Business Processes Modelling

MPB (6 cfu, 295AA)

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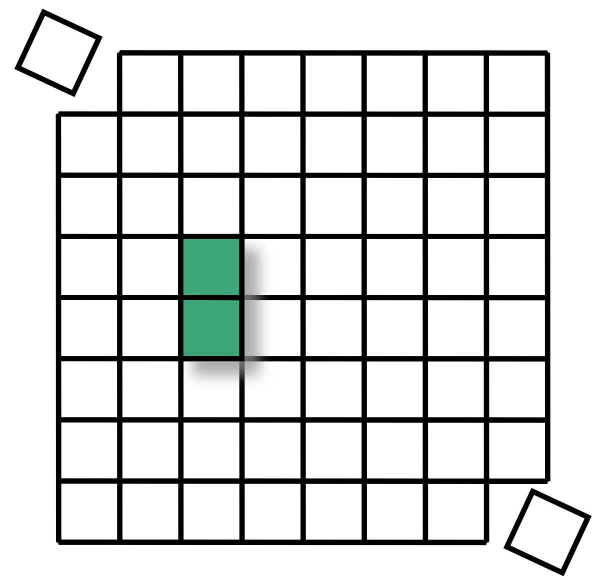
http://www.di.unipi.it/~bruni

11 - Invariants

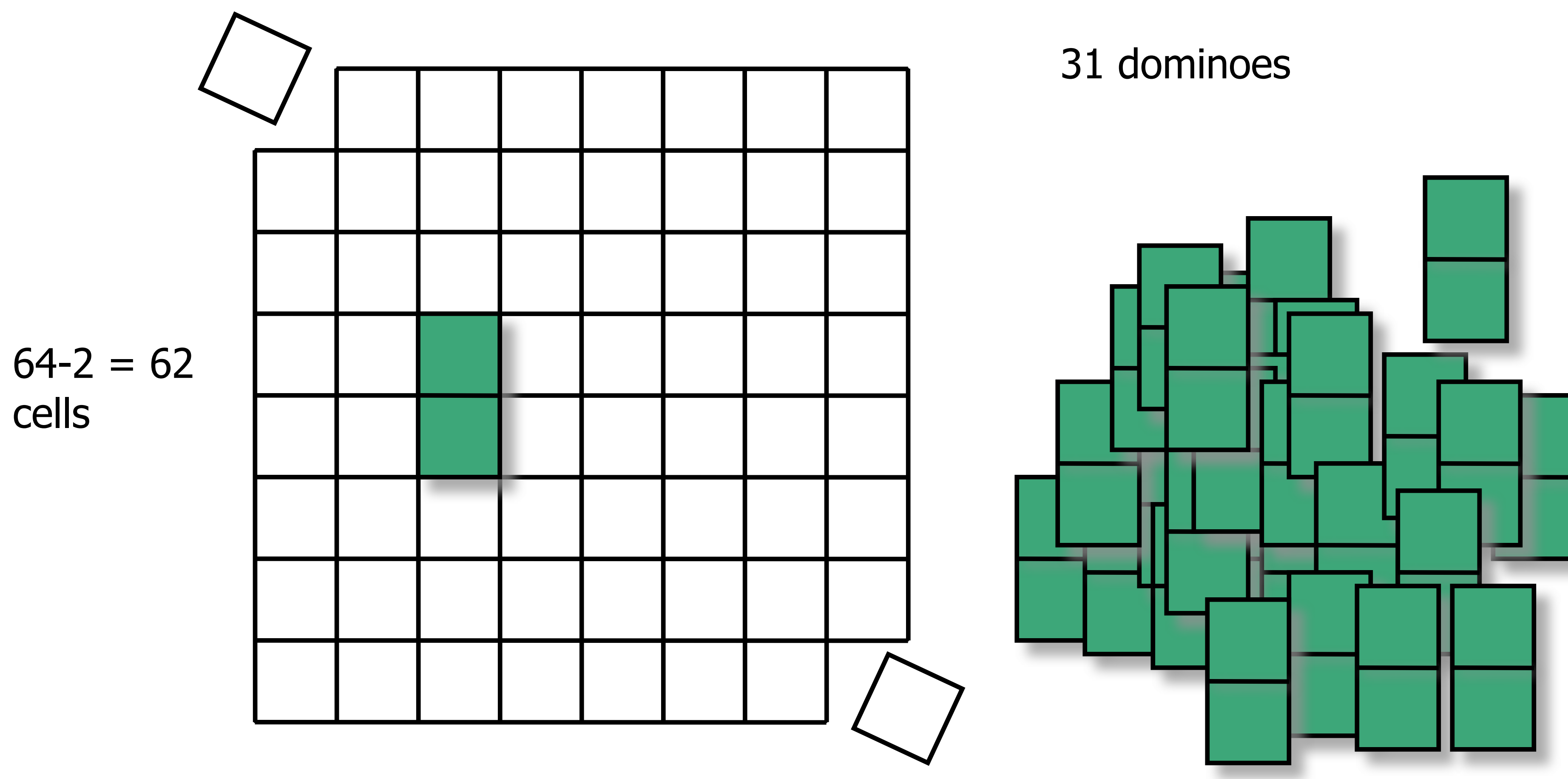
Object



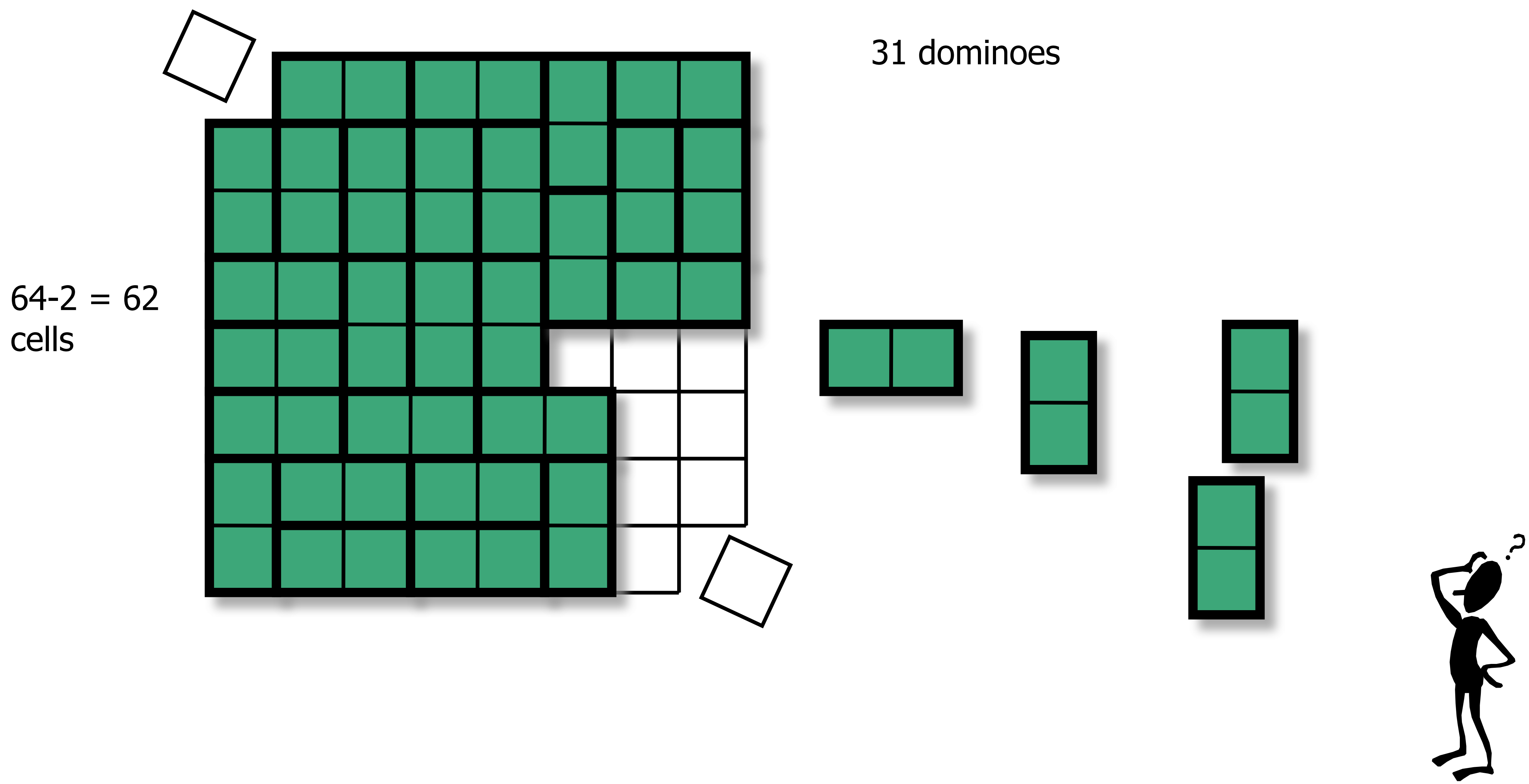
We introduce two relevant kinds of invariants for Petri nets

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Puzzle time: tiling a chessboard with dominoes

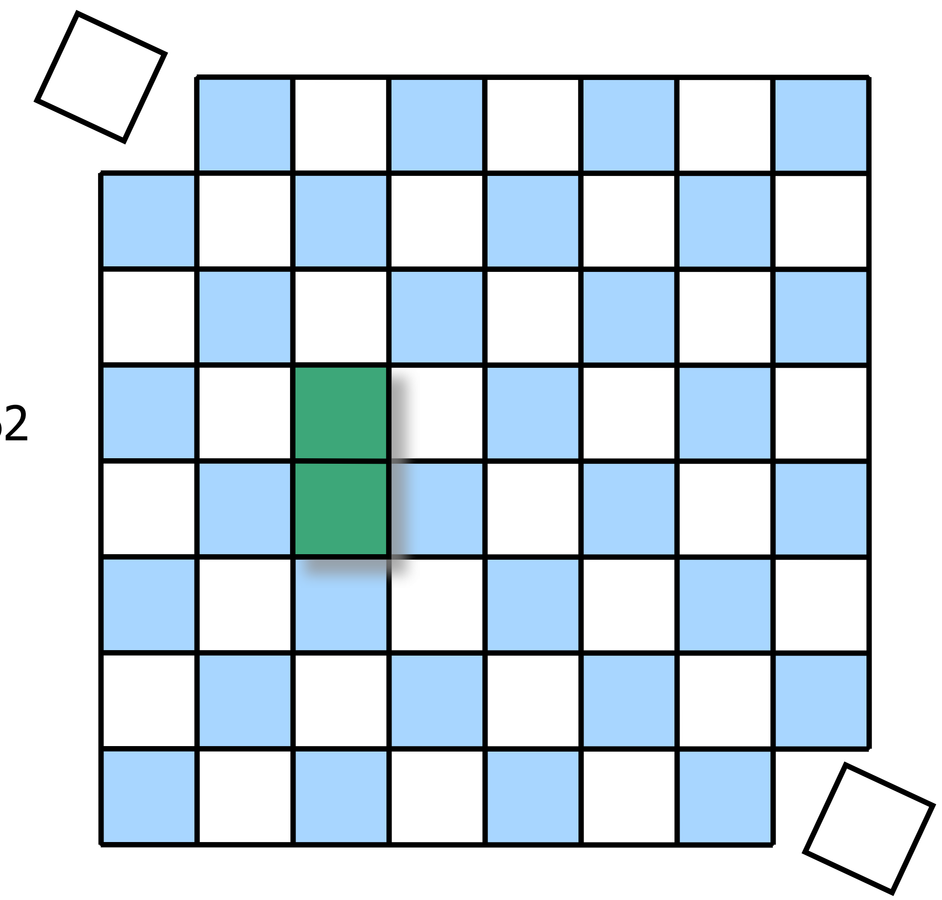


Puzzle time: tiling a chessboard with dominoes

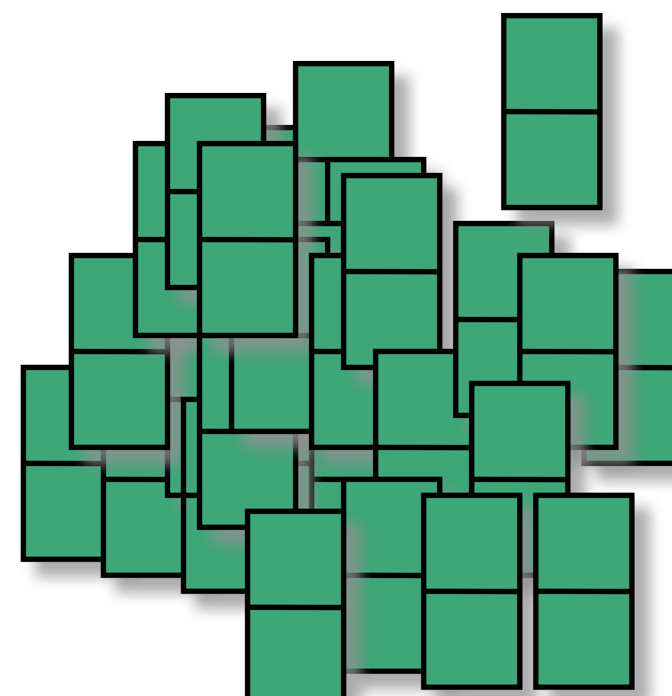


Puzzle time: tiling a chessboard with dominoes

64-2 = 62
cells



31 dominoes

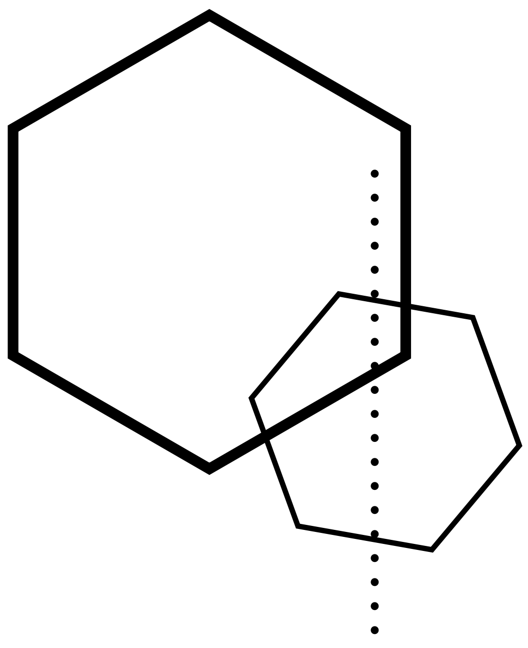


Invariant

An invariant of a dynamic system is an assertion that holds at every reachable state

Example

You have a polygon



You can rotate it

You can move it

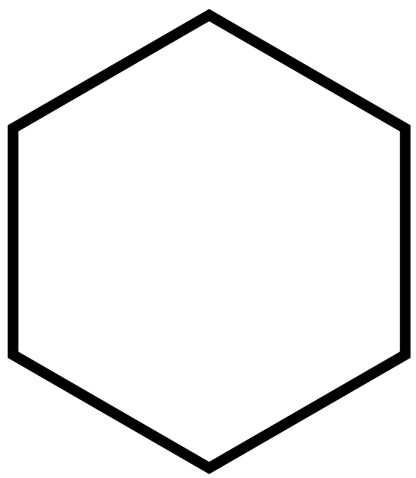
You can scale it

You can mirror it

Which invariants?

Example

You have a polygon



You can rotate it

You can move it

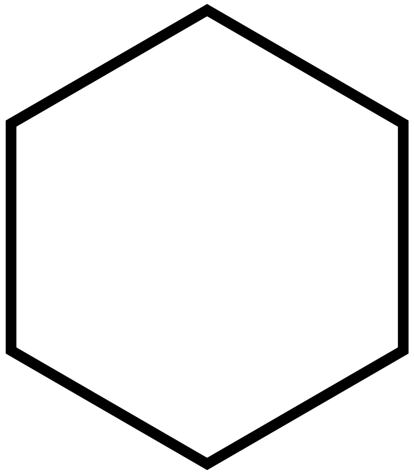
You can scale it

You can mirror it

Which invariants? perimeter

Example

You have a polygon



You can rotate it

You can move it

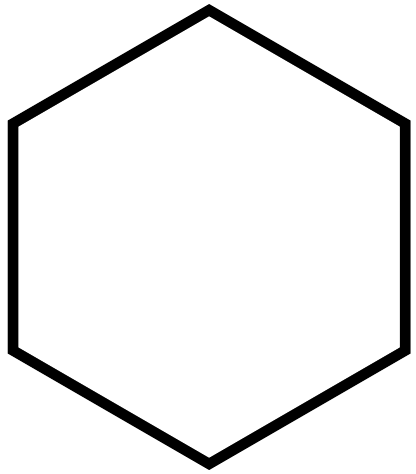
You can scale it

You can mirror it

Which invariants? area

Example

You have a polygon



You can rotate it

You can move it

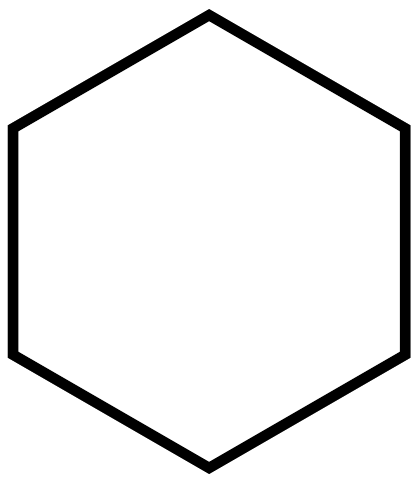
You can scale it

You can mirror it

Which invariants? number of vertices

Example

You have a polygon



You can rotate it

You can move it

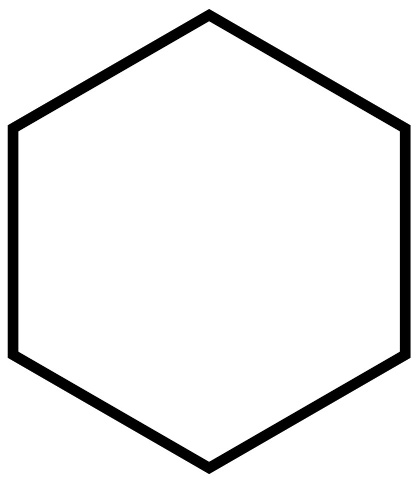
You can scale it

You can mirror it

Which invariants? number of sides

Example

You have a polygon



You can rotate it

You can move it

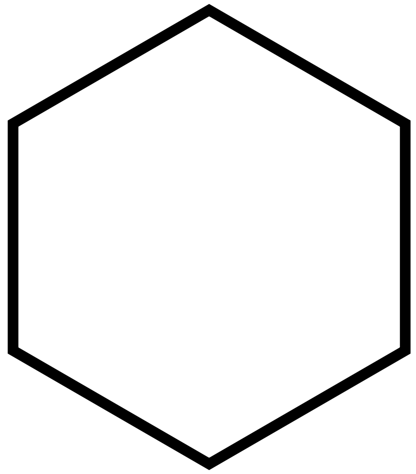
You can scale it

You can mirror it

Which invariants? vertex degrees

Example

You have a polygon



You can rotate it

You can move it

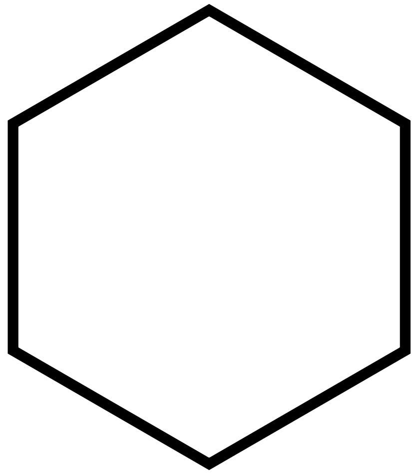
You can scale it

You can mirror it

Which invariants? convexity

Example

You have a polygon



You can rotate it

You can move it

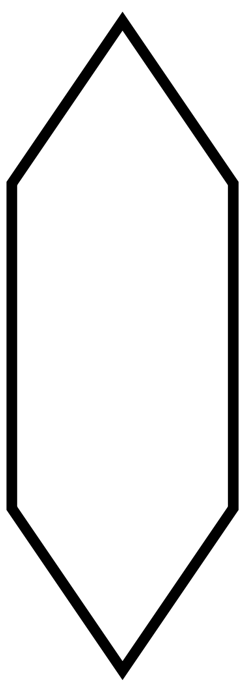
You can scale it

You can mirror it

Which invariants? color

Question time

You have a polygon



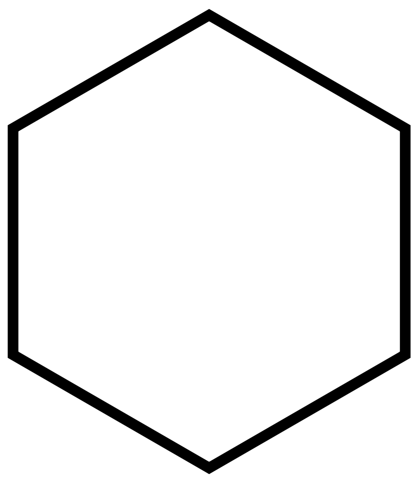
You can rotate it
You can move it
You can scale it
You can mirror it
You can stretch it

Which invariants?

color
convexity?
vertex degrees?
number of sides?
number of vertices?
area
perimeter

Question time

You have a polygon



You can rotate it

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Which invariants?

color

convexity

vertex degrees?

number of sides?

number of vertices?

