M_0$
S-invariant I and $I \cdot M \neq I \cdot M_0 \Rightarrow M$ not reachable

S-invariants: pay attention to implication

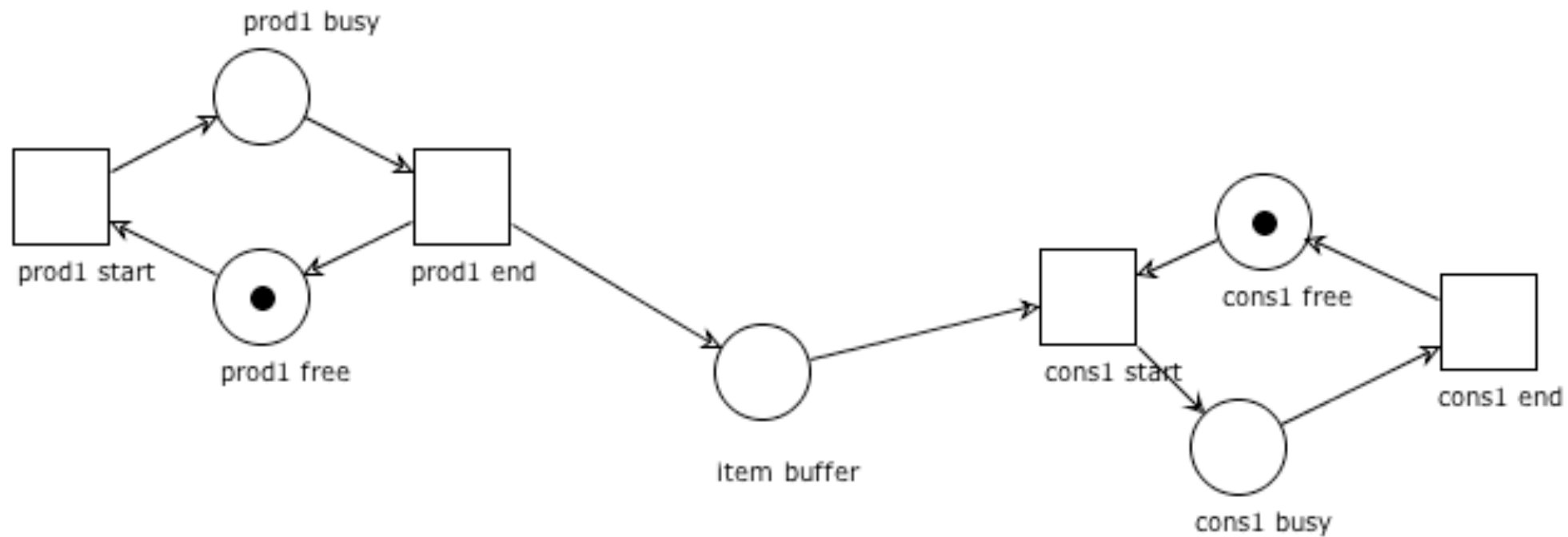
No positive S-invariant \Rightarrow **maybe** unbounded