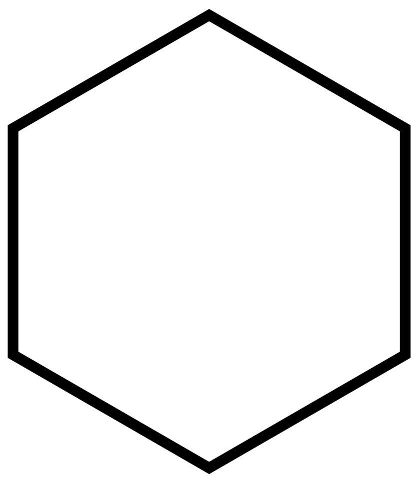
area

perimeter

Question time

You have a polygon

Which invariants?



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color

convexity

vertex degrees

number of sides?

number of vertices?

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perimeter

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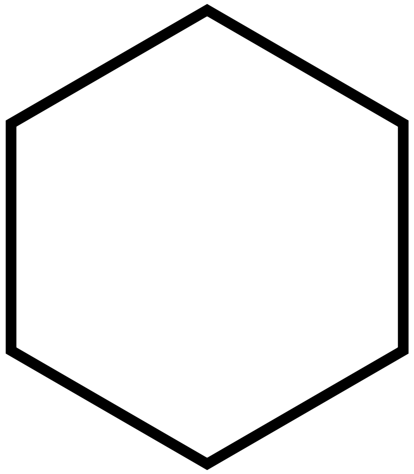

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Which invariants?

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number of sides

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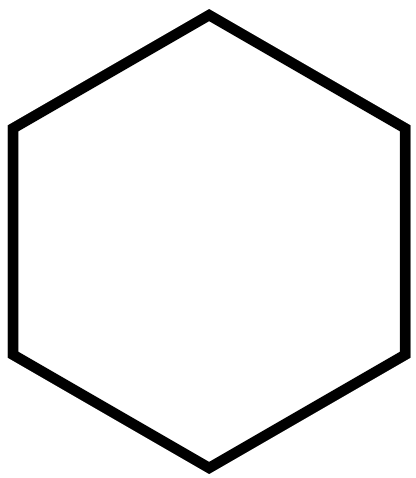

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Which invariants?

color

convexity

vertex degrees

number of sides

number of vertices

area

perimeter

Puzzle: from MI to MU

You can compose words using symbols **M**, **I**, **U**

Given the initial word **MI**, you can apply the following transformations, in any order, as many times as you like:

1. Add a **U** to the end of any string ending in **I** (e.g., **MI** to **MIU**).
2. Double the string after the **M** (e.g., **MIU** to **MIUIU**).
3. Replace any **III** with a **U** (e.g., **MUIIIU** to **MUUU**).
4. Remove any **UU** (e.g., **MUUU** to **MU**).

Puzzle: from MI to MU

You can compose words using symbols **M**, **I**, **U**

Given the initial word **MI**, you can apply the following transformations, in any order, as many times as you like:

1. Add a **U** to the end of any string ending in **I**. $w\mathbf{I} \rightarrow w\mathbf{IU}$
2. Double the string after the **M**. $\mathbf{M}w \rightarrow \mathbf{M}ww$
3. Replace any **III** with a **U**. $w_1\mathbf{III}w_2 \rightarrow w_1\mathbf{U}w_2$
4. Remove any **UU**. $w_1\mathbf{UU}w_2 \rightarrow w_1w_2$

$$\mathbf{MI} \xrightarrow[2]{} \mathbf{MII} \xrightarrow[2]{} \mathbf{MIIII} \xrightarrow[1]{} \mathbf{MIIIIU} \xrightarrow[3]{} \mathbf{MIUU} \xrightarrow[4]{} \mathbf{MI}$$

Puzzle: from MI to MU

You can compose words using symbols **M**, **I**, **U**

Given the initial word **MI**, you can apply the following transformations, in any order, as many times as you like:

1. Add a **U** to the end of any string ending in **I**.
2. Double the string after the **M**.
3. Replace any **III** with a **U**.
4. Remove any **UU**.

Can you transform **MI** to **MU**?
(*Hint*: count the number of **I** modulo 3)

Modular arithmetic

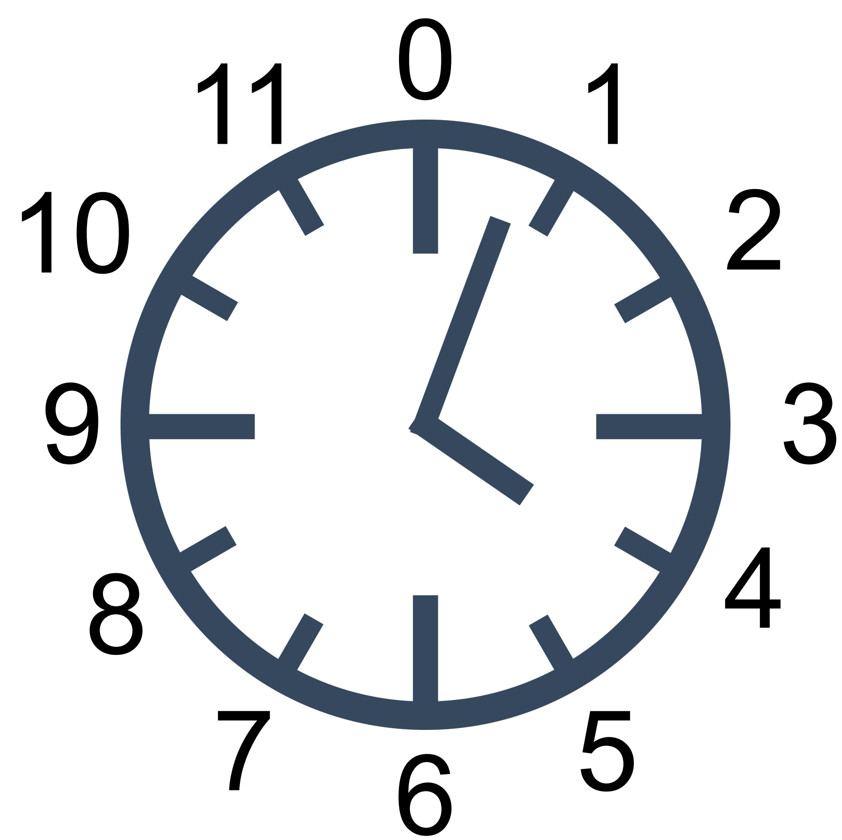
Numbers where the counting “wrap around”
when reaching a certain bound, called the modulus

counting modulo k: only numbers from 0 to k-1

n modulo k = remainder of integer division n over k
(often denoted n%k)

9 % 2 = 1
9 % 3 = 0
9 % 5 = 4

Modular arithmetic: example

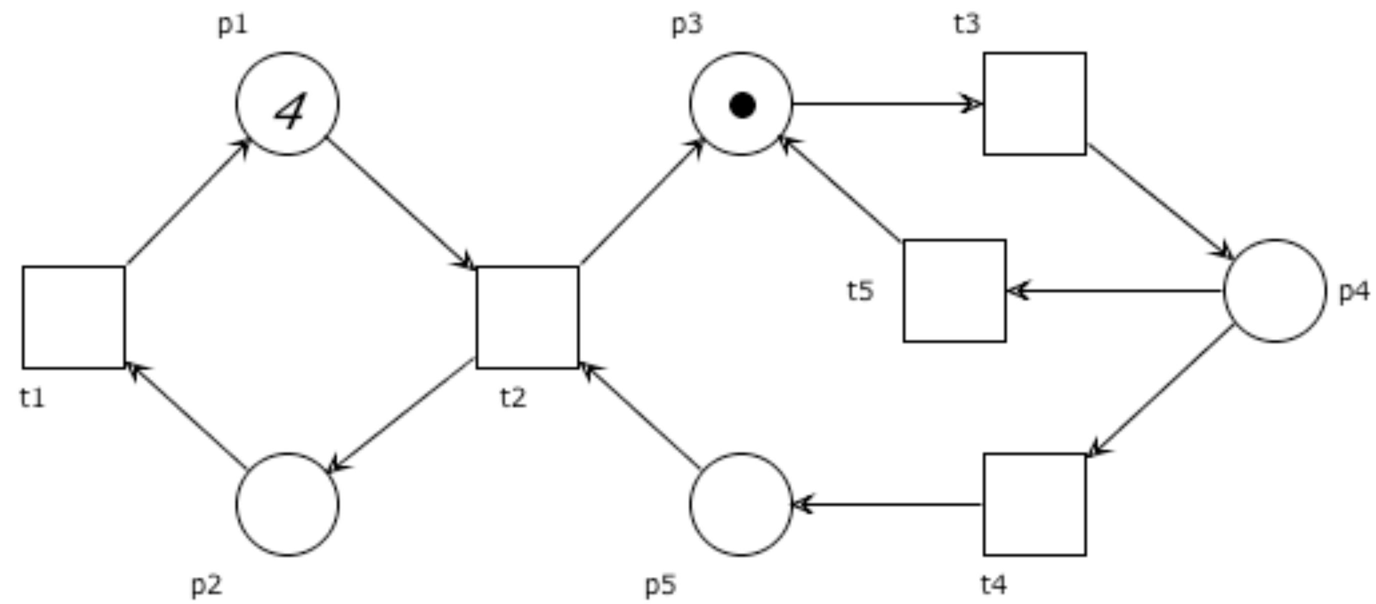


31 % 12 = 7

Example

You have a Petri net

$$(P, T, F, M_0)$$



You can fire any currently enabled transition

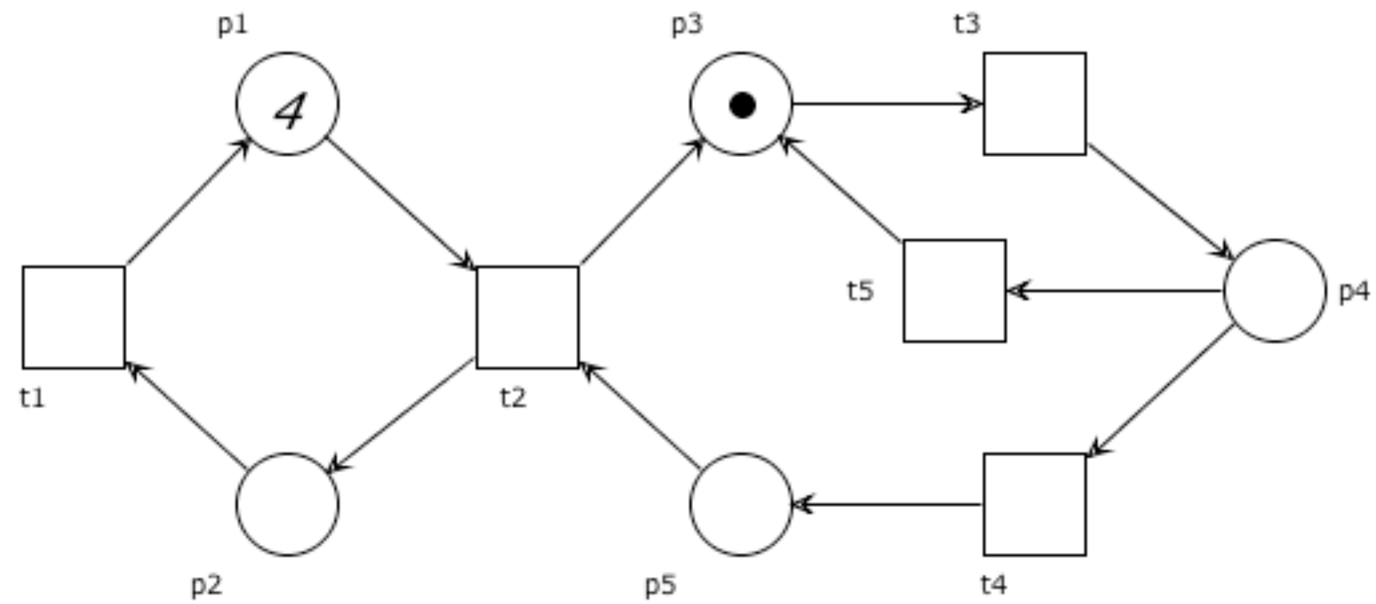
Which invariants?

Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any currently enabled transition

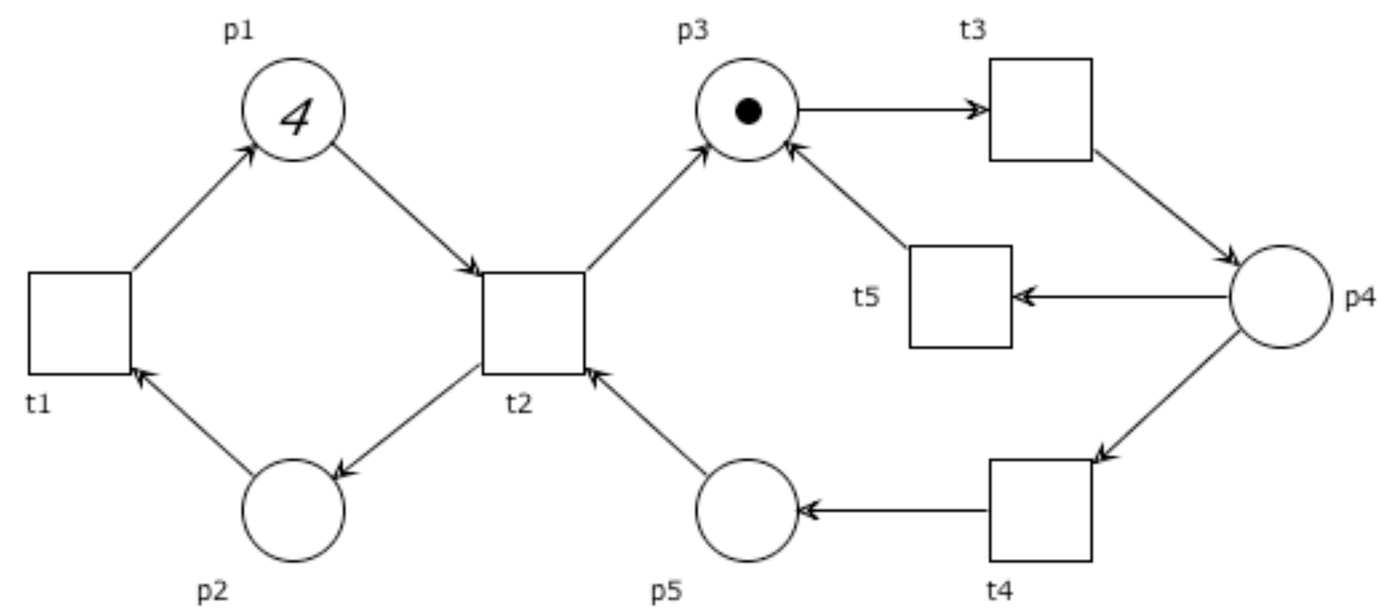


Which invariants? color

Example

You have a Petri net

$$(P, T, F, M_0)$$



You can fire any currently enabled transition

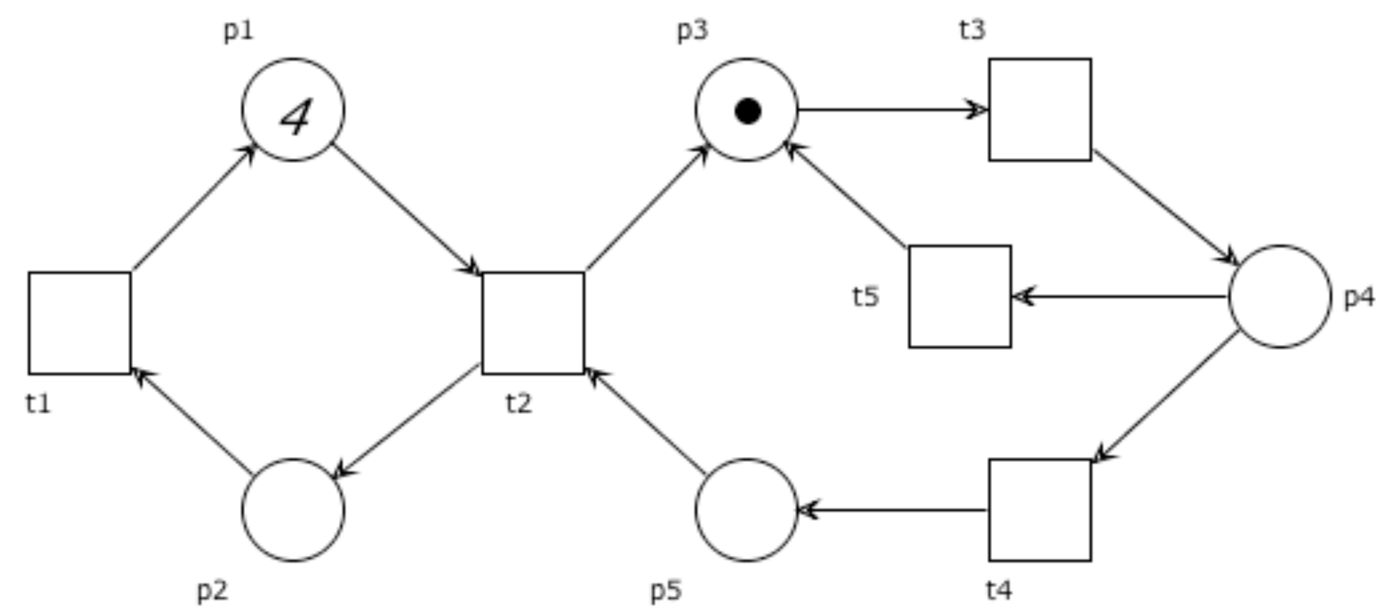
Which invariants? P, T, F

Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any currently enabled transition

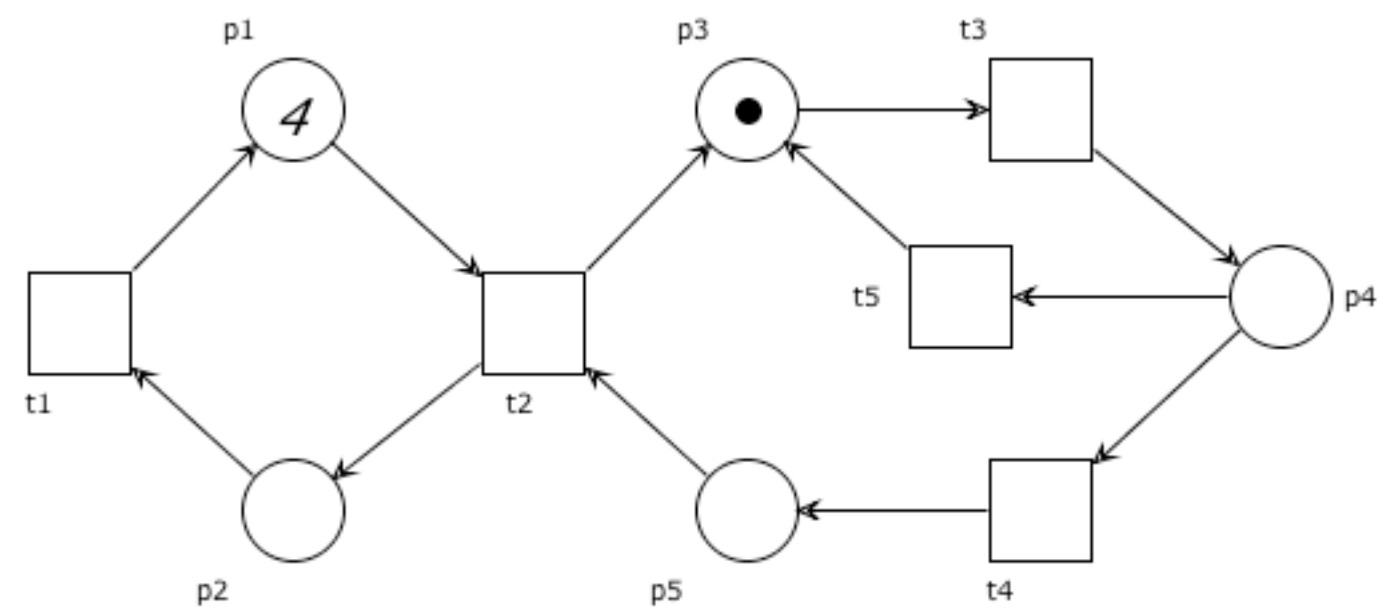


Which invariants? number of tokens in p3

Example

You have a Petri net

$$(P, T, F, M_0)$$



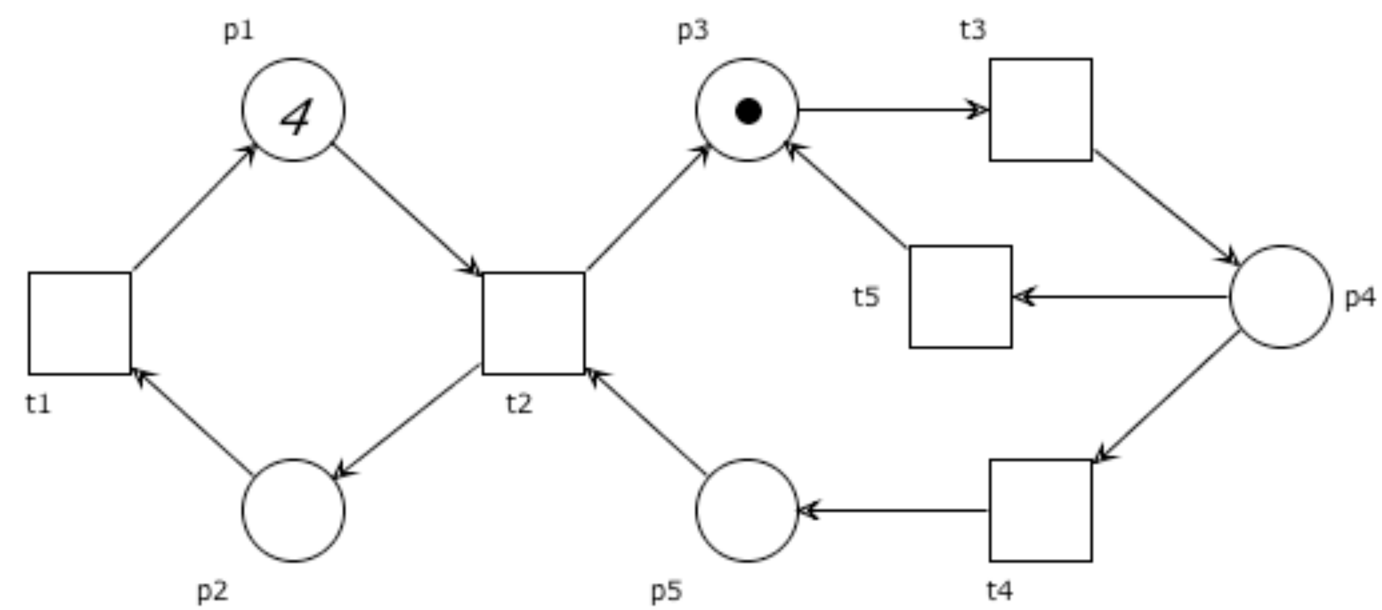
You can fire any currently enabled transition

Which invariants? number of tokens in a dead place

Example

You have a Petri net

$$(P, T, F, M_0)$$



You can fire any currently enabled transition

Which invariants?

Any property that holds for any reachable marking

Recall: Liveness, formally

$$(P, T, F, M_0)$$

$$\forall t \in T, \quad \forall M \in [\, M_0 \,\rangle, \quad \exists M' \in [\, M \,\rangle, \quad M' \xrightarrow{t}$$

Liveness as invariant

Lemma
If (P, T, F, M_0) is live and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is live.

Let $t \in T$ and $M' \in [\, M \,\rangle$.

Since $M \in [\, M_0 \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is live, $\exists M'' \in [\, M' \,\rangle$ with $M'' \xrightarrow{t}$.

Therefore (P, T, F, M) is live.

Recall: Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [\, M_0 \,\rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

Deadlock freedom as invariant

Lemma: If (P, T, F, M_0) is deadlock-free and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is deadlock-free.

Let $M' \in [\, M \,\rangle$.

Since $M \in [\, M_0 \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is deadlock-free, $\exists t \in T$ with $M' \xrightarrow{t}$.

Therefore (P, T, F, M) is deadlock-free.

Exercise

Give the formal definition of Boundedness

Then prove that Boundedness is an invariant

Or give a counter-example

Exercise

Give the formal definition of Cyclicity

Then prove that Cyclicity is an invariant

Or give a counter-example

Structural invariants

In the case of Petri nets, it is possible to compute certain vectors of **rational** numbers(*) (directly from the structure of the net) (independently from the initial marking) which induce nice invariants, called

S-invariants

T-invariants

(*) it is not necessary to consider real-valued solutions, because incidence matrices only have integer entries

Why invariants?

Can be calculated efficiently
(polynomial time for a basis)

Independent of initial marking

Structural property with behavioural consequences

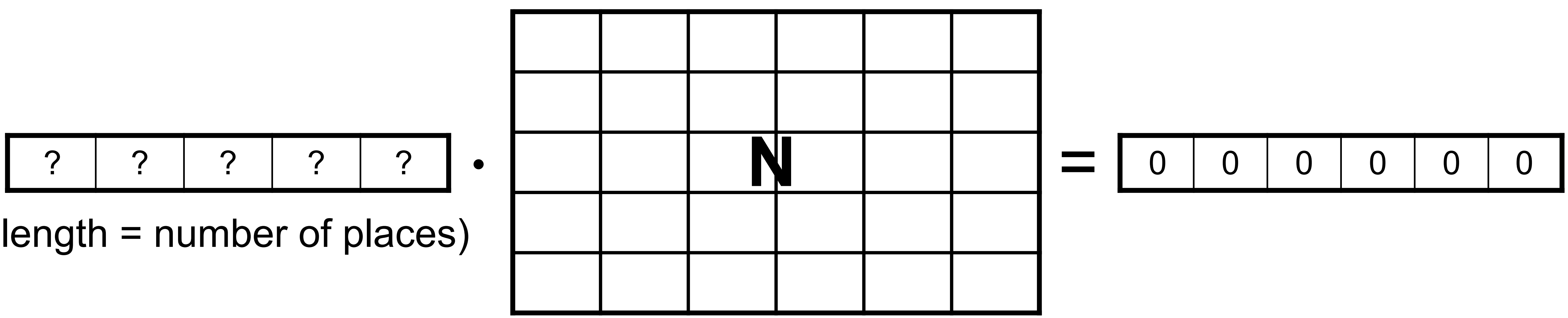
However, the main reason is didactical!
You only truly understand a model if you think
about it in terms of invariants!

S-invariants

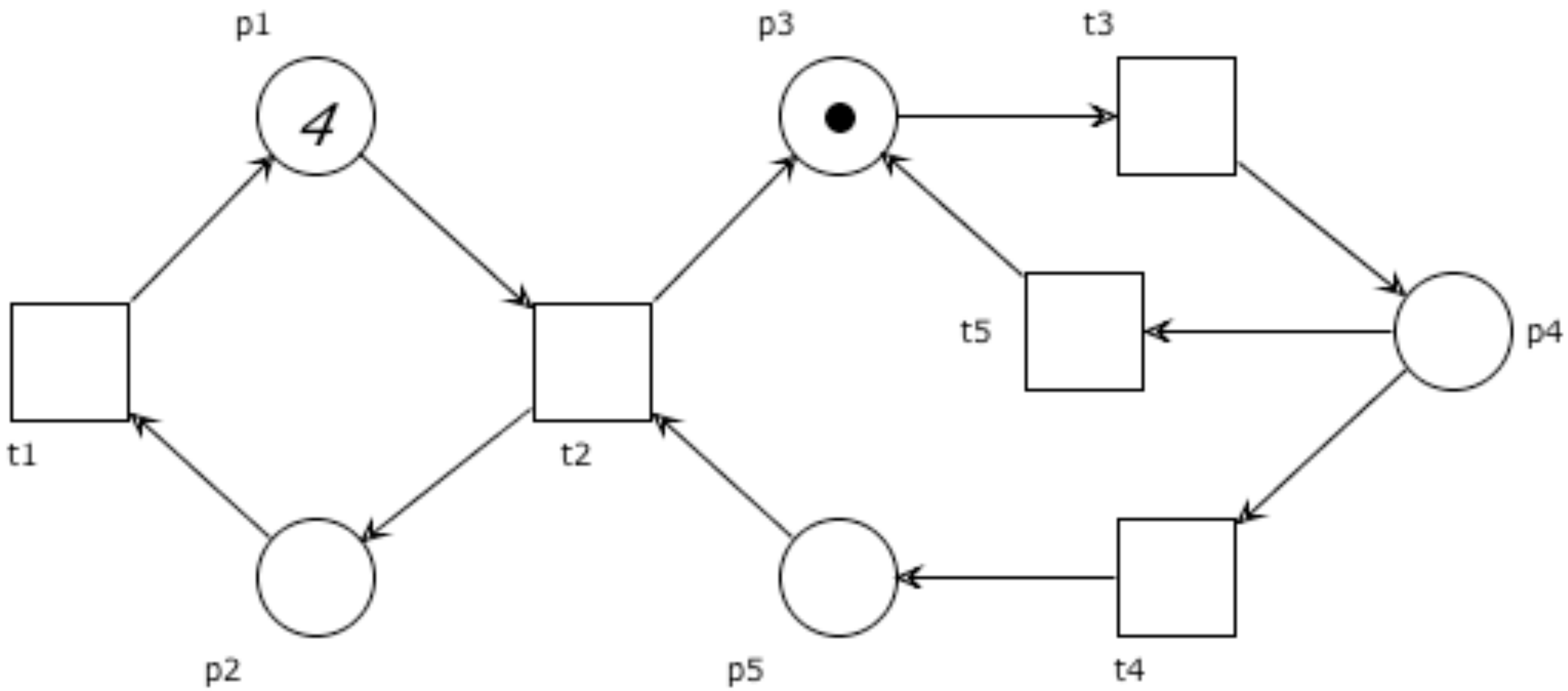
S-invariant
(aka place-invariant)

Definition: An **S-invariant** of a net N=(P,T,F) is a rational-valued solution **x** of the equation

$$\mathbf{x} \cdot \mathbf{N} = \mathbf{0}$$

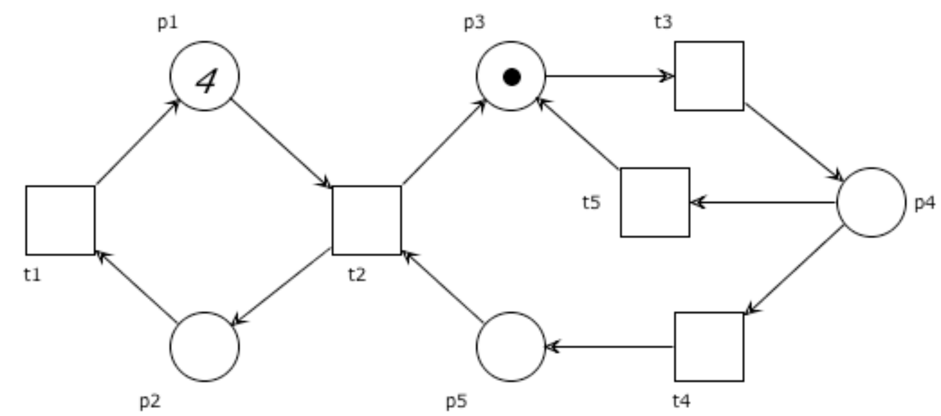


Example



Find some/all S-invariants for the net above

Example



Find some/all S-invariants for the net above

$$\overset{\mathbf{x}}{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}} \cdot \overset{N}{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}} = \mathbf{0}$$

$$\begin{cases} \begin{array}{cccccc} x_1 & -x_2 & & & & = 0 \\ -x_1 & +x_2 & +x_3 & & -x_5 & = 0 \\ & & -x_3 & +x_4 & & = 0 \\ & & & -x_4 & +x_5 & = 0 \\ & & x_3 & -x_4 & & = 0 \end{array} \end{cases}$$

$$x_1 = x_2$$

$$x_3 = x_4$$

$$x_4 = x_5$$

$$\begin{bmatrix} n & n & m & m & m \end{bmatrix}$$

Homogeneous systems of linear equations

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,n}x_n & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,n}x_n & = & 0 \\ \cdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & a_{m,n}x_n & = & 0 \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the "unknowns"

trivial solution: $x_1 = x_2 = \ldots = x_n = 0$

if $\mathbf{x}$ and $\mathbf{x}'$ are solutions, then $\mathbf{x} + \mathbf{x}'$ is a solution

if $\mathbf{x}$ is a solution, then $k\mathbf{x}$ is a solution

Linear combination

Proposition:
Any linear combination of S-invariants is an S-invariant

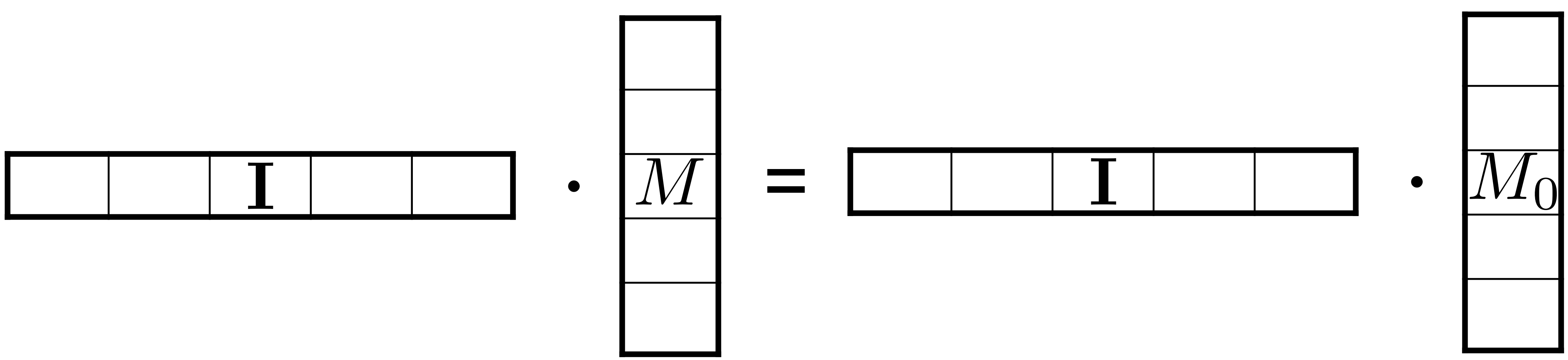
Take any two S-Invariants $\mathbf{I}_1$ and $\mathbf{I}_2$ and any two values k_1, k_2.
We want to prove that $k_1\,\mathbf{I}_1 + k_2\,\mathbf{I}_2$ is an S-invariant.

$$
\begin{aligned}
(k_1\,\mathbf{I}_1 + k_2\,\mathbf{I}_2) \cdot \mathbf{N} &= k_1\,\mathbf{I}_1 \cdot \mathbf{N} + k_2\,\mathbf{I}_2 \cdot \mathbf{N} \\
&= k_1\,\mathbf{0} + k_2\,\mathbf{0} \\
&= \mathbf{0}
\end{aligned}
$$

Fundamental property of S-invariants

Proposition: Let $\mathbf{I}$ be an invariant of N.

For any $M \in [\, M_0 \,\rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$



Fundamental property of S-invariants

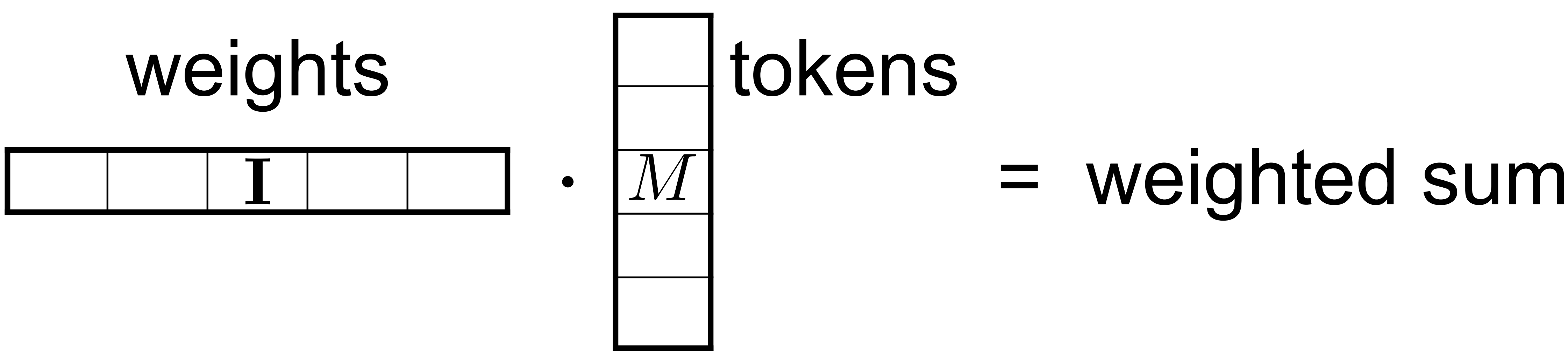
Proposition: Let $\mathbf{I}$ be an invariant of N.