Semi-positive S-invariant I and $I \cdot M_0 > 0 \Rightarrow$ **maybe** live

S-invariant I and $I \cdot M = I \cdot M_0 \Rightarrow$ **maybe** M reachable

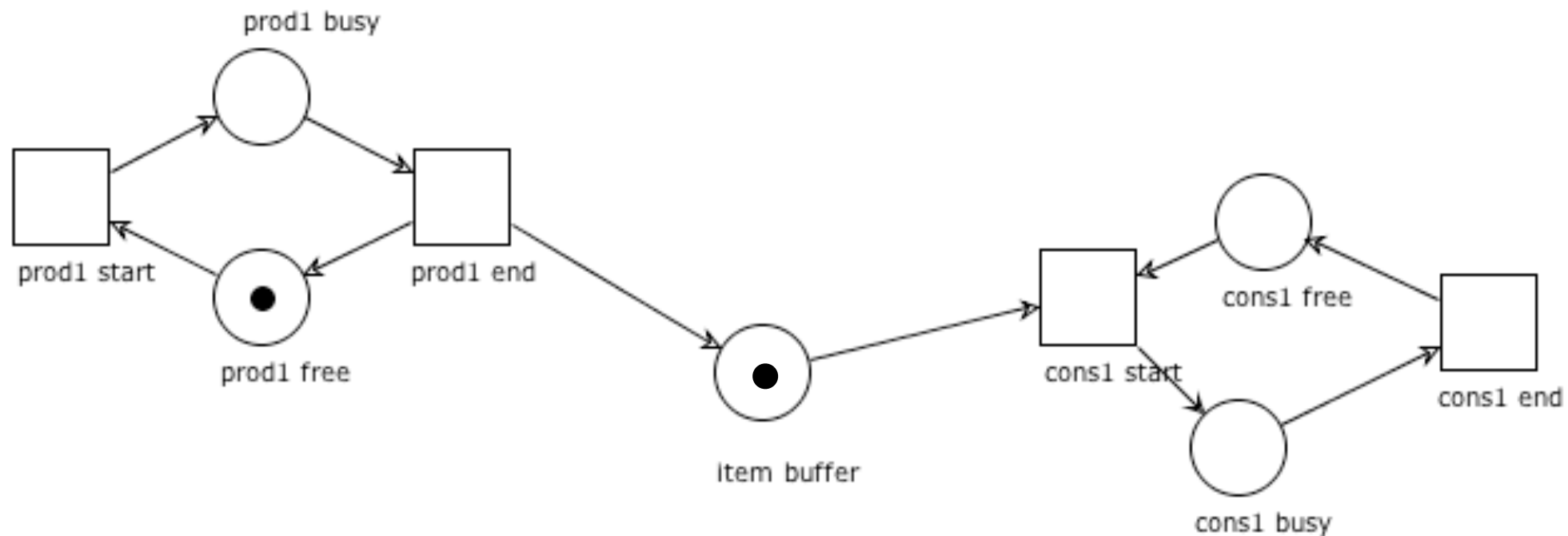
Exercises

Can you find a positive S-invariant?



Exercises

Prove that the system is not live by exhibiting a suitable S-invariant



T-invariants

Dual reasoning

The S-invariants of a net N are vectors satisfying the equation

$$\mathbf{x} \cdot \mathbf{N} = \mathbf{0}$$

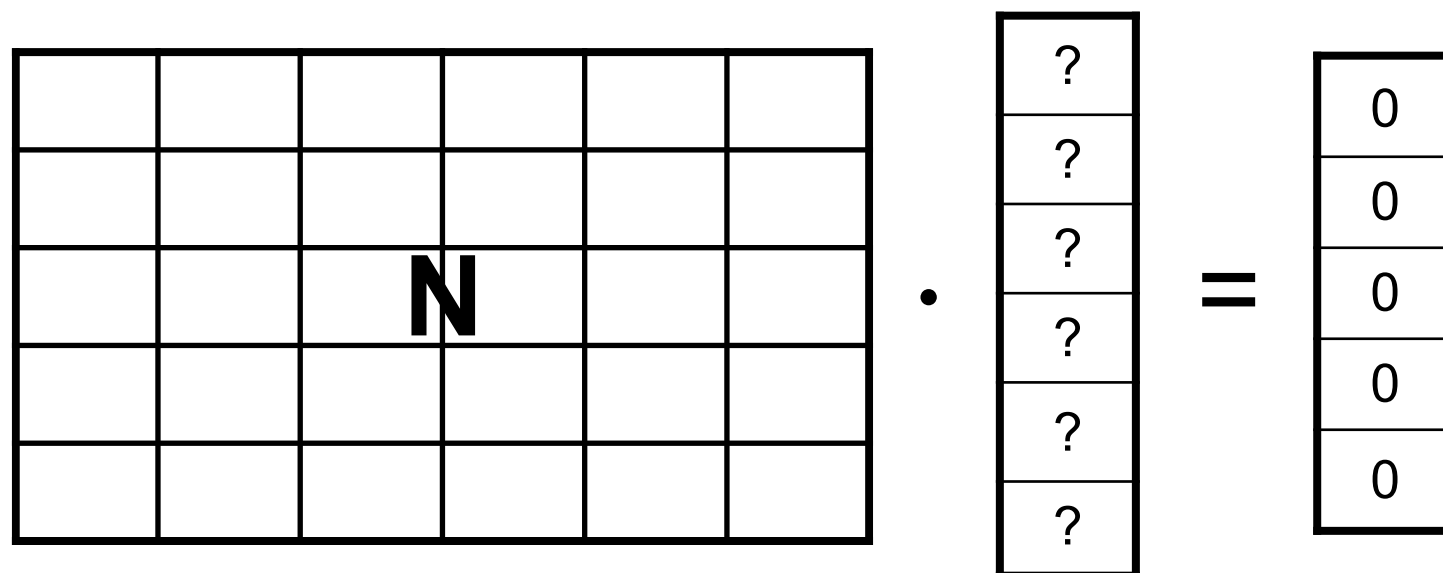
It seems natural to ask if we can find some interesting properties also for the vectors satisfying the equation