For any $M \in [\, M_0 \,\rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since $M \in [\, M_0 \,\rangle$, there is σ s.t. $M_0 \xrightarrow{\sigma} M$
By the marking equation: $M = M_0 + \mathbf{N} \cdot \vec{\sigma}$

$$
\text{Therefore:} \quad \begin{aligned}
\mathbf{I} \cdot M &= \mathbf{I} \cdot (M_0 + \mathbf{N} \cdot \vec{\sigma}) \\
&= \mathbf{I} \cdot M_0 + \mathbf{I} \cdot \mathbf{N} \cdot \vec{\sigma} \\
&= \mathbf{I} \cdot M_0 + \mathbf{0} \cdot \vec{\sigma} \\
&= \mathbf{I} \cdot M_0
\end{aligned}
$$

Place-invariant, intuitively



Place-invariant, intuitively

A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

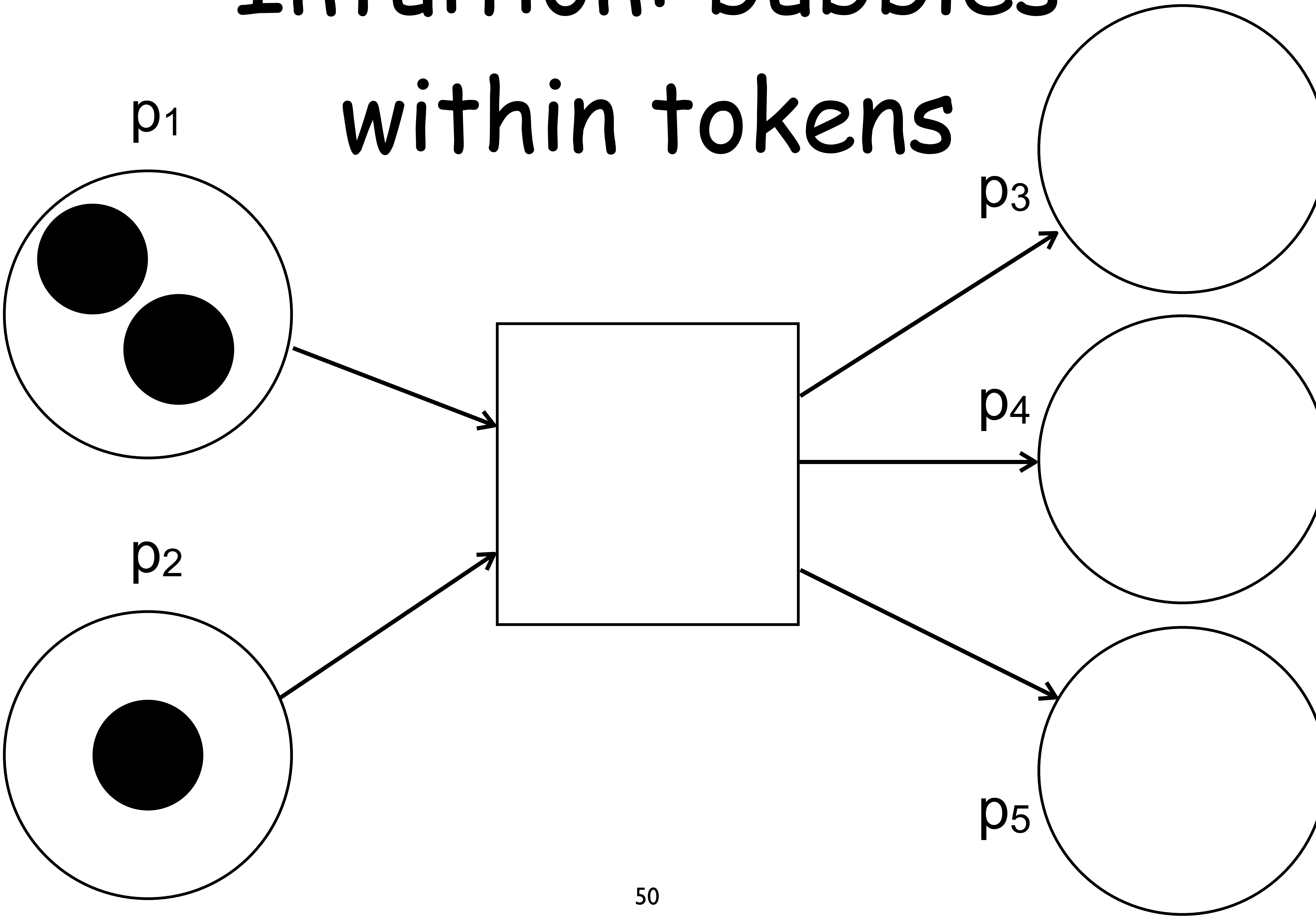
For example, you can imagine that tokens are coins,
places are the different kinds of available coins,
the S-invariant assigns a value to each coin:
the value of a marking is the sum of the values of the tokens/coins in it and it is not changed by firings

Place-invariant, intuitively

A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

For example, you can imagine that tokens are molecules, places are different kinds of molecules, the S-invariant assigns the number of atoms needed to form each molecule: the overall number of atoms is not changed by firings

Intuition: bubbles within tokens



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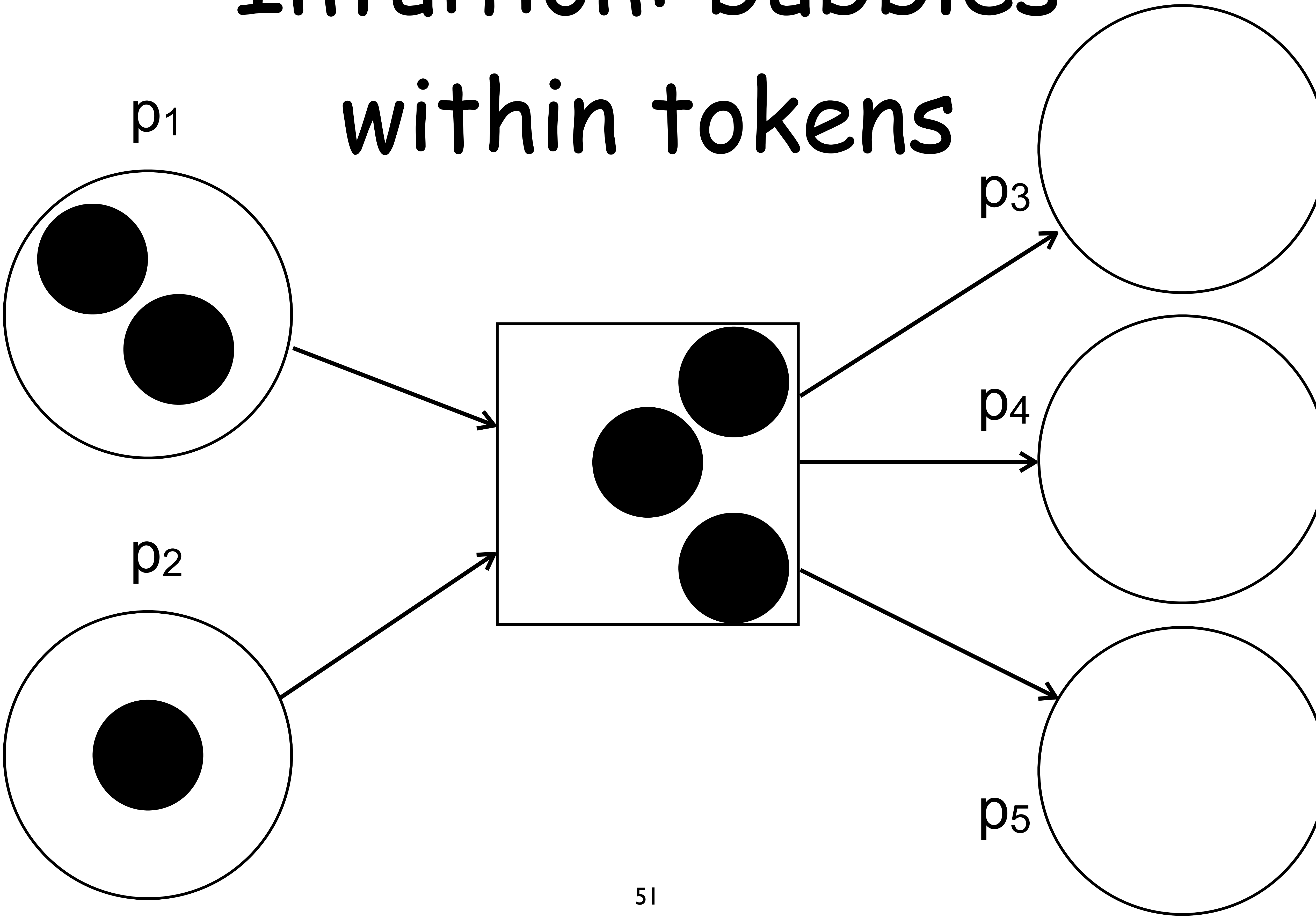

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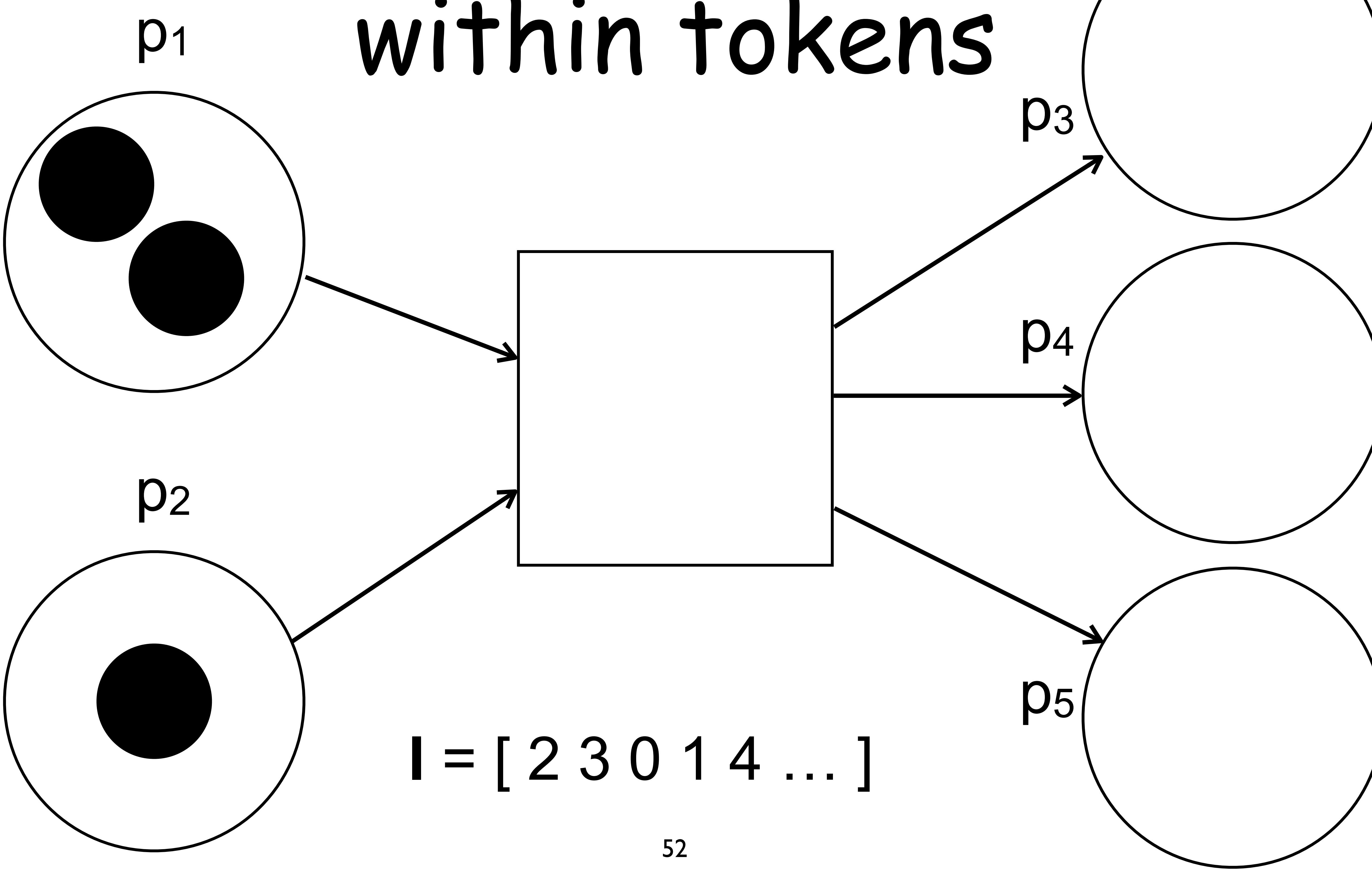
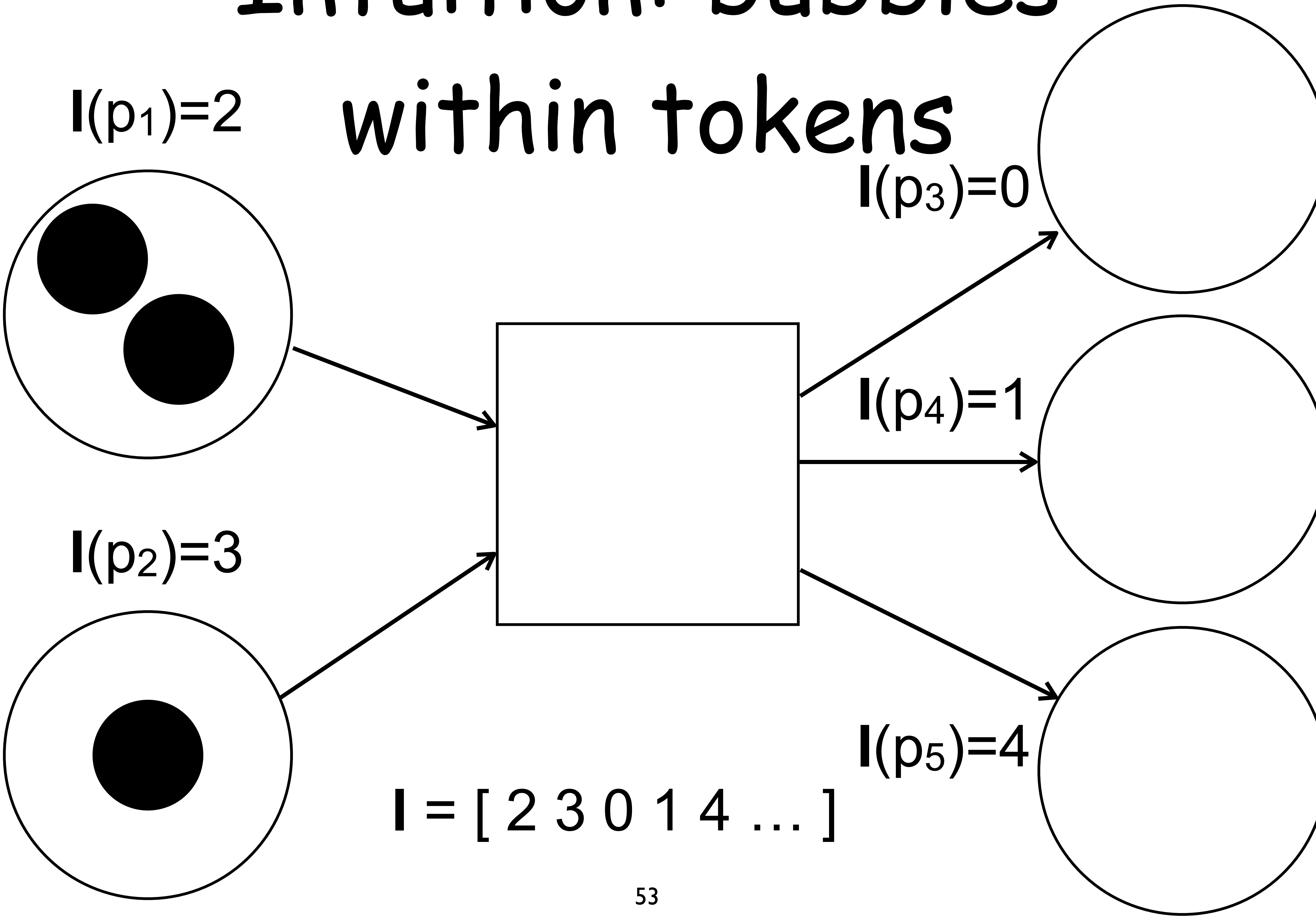


I = [2 3 0 1 4 …]

Intuition: bubbles within tokens



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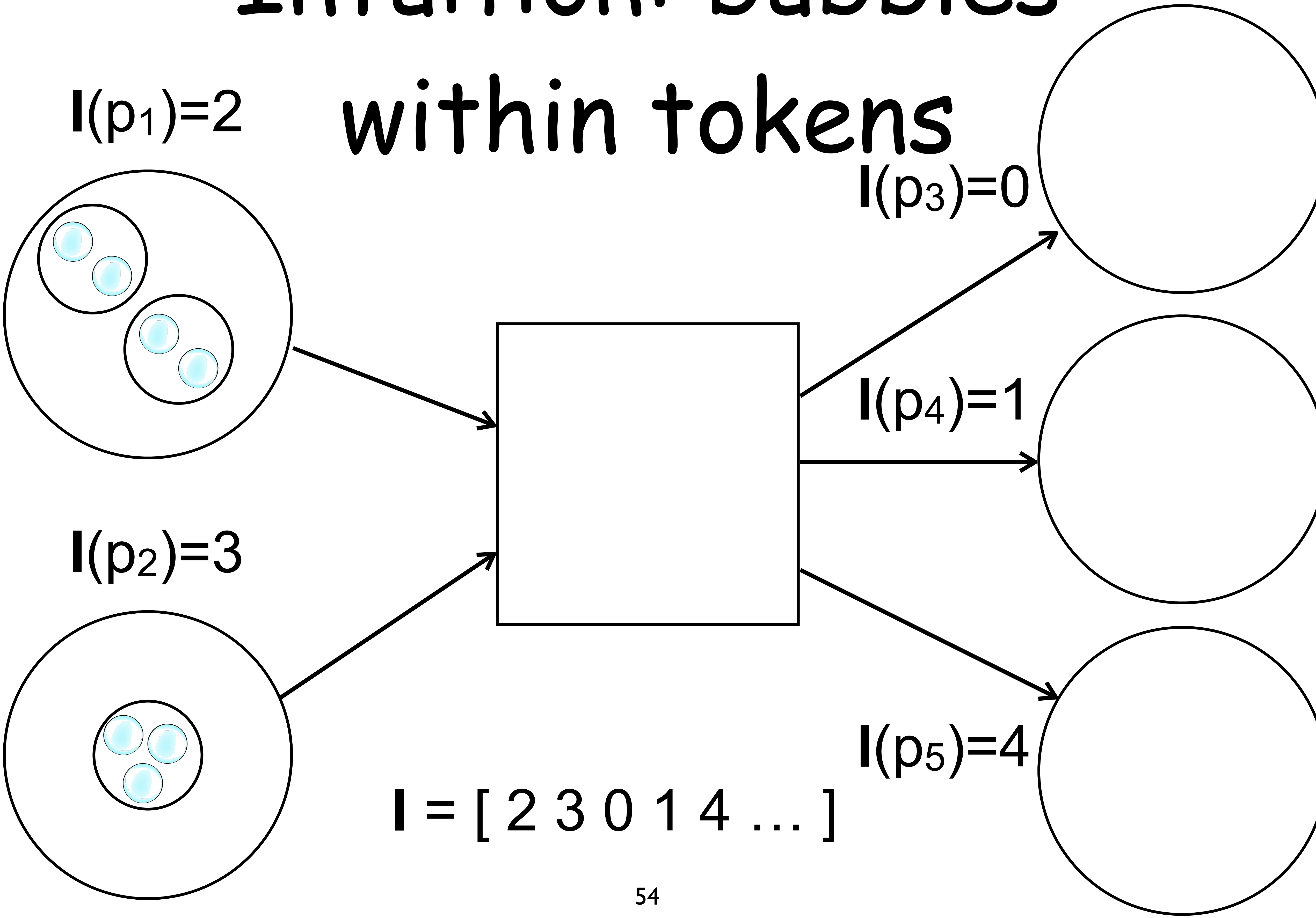

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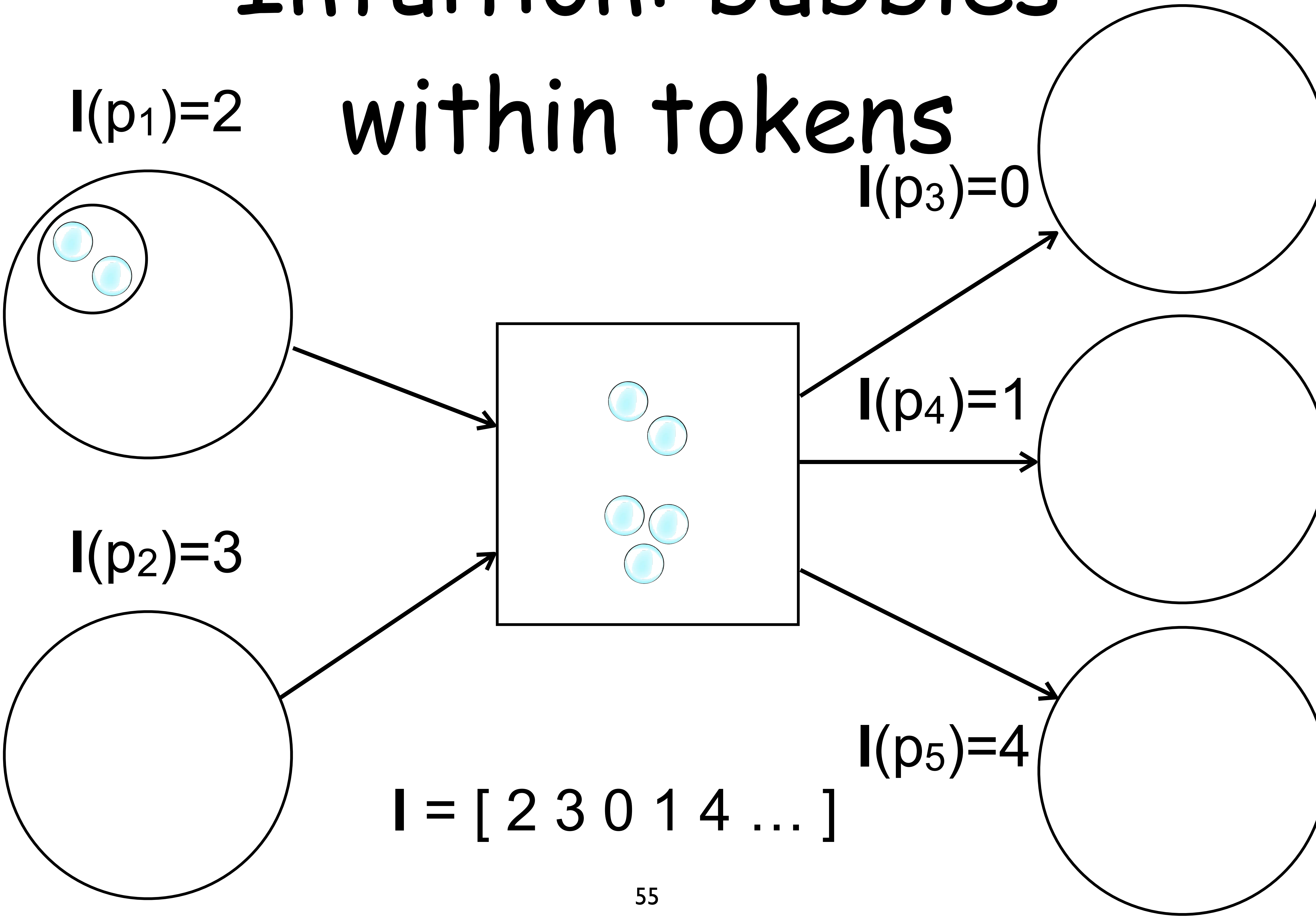

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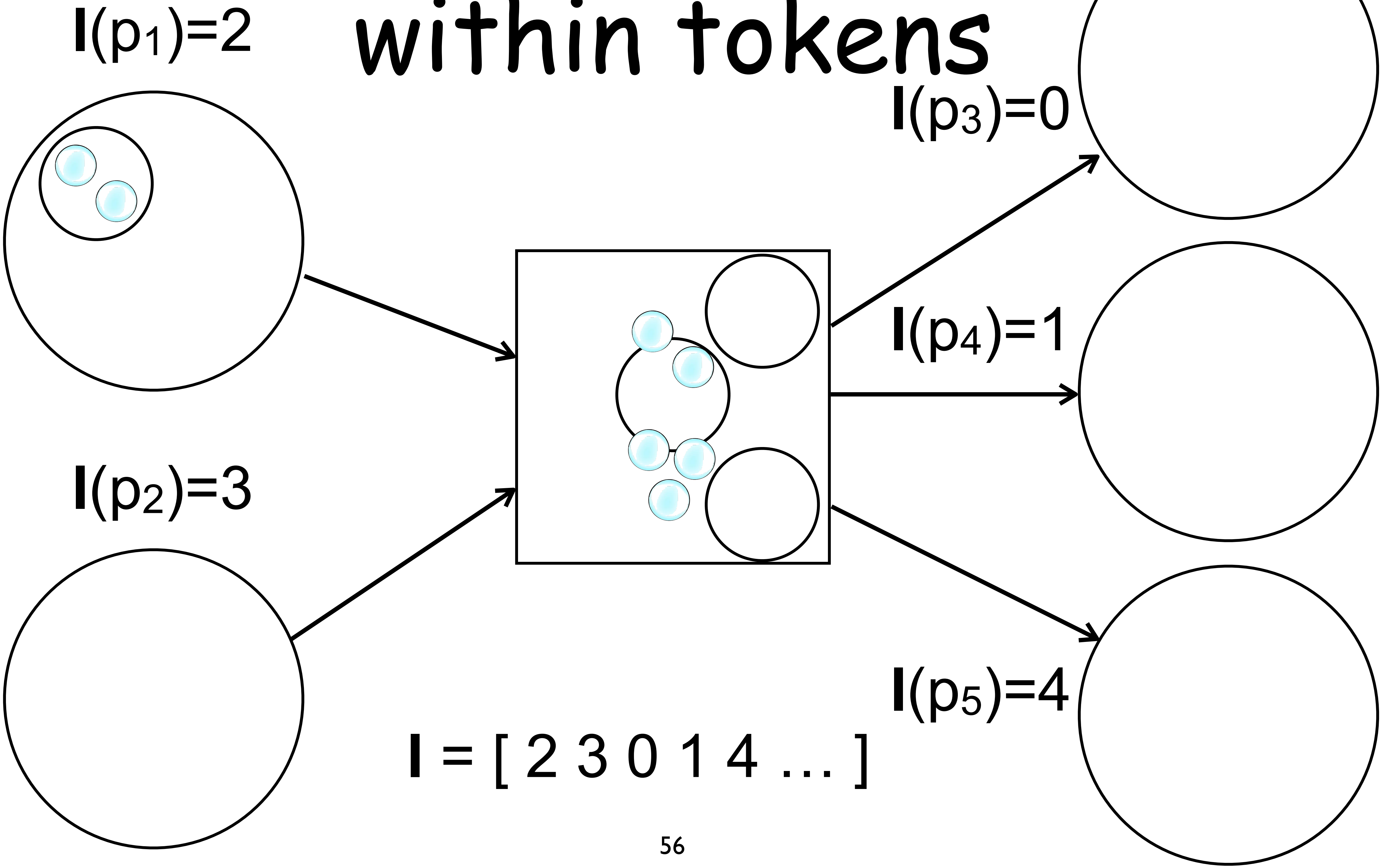
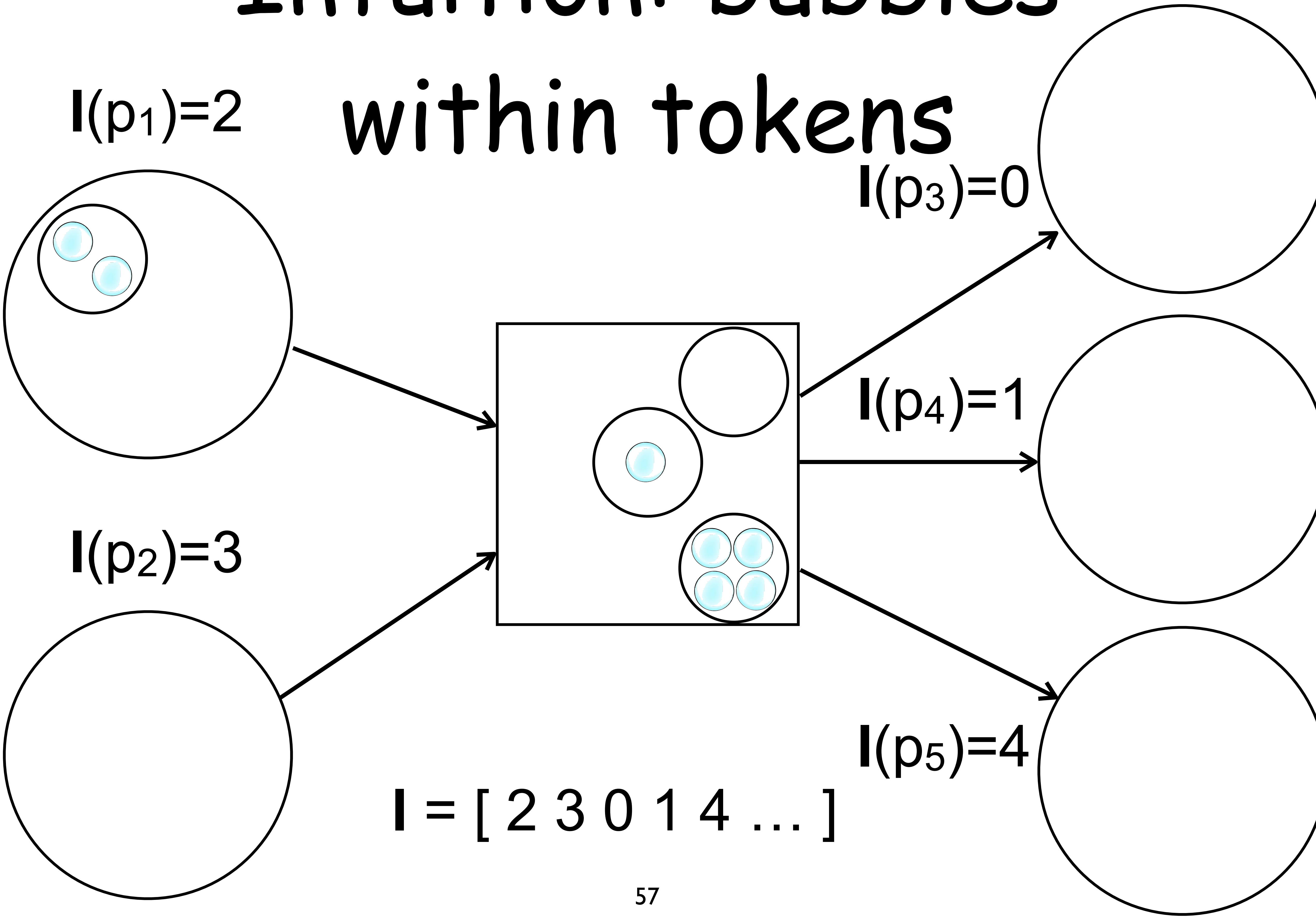
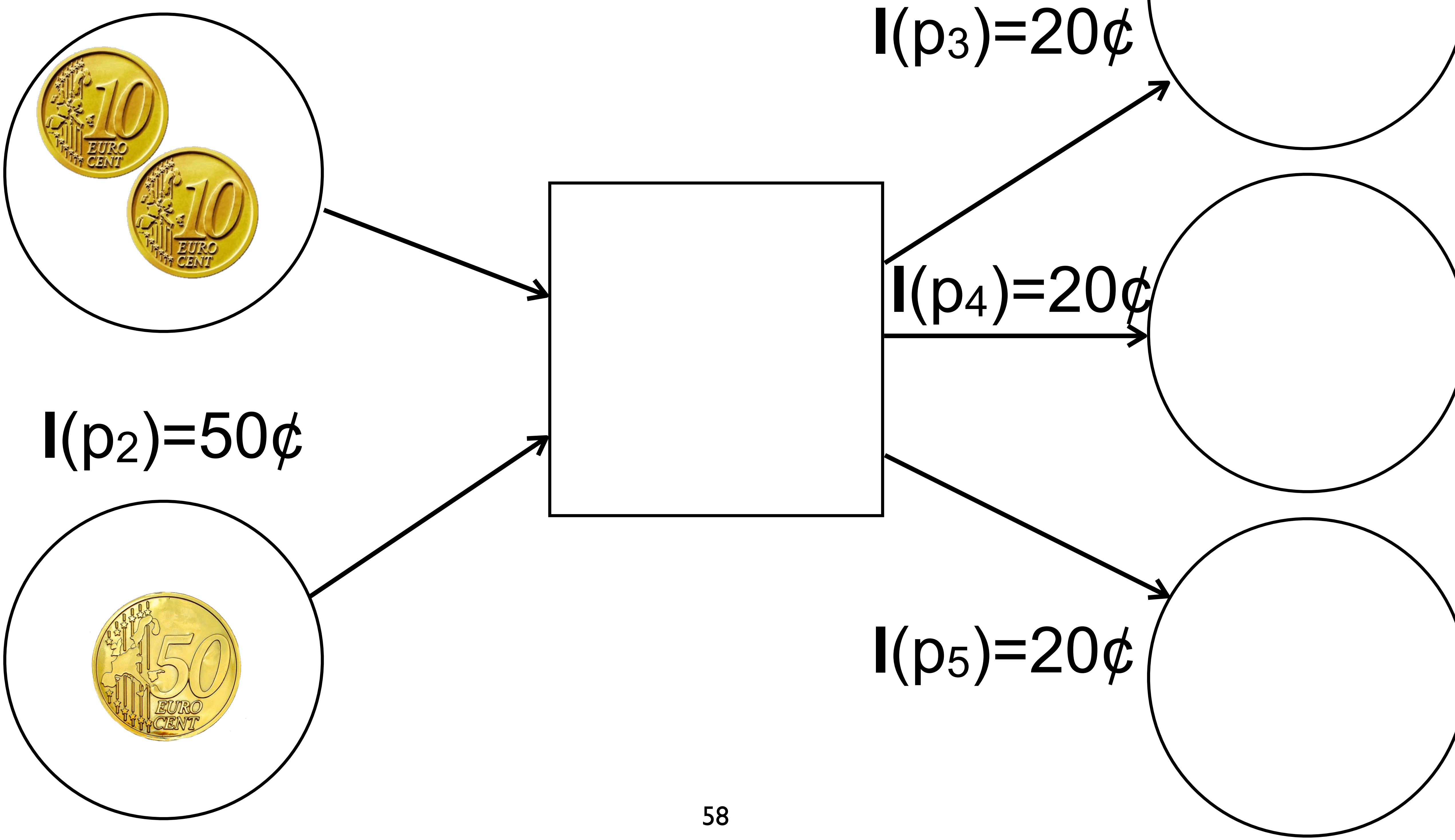


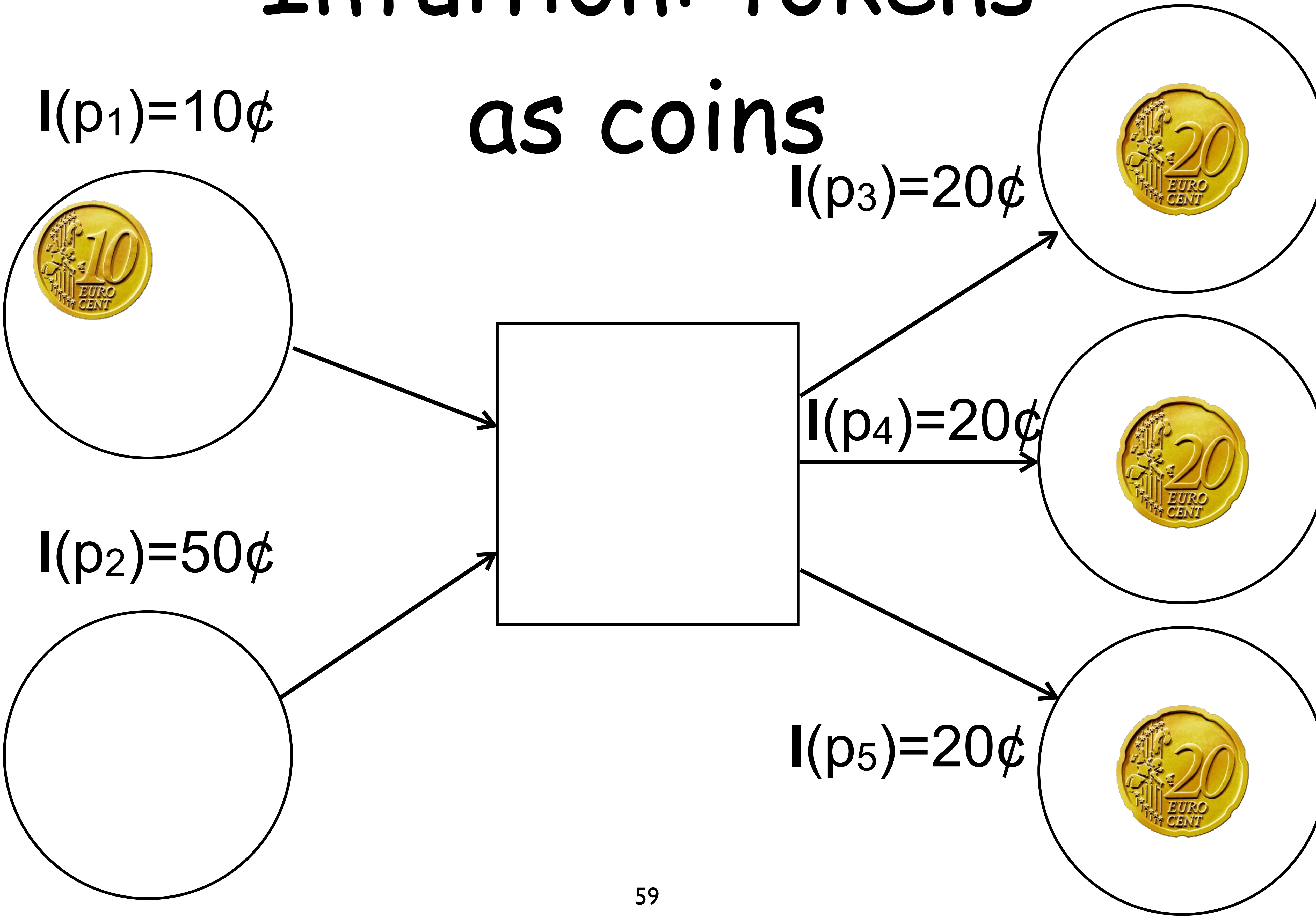
Intuition: bubbles within tokens



Intuition: tokens as coins



Intuition: tokens as coins



Alternative definition of S-invariant

Proposition:

A mapping $\mathbf{I} : P \to \mathbb{Q}$ is an S-invariant of N iff for any $t \in T$:

$$\sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$

Consequence of alternative definition

Very useful in proving S-invariance!

The check is possible without constructing the incidence matrix

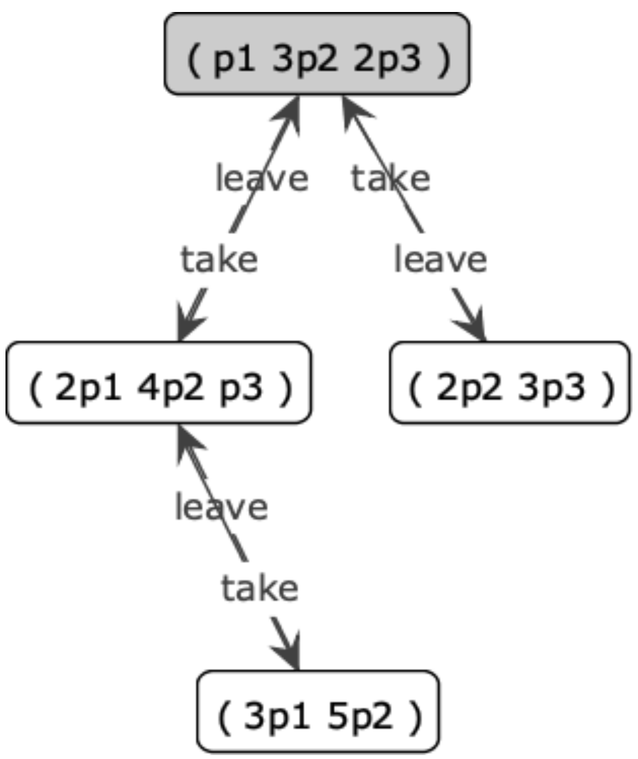
It can also help to build S-invariants directly over the picture

Exercise

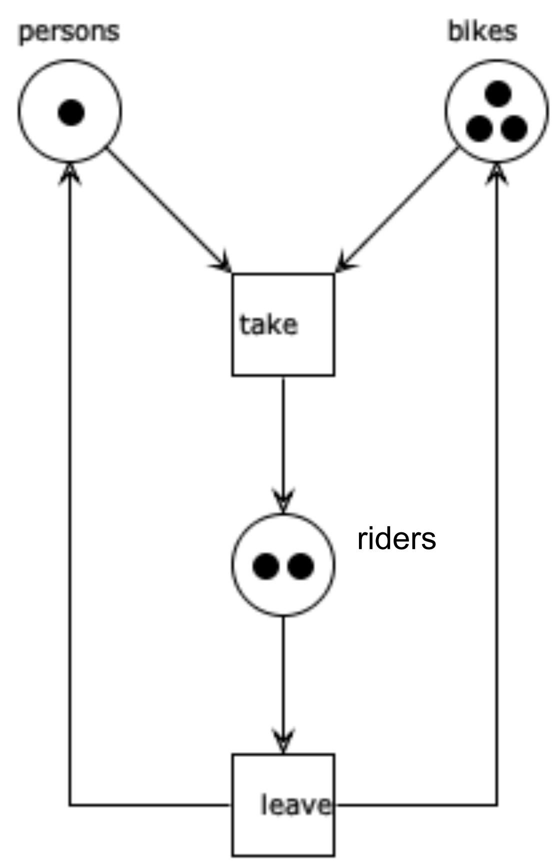
Prove the proposition about the alternative characterization of S-invariants

Question time

Which of the following are S-invariants?



p1 - persons
p2 - bikes
p3 - riders

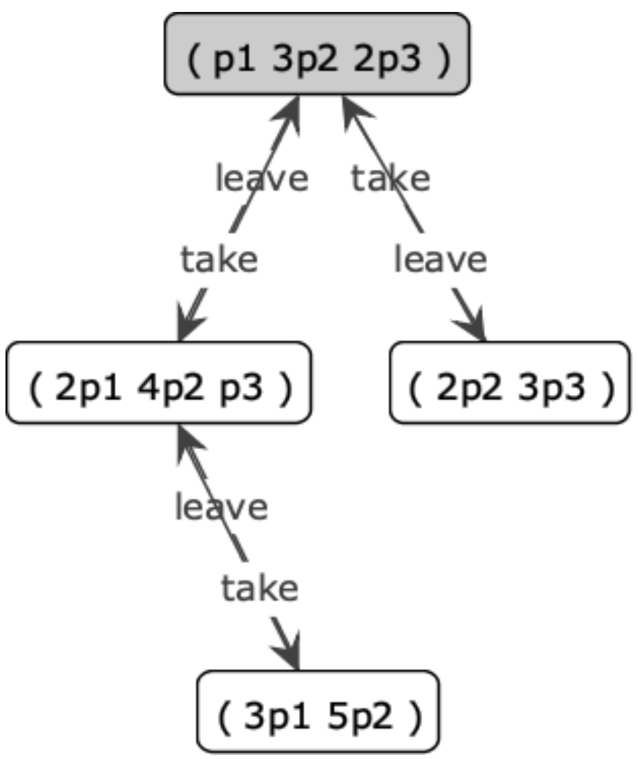


p	b	r
1	1	-1
1	0	1
0	1	1
1	1	1
1	-1	0
1	1	2
1	2	2

$$\forall t \in T,\ \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t\bullet} \mathbf{I}(p)$$

Question time

Which of the following are S-invariants?