$$\mathbf{N} \cdot \mathbf{y} = \mathbf{0}$$

T-invariant (aka transition-invariant)

Definition: A **T-invariant** of a net $N=(P,T,F)$ is a rational-valued solution y of the equation

$$N \cdot y = 0$$



Fundamental property of T-invariants

Proposition: Let $M \xrightarrow{\sigma} M'$.

The Parikh vector $\vec{\sigma}$ is a T-invariant iff $M' = M$

\Rightarrow) By the marking equation lemma $M' = M + \mathbf{N} \cdot \vec{\sigma}$
Since $\vec{\sigma}$ is a T-invariant $\mathbf{N} \cdot \vec{\sigma} = \mathbf{0}$, thus $M' = M$.

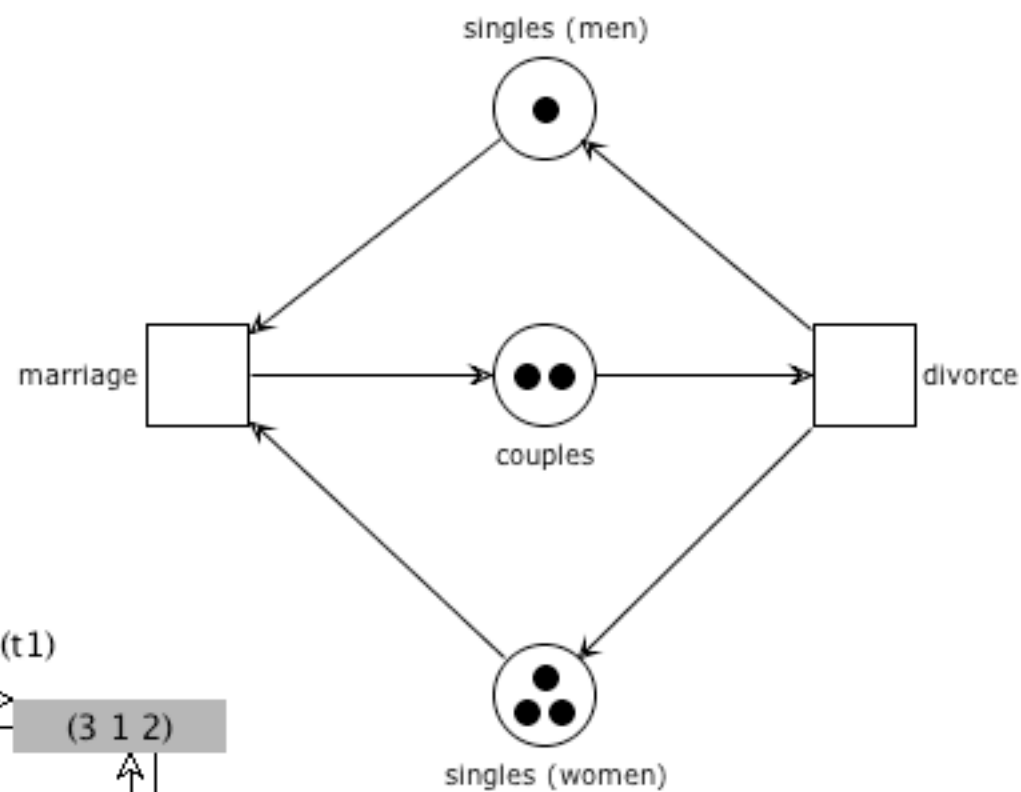
\Leftarrow) If $M \xrightarrow{\sigma} M$, by the marking equation lemma $M = M + \mathbf{N} \cdot \vec{\sigma}$
Thus $\mathbf{N} \cdot \vec{\sigma} = M - M = \mathbf{0}$ and $\vec{\sigma}$ is a T-invariant