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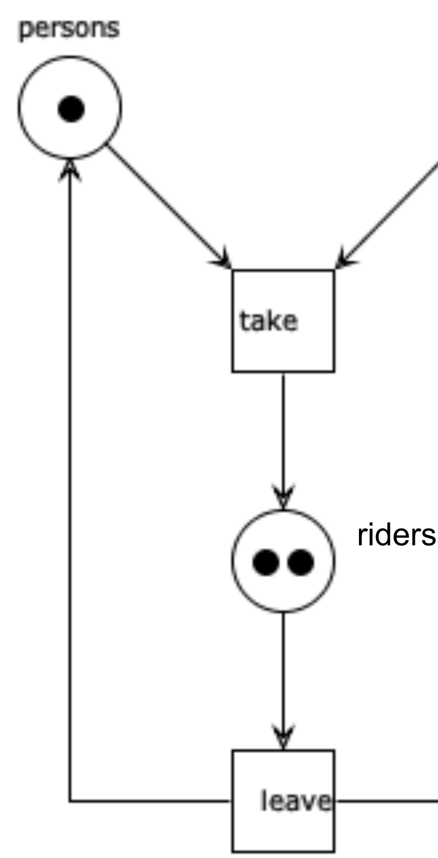

$$\begin{array}{ccc} p & b & r \\ {[\,1} & 1 & -1\,] \\ {[\,1} & 0 & 1\,] \\ {[\,0} & 1 & 1\,] \\ {[\,1} & 1 & 1\,] \\ {[\,1} & -1 & 0\,] \\ {[\,1} & 1 & 2\,] \\ {[\,1} & 2 & 2\,] \end{array}$$

$$\forall t \in T, \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t \bullet} \mathbf{I}(p)$$

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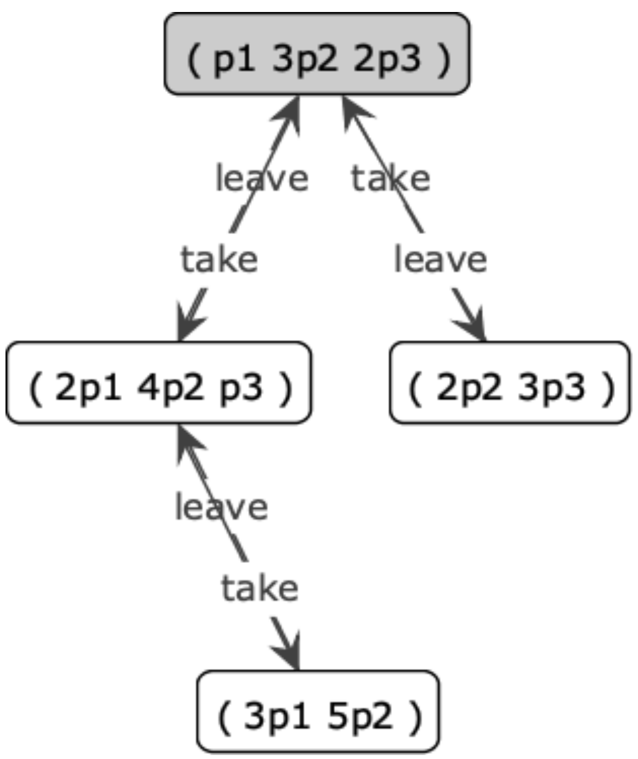

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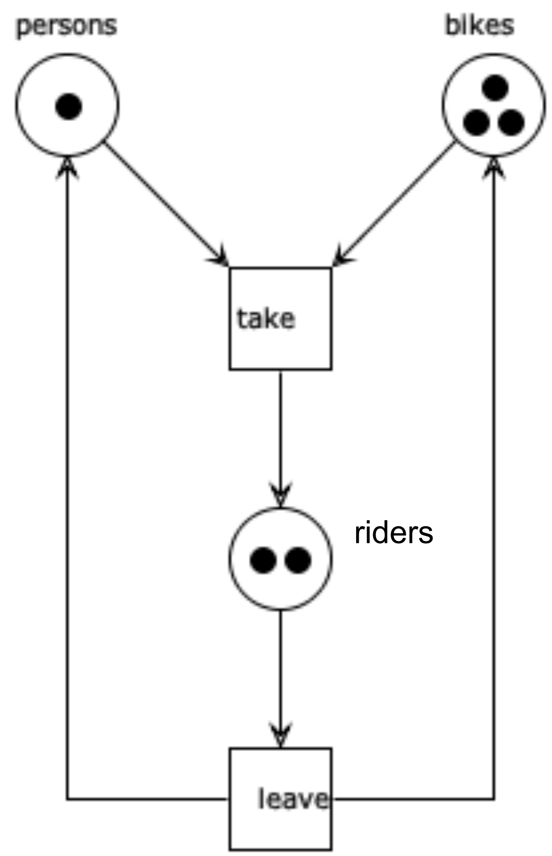

$$\begin{array}{ccc} p & b & r \\ {[\ 1} & 1 & -1\ ] \\ {[\ 1} & 0 & 1\ ] \\ {[\ 0} & 1 & 1\ ] \\ {[\ 1} & 1 & 1\ ] \\ {[\ 1} & -1 & 0\ ] \\ {[\ 1} & 1 & 2\ ] \\ {[\ 1} & 2 & 2\ ] \end{array}$$

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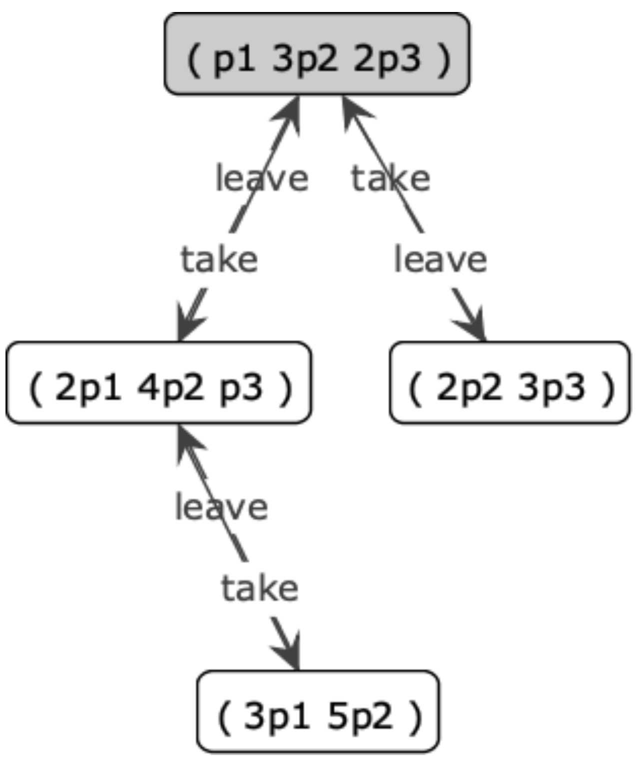

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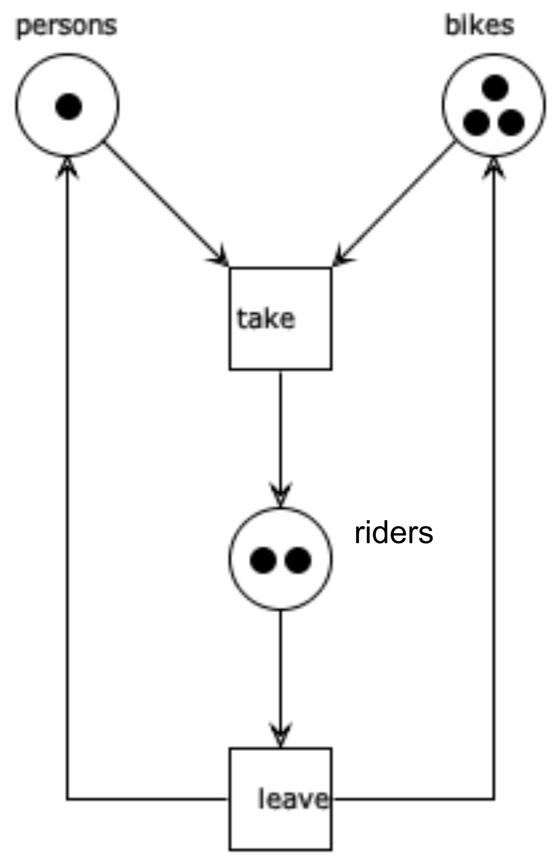

$$\begin{array}{ccc} p & b & r \end{array}$$
$$[\;1 \quad 1 \quad -1\;]$$
$$[\;1 \quad 0 \quad 1\;]$$
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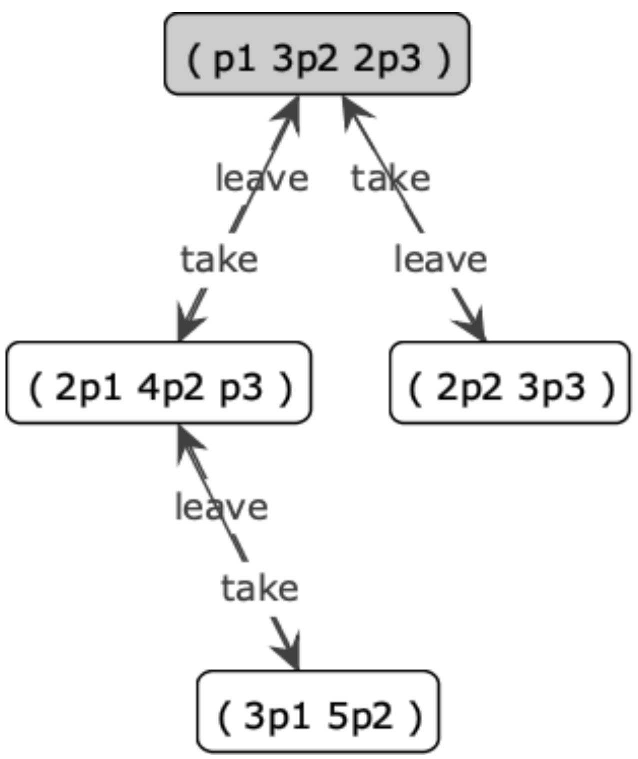

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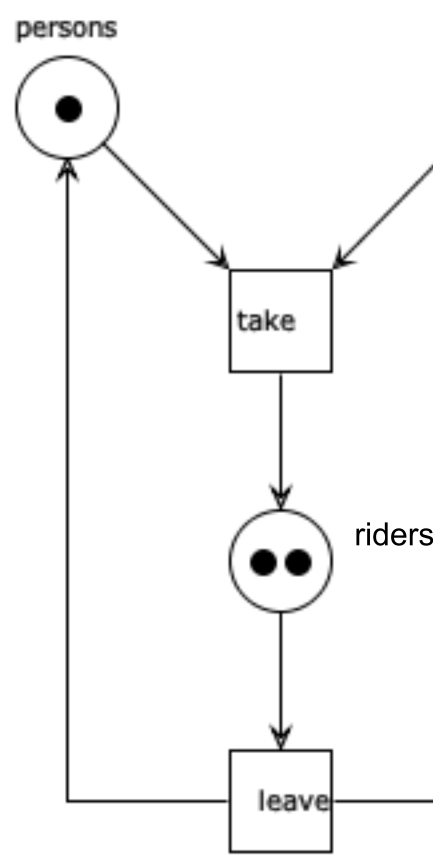

$$\begin{array}{ccc} p & b & r \\ [\;1 & 1 & -1\;] \\ [\;1 & 0 & 1\;] \\ [\;0 & 1 & 1\;] \\ [\;1 & 1 & 1\;] \\ [\;1 & -1 & 0\;] \\ [\;1 & 1 & 2\;] \\ [\;1 & 2 & 2\;] \end{array}$$

$$\forall t \in T,\ \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t\bullet} \mathbf{I}(p)$$

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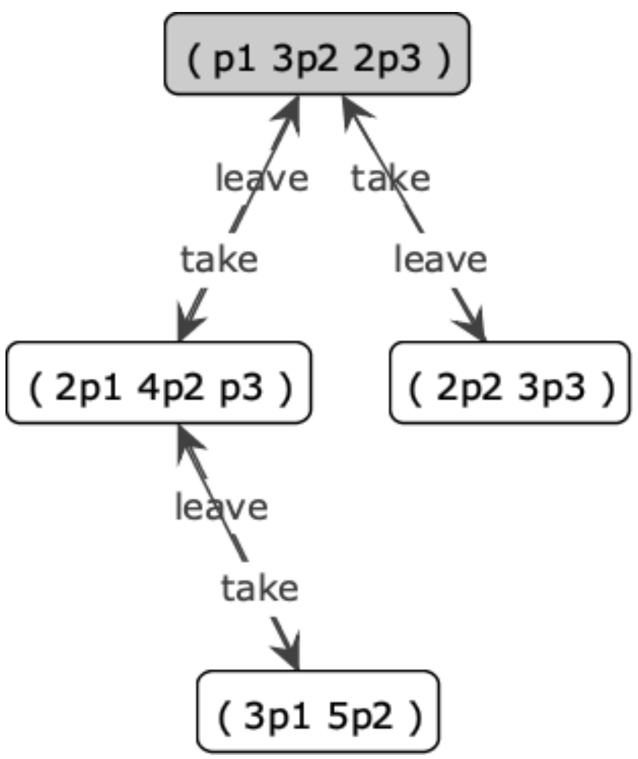

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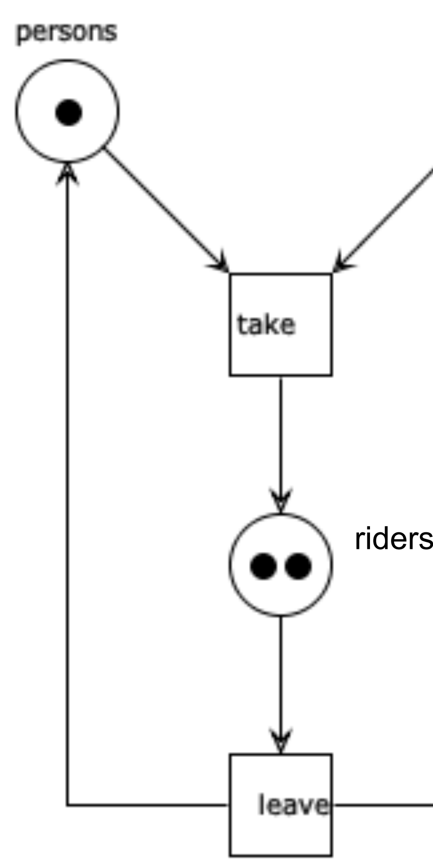

$$\begin{array}{ccc} p & b & r \\ \left[\begin{array}{ccc} 1 & 1 & -1 \end{array}\right] \\ \left[\begin{array}{ccc} 1 & 0 & 1 \end{array}\right] \\ \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right] \\ \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right] \\ \left[\begin{array}{ccc} 1 & -1 & 0 \end{array}\right] \\ \left[\begin{array}{ccc} 1 & 1 & 2 \end{array}\right] \\ \left[\begin{array}{ccc} 1 & 2 & 2 \end{array}\right] \end{array}$$

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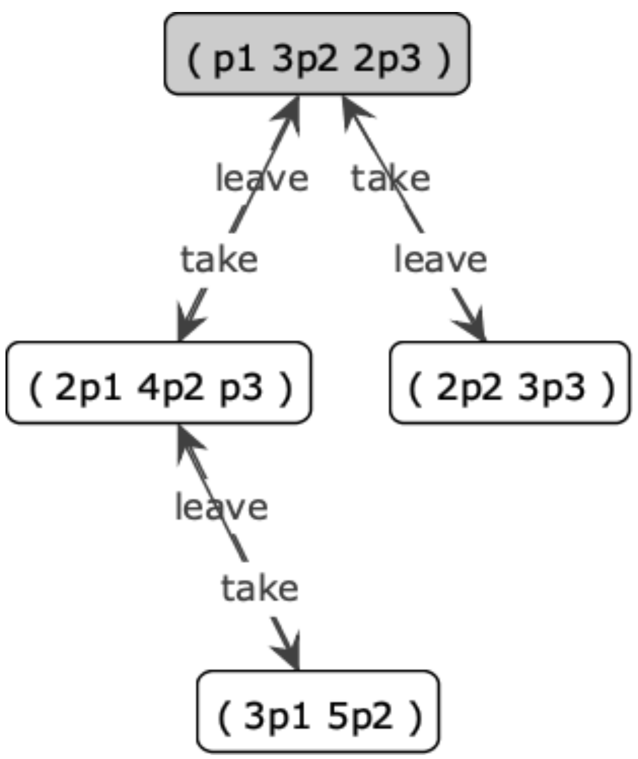

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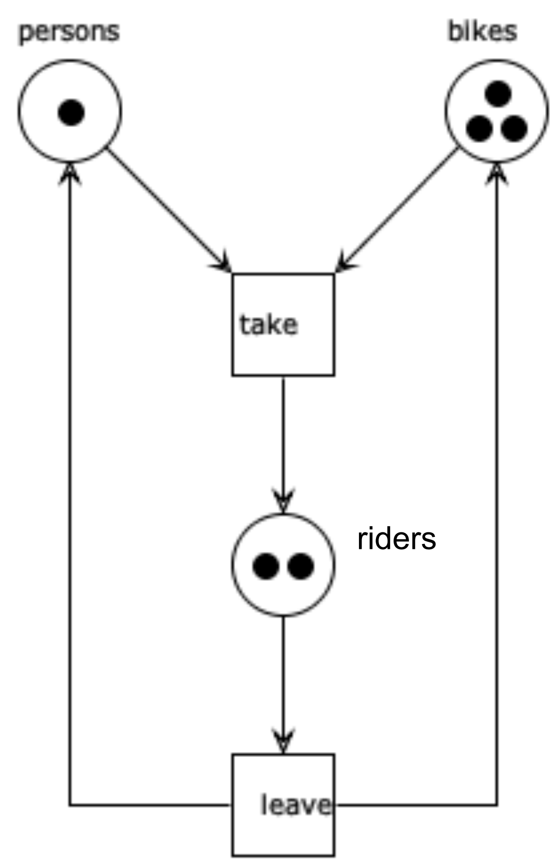

$$\begin{array}{ccc} p & b & r \end{array}$$

[1 1 -1]
[1 0 1]
[0 1 1]
[1 1 1]
[1 -1 0]
[1 1 2]
[1 2 2]

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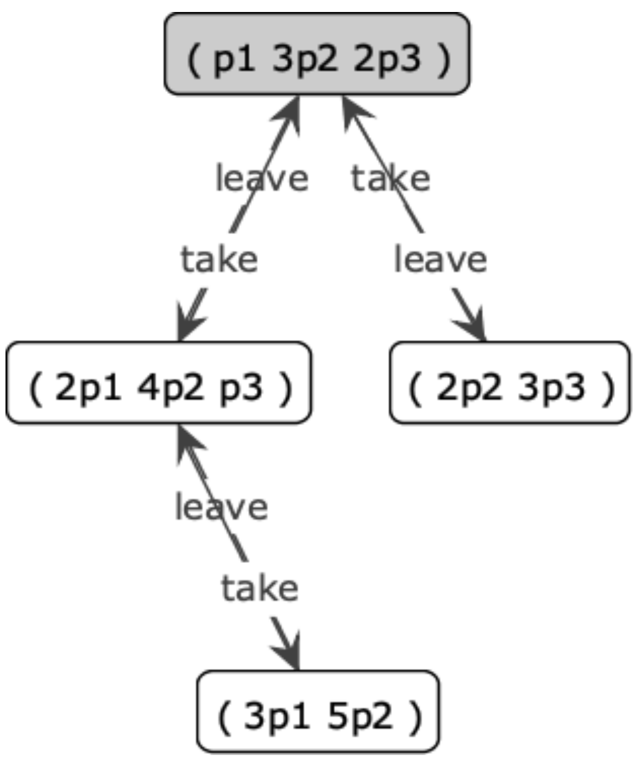

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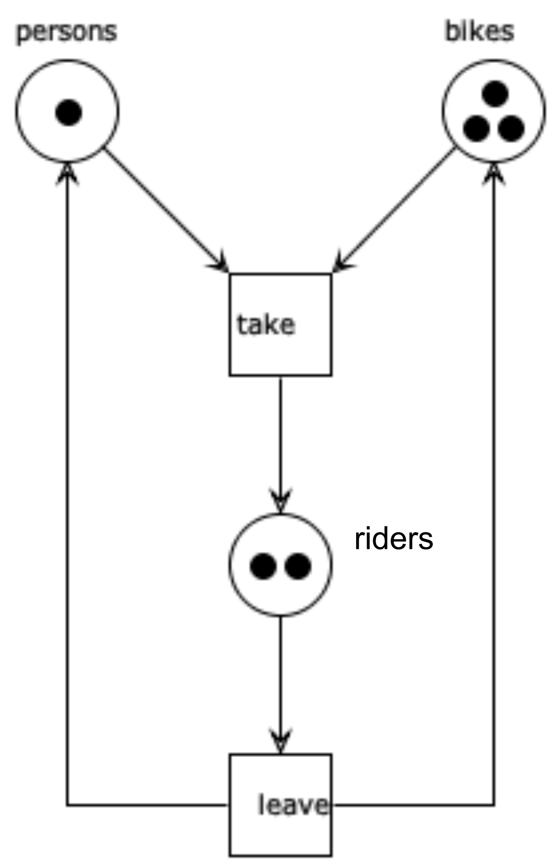

$$
\begin{array}{ccc}
p & b & r \\
{[}\ 1 & 1 & -1\ {]} \\
{[}\ 1 & 0 & 1\ {]} \\
{[}\ 0 & 1 & 1\ {]} \\
{[}\ 1 & 1 & 1\ {]} \\
{[}\ 1 & -1 & 0\ {]} \\
{[}\ 1 & 1 & 2\ {]} \\
{[}\ 1 & 2 & 2\ {]}
\end{array}
$$

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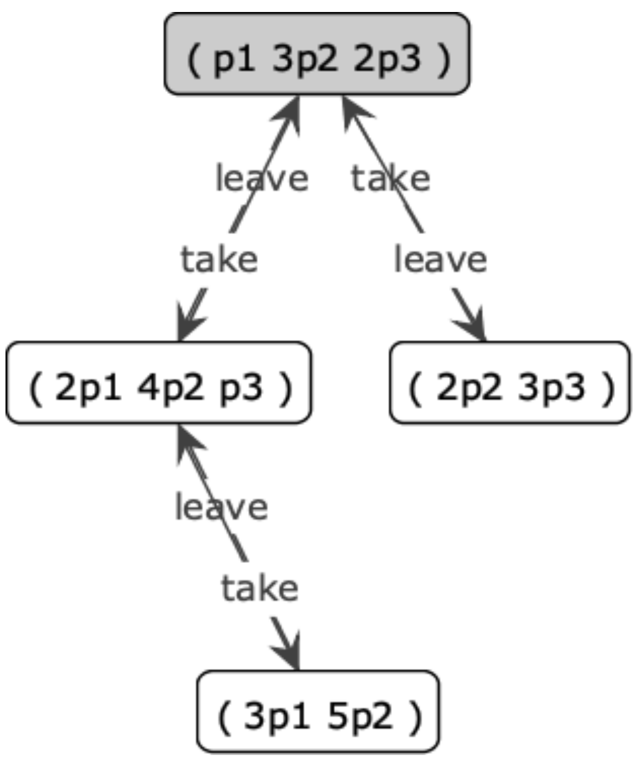

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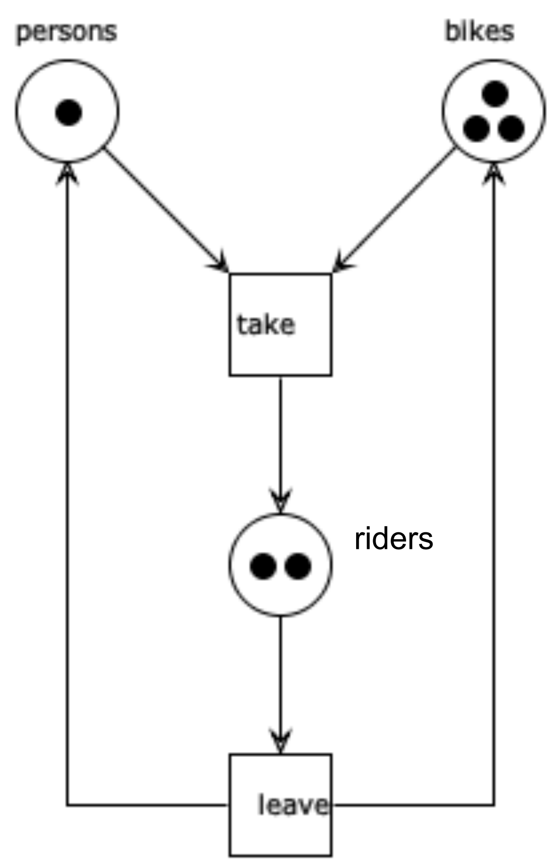

$$
\begin{array}{rcccc}
 & & p & b & r \\
 & [& 1 & 1 & -1 \;] \\
 & [& 1 & 0 & 1 \;] \\
+ & [& 0 & 1 & 1 \;] \\
 & [& 1 & 1 & 1 \;] \\
 & [& 1 & -1 & 0 \;] \\
= & [& 1 & 1 & 2 \;] \\
 & [& 1 & 2 & 2 \;]
\end{array}
$$

$$\forall t \in T, \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t \bullet} \mathbf{I}(p)$$

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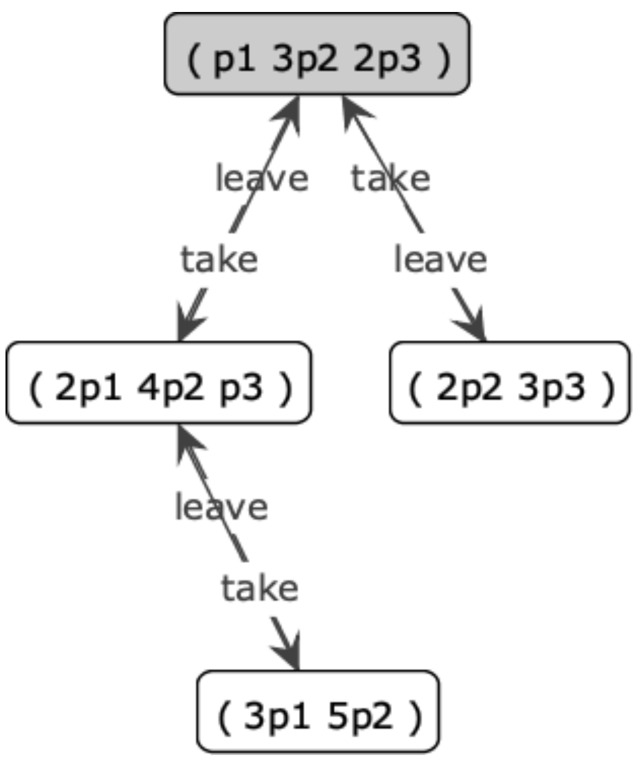

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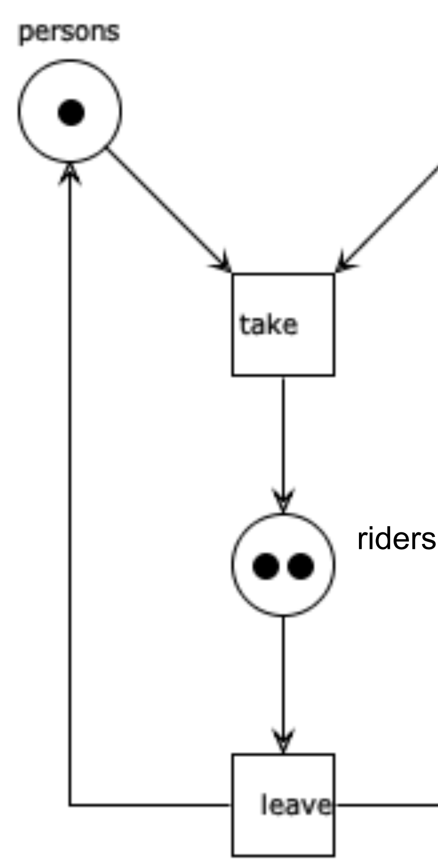
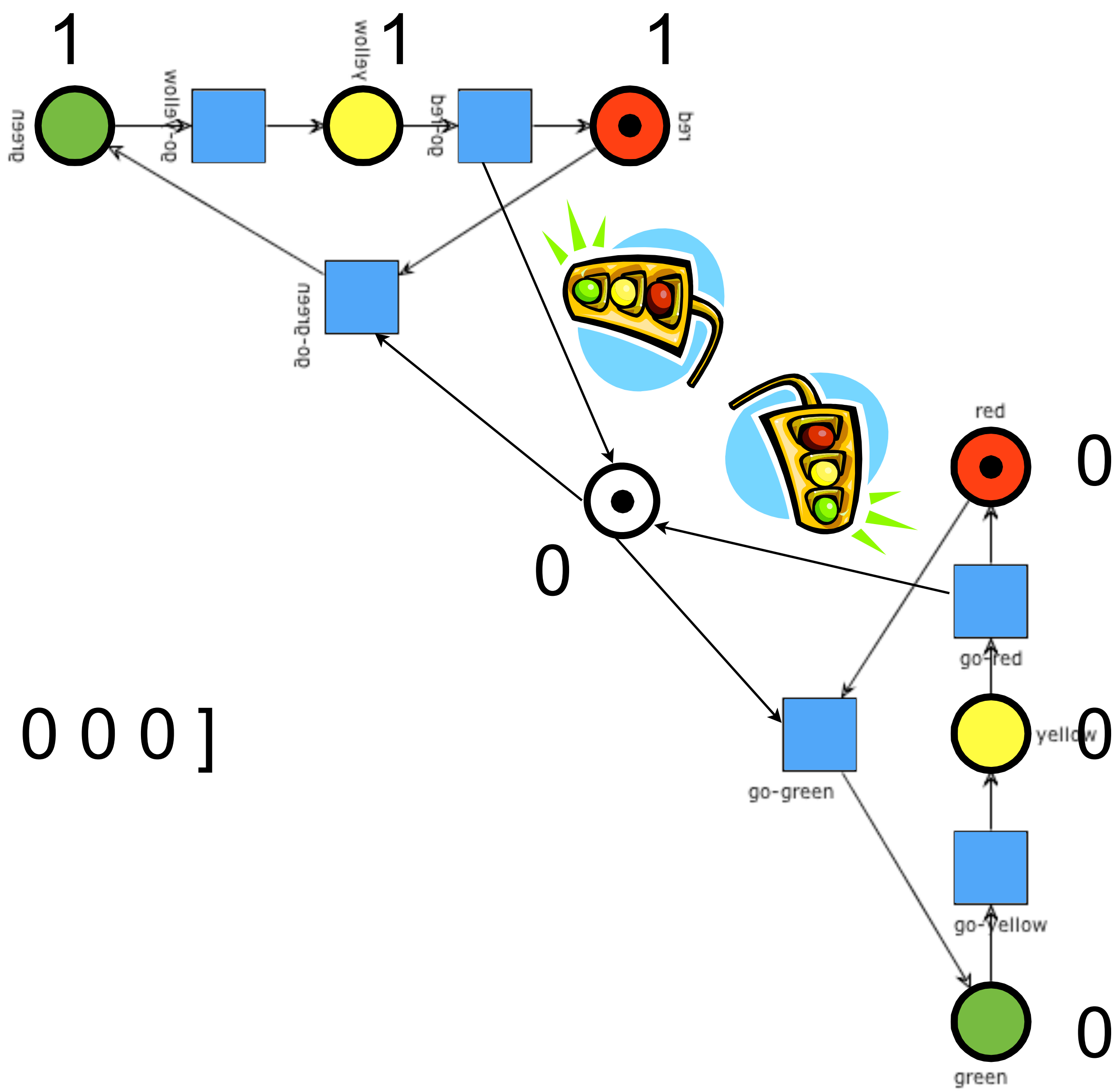


$$\begin{array}{cccc} & p & b & r \\ & [\,1 & 1 & -1\,] \\ & [\,1 & 0 & 1\,] \\ - & [\,0 & 1 & 1\,] \\ & [\,1 & 1 & 1\,] \\ = & [\,1 & -1 & 0\,] \\ & [\,1 & 1 & 2\,] \\ & [\,1 & 2 & 2\,] \end{array}$$

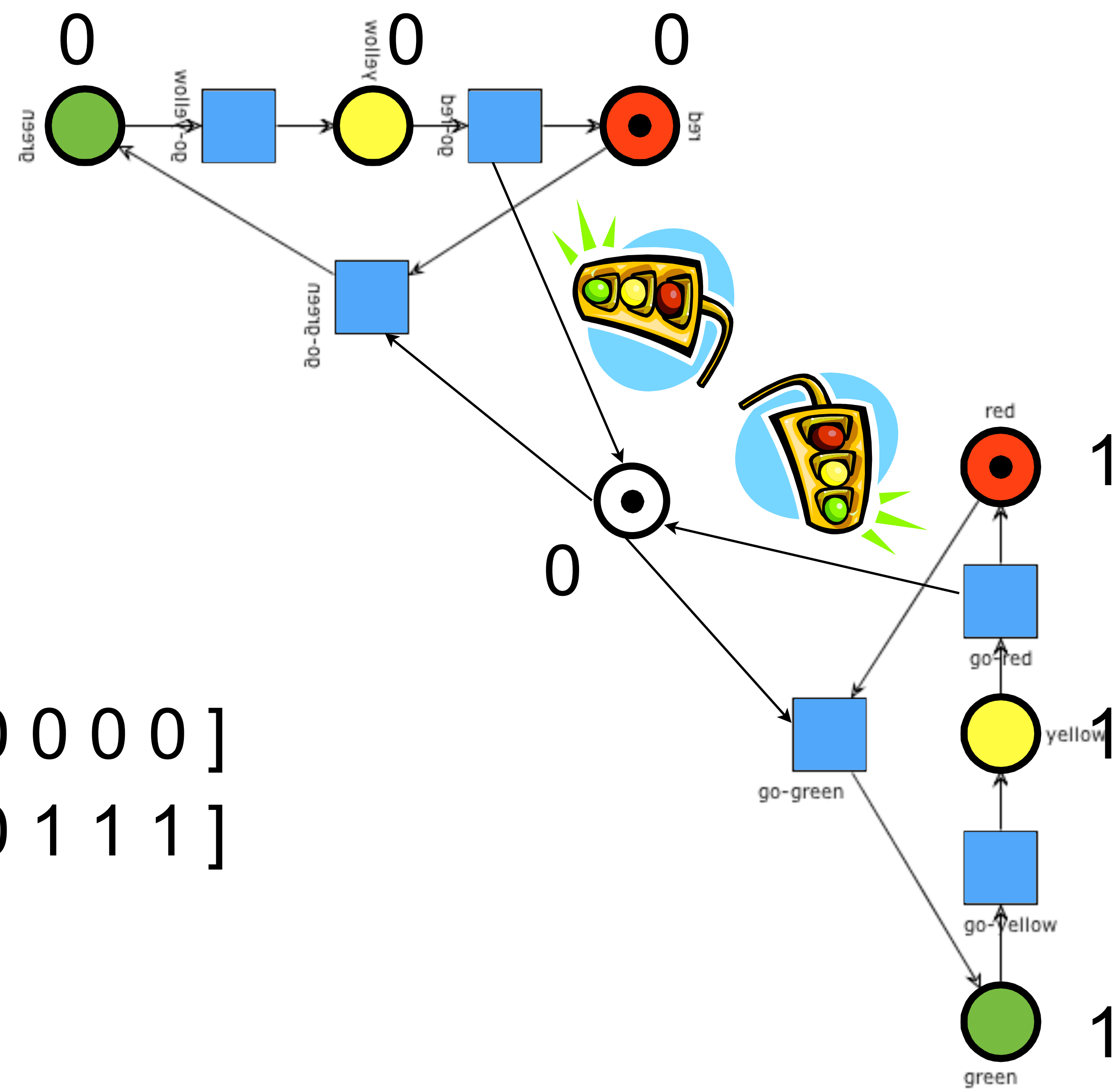
$$\forall t \in T, \sum_{p \in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p \in t \bullet} \mathbf{I}(p)$$

Traffic-lights example



[1 1 1 0 0 0 0]

Traffic-lights example



[1 1 1 0 0 0 0]
[0 0 0 0 1 1 1]