Transition-invariant, intuitively

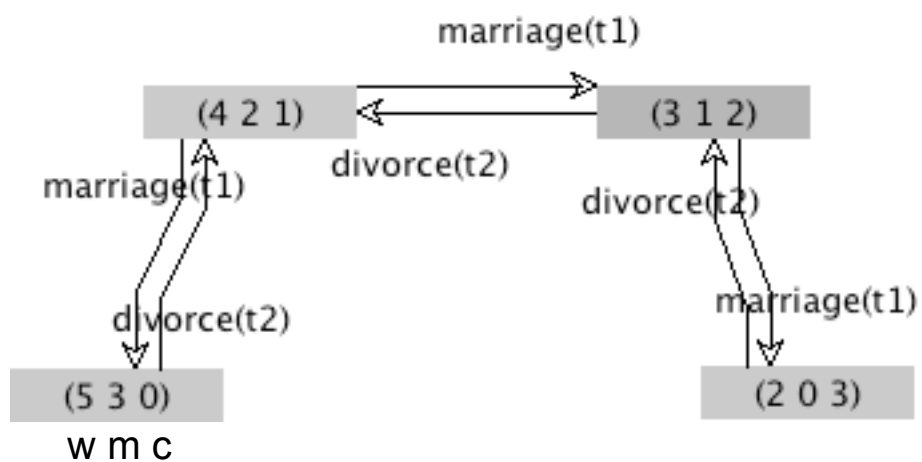
A transition-invariant assigns a **number of occurrences to each transition** such that any occurrence sequence comprising exactly those transitions leads to the same marking where it started (independently from the order of execution)

Example

An easy-to-be-found T-invariant



$$\begin{matrix} m & d \\ [& 1 & 1] \end{matrix}$$



Alternative definition of T-invariant

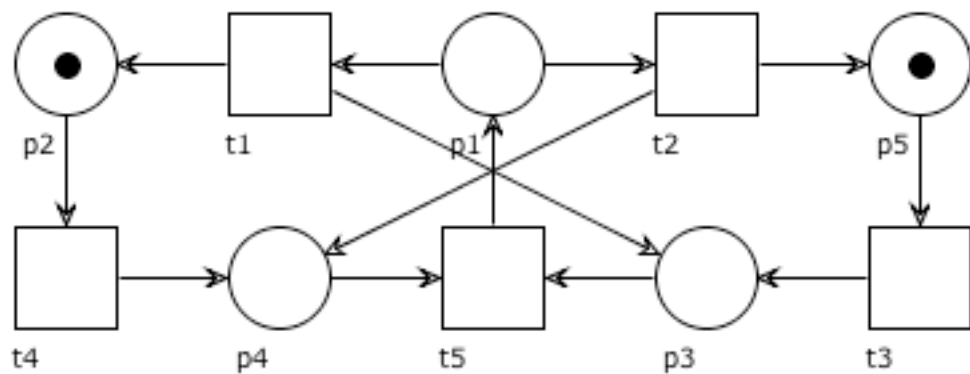
Proposition:

A mapping $\mathbf{J} : T \rightarrow \mathbb{Q}$ is a T-invariant of N iff for any $p \in P$:

$$\sum_{t \in \bullet p} \mathbf{J}(t) = \sum_{t \in p \bullet} \mathbf{J}(t)$$

Question time

Which of the following are T-invariants?



t_1	t_2	t_3	t_4	t_5
$[1$	0	0	1	$1]$
$[1$	1	2	1	$2]$
$[1$	1	1	0	$2]$
$[1$	1	1	1	$2]$
$[0$	1	1	0	$1]$

$$\forall p \in P, \sum_{t \in \bullet p} \mathbf{J}(t) \stackrel{?}{=} \sum_{t \in p \bullet} \mathbf{J}(t)$$

T-invariants and system properties

Pigeonhole principle

If n items are put into m slots, with $n > m$, then at least one slot must contain more than one item



Reproduction lemma

Lemma: Let (P, T, F, M_0) be a bounded system.

If $M_0 \xrightarrow{\sigma}$ for some infinite sequence σ , then

there is a semi-positive T-invariant \mathbf{J} such that $\langle \mathbf{J} \rangle \subseteq \{t \mid t \in \sigma\}$.

Assume $\sigma = t_1 t_2 t_3 \dots$ and $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \dots$

By boundedness: $[M_0]$ is finite.

By the pigeonhole principle, there are $0 \leq i < j$ s.t. $M_i = M_j$

Let $\sigma' = t_{i+1} \dots t_j$. Then $M_i \xrightarrow{\sigma'} M_j = M_i$

By the marking equation lemma: $\vec{\sigma}'$ is a T-invariant. (fund. prop. of T-inv.)

It is semi-positive, because σ' is not empty ($i < j$).

Clearly, $\langle \mathbf{J} \rangle$ only includes transitions in σ .

Boundedness, liveness and positive T-invariant

Theorem: If a bounded system is live,
then it has a positive T-invariant

By boundedness: $[M_0 \rangle$ is finite and we let $k = |[M_0 \rangle|$.

By liveness: $M_0 \xrightarrow{\sigma_1} M_1$ with $\vec{\sigma}_1(t) > 0$ for any $t \in T$

Similarly: $M_1 \xrightarrow{\sigma_2} M_2$ with $\vec{\sigma}_2(t) > 0$ for any $t \in T$

Similarly: $M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \dots \xrightarrow{\sigma_k} M_k$

By the pigeonhole principle, there are $0 \leq i < j \leq k$ s.t. $M_i = M_j$

Let $\sigma = \sigma_{i+1} \dots \sigma_j$. Then $M_i \xrightarrow{\sigma} M_j = M_i$

By the marking equation lemma: $\vec{\sigma}$ is a T-invariant. (fund. prop. of T-inv.)

It is positive, because $\vec{\sigma}(t) \geq \vec{\sigma}_j(t) > 0$ for any $t \in T$.

Corollary of previous theorem

Every live and bounded system has:

a reachable marking M and

an occurrence sequence $M \xrightarrow{\sigma} M$

such that all transitions of N occur in σ .

T-invariants: recap

Boundedness + liveness \Rightarrow positive T-invariant

No positive T-invariant \Rightarrow non (live + bounded)

No positive T-invariant \Rightarrow non-live OR unbounded

No positive T-invariant + liveness \Rightarrow unbounded

No positive T-invariant + boundedness \Rightarrow non-live

No positive T-inv. + positive S-inv. \Rightarrow non-live

T-invariants: pay attention to implication

No positive T-invariant

=> **maybe** non live

Exercises

Exhibit a system that has a positive T-invariant
but is not
live and bounded

Exhibit a live system that has a positive T-invariant
but is not bounded