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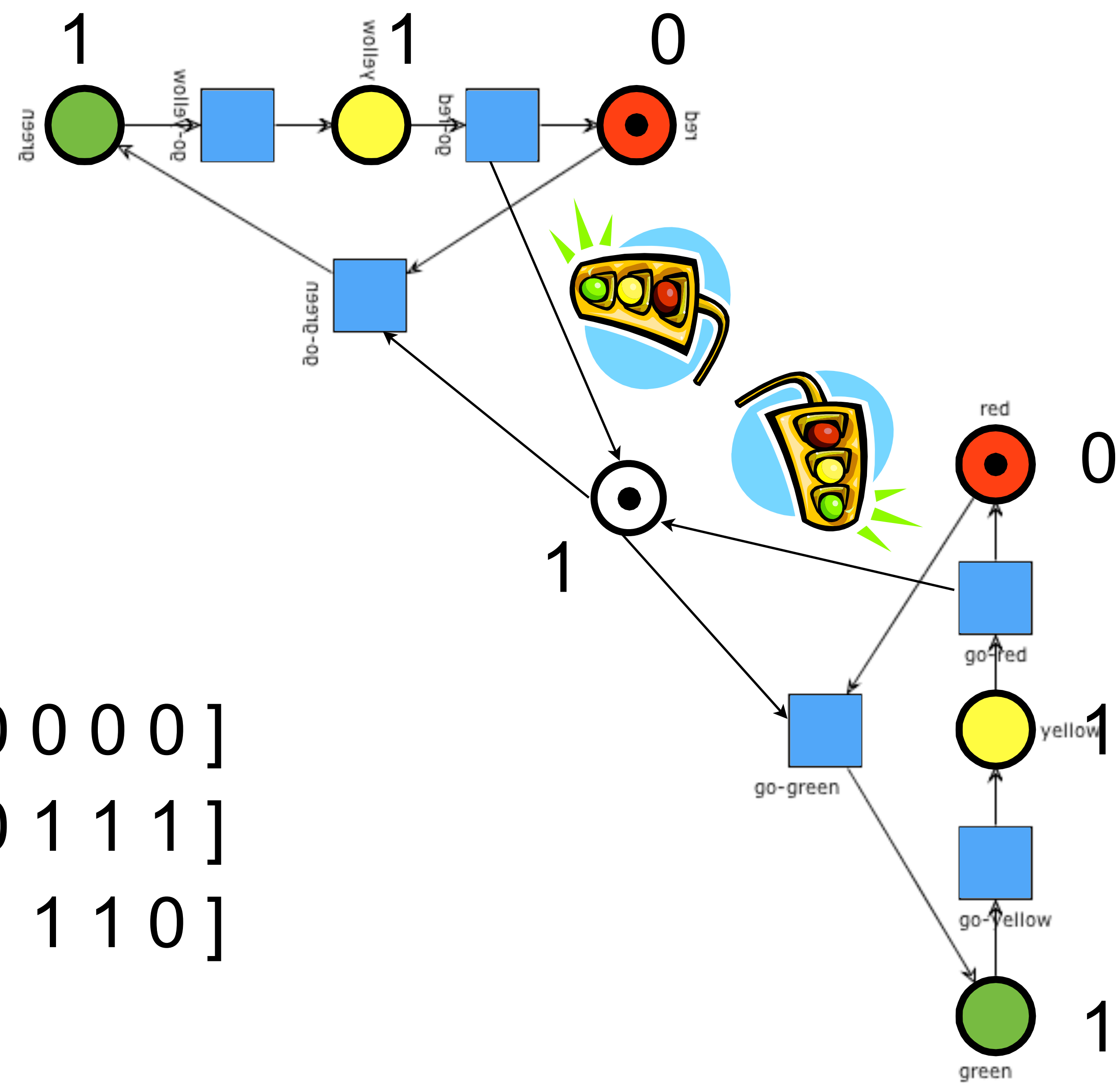
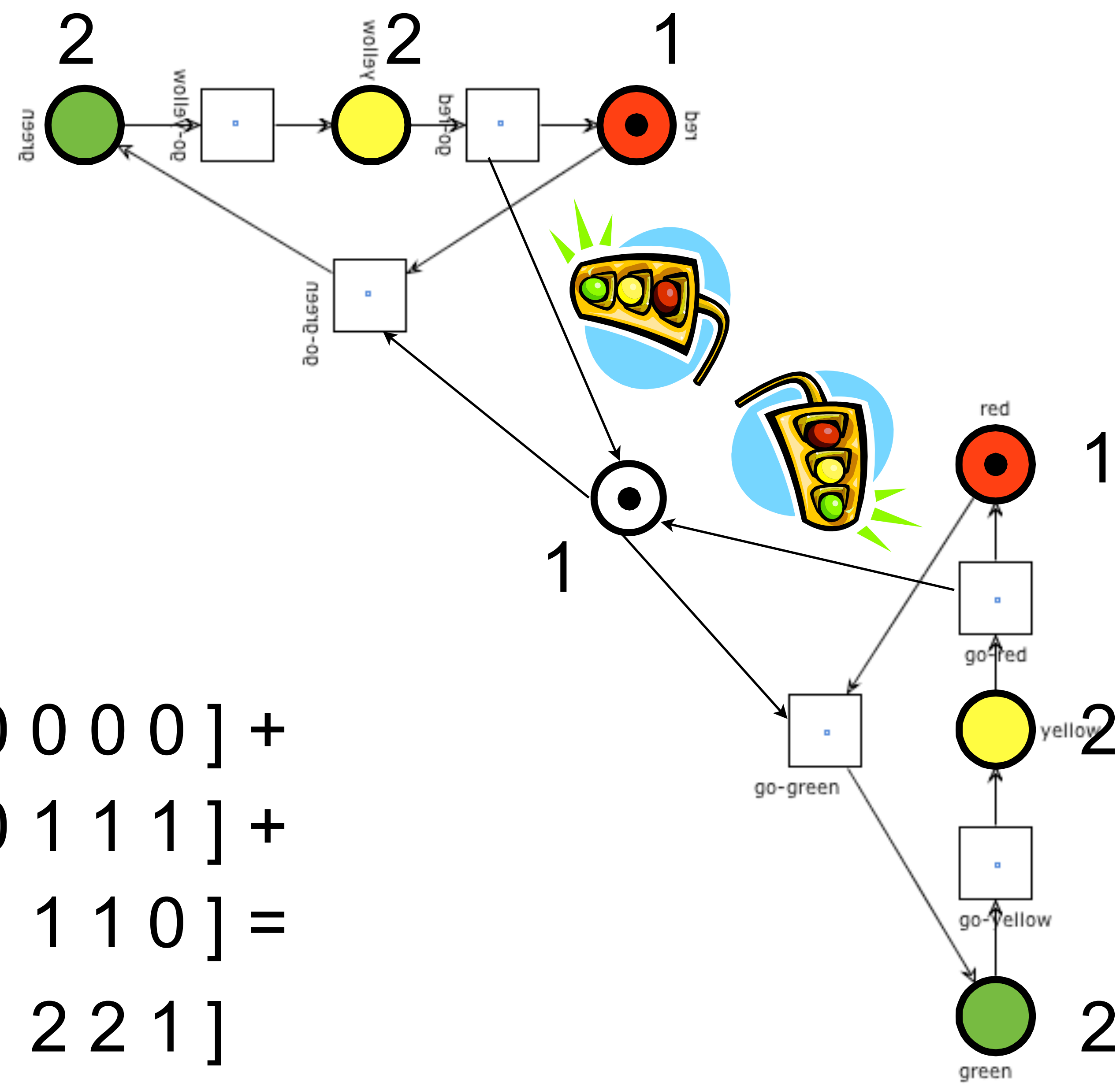


[1 1 1 0 0 0 0]
[0 0 0 0 1 1 1]
[1 1 0 1 1 1 0]

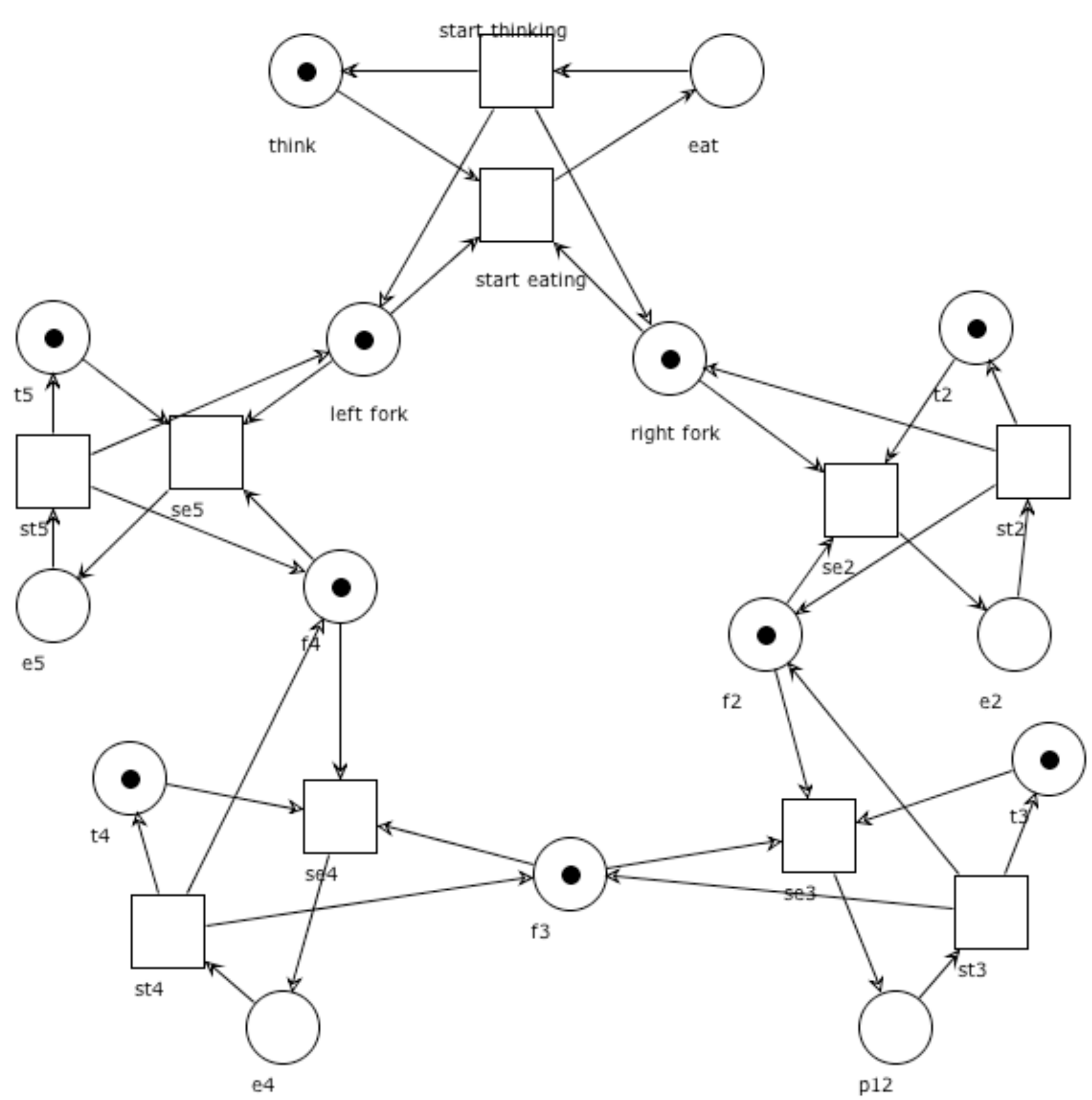
Traffic-lights example

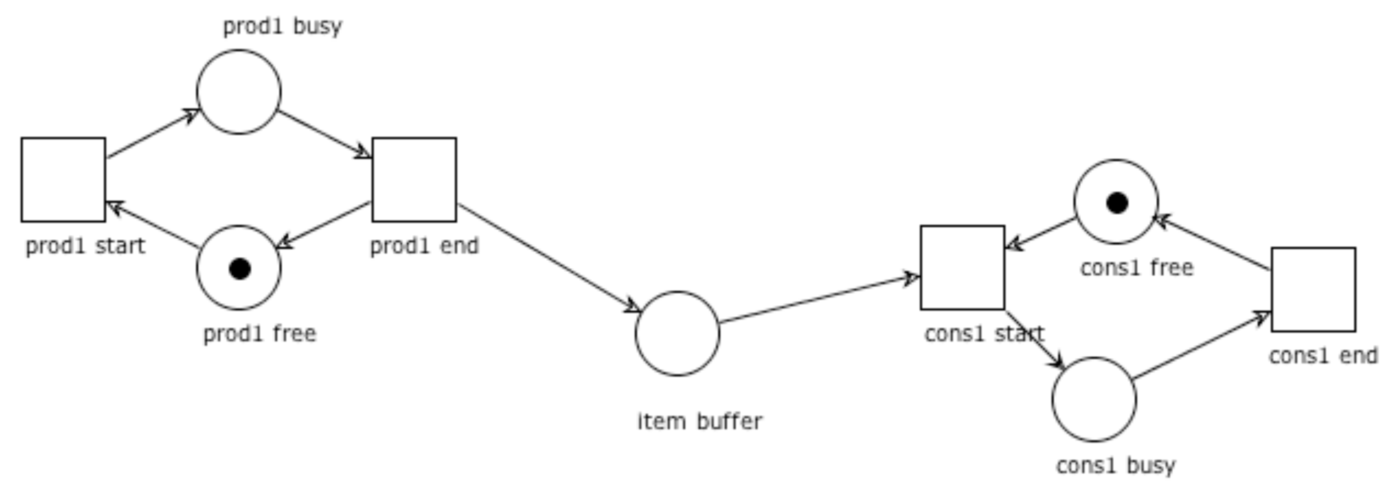


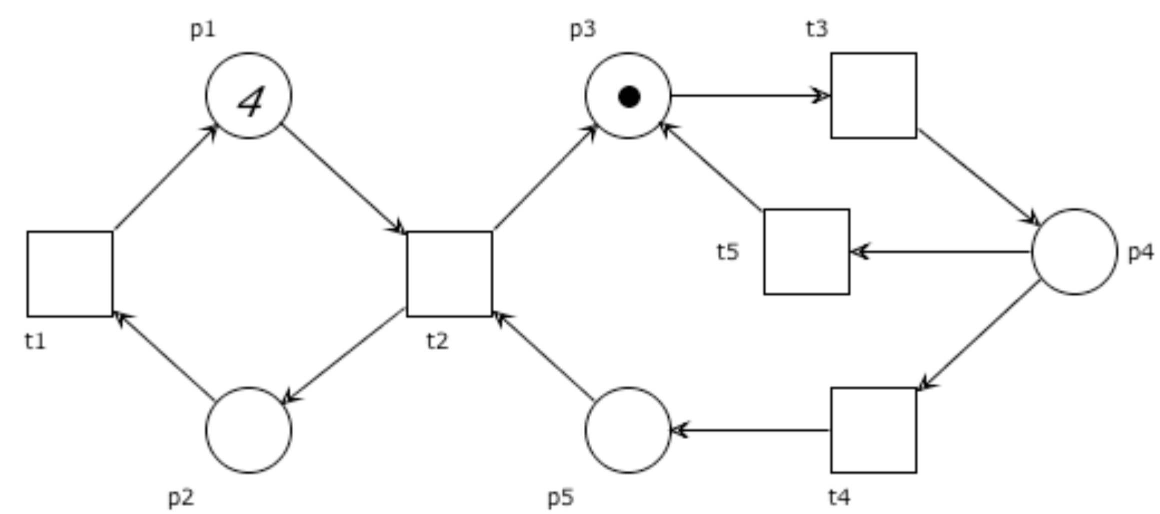
[1 1 1 0 0 0 0] +
[0 0 0 0 1 1 1] +
[1 1 0 1 1 1 0] =
[2 2 1 1 2 2 1]

Exercises

Define two (linearly independent) S-invariants for each of the nets below







S-invariants and system properties

(Semi-)Positive S-invariants

The S-invariant $\mathbf{I}$ is **semi-positive** if $\mathbf{I} > \mathbf{0}$
(i.e. $\mathbf{I} \geq \mathbf{0}$ and $\mathbf{I} \neq \mathbf{0}$)

all entries are non-negative and at least one is positive

The **support** of $\mathbf{I}$ is: $\langle \mathbf{I} \rangle = \{ p \mid \mathbf{I}(p) > 0 \}$

set of places with positive weights

The S-invariant $\mathbf{I}$ is **positive** if $\mathbf{I} \succ \mathbf{0}$ **all entries are positive**
(i.e. $\mathbf{I}(p) > 0$ for any place $p \in P$)
(i.e. $\langle \mathbf{I} \rangle = P$)

A (semi-positive) S-invariant whose coefficients are all 0 and 1 is called **uniform**

Note

Notation: $\bullet S = \bigcup_{s \in S} \bullet s$

Every semi-positive invariant
satisfies the equation

transitions that produce tokens in some places of the support

$$\bullet\langle \mathbf{I} \rangle = \langle \mathbf{I} \rangle \bullet$$

transitions that consume tokens from some places of the support

pre-sets of support equal post-sets of support

(the result holds for both S-invariants and T-invariants)

A sufficient condition for boundedness

Theorem:
If (P, T, F, M_0) has a positive S-invariant then it is bounded

Let $M \in [\, M_0 \,\rangle$ and let $\mathbf{I}$ be a positive S-invariant.

Let $p \in P$. Then $\mathbf{I}(p)M(p) \leq \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since $\mathbf{I}$ is positive, we can divide by $\mathbf{I}(p)$:
$M(p) \leq (\mathbf{I} \cdot M_0)/\mathbf{I}(p)$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q)M(q)$$

Consequences of previous theorem

By exhibiting a positive S-invariant we can prove that the system is **bounded for any initial marking**

Note that all places in the support of a semi-positive S-invariant are **bounded for any initial marking**

$$M(p) \leq \frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p)}$$

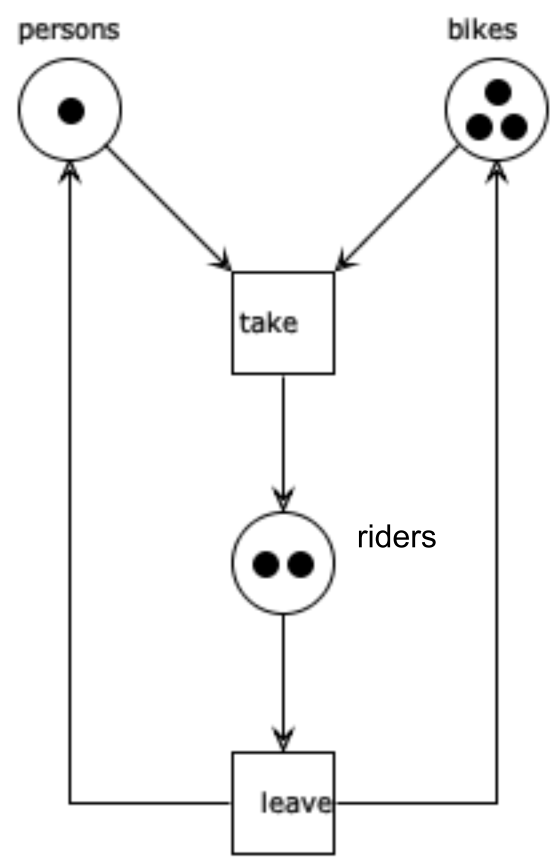
this value is independent from the reachable marking M

Example

To prove that the system is bounded we can just exhibit a positive S-invariant

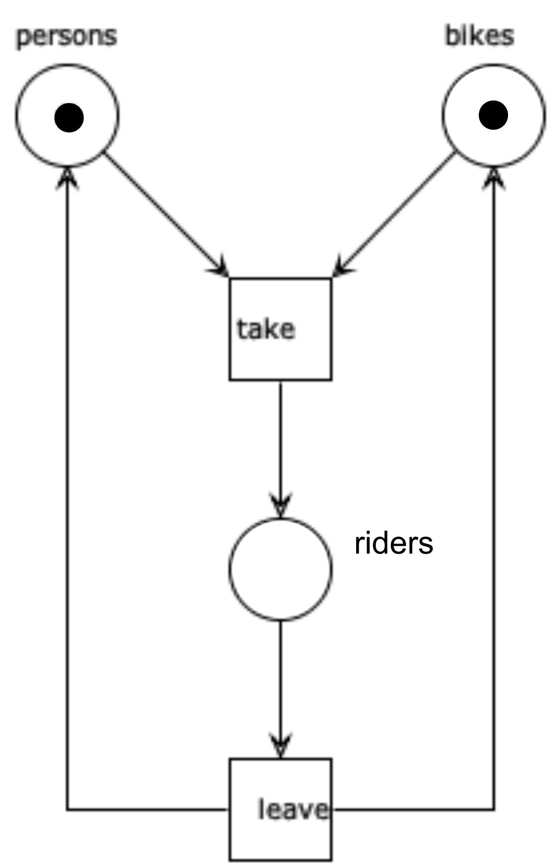
I = [1 1 2]

p1 - persons
p2 - bikes
p3 - riders



Example

How many tokens are at most in p_3?



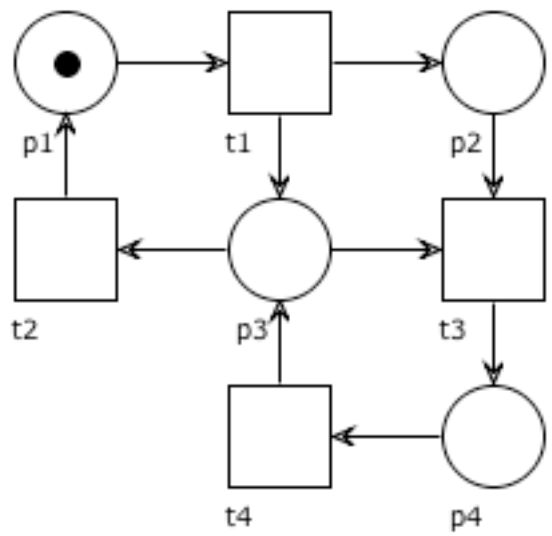
p1 - persons
p2 - bikes
p3 - riders

I = [1 1 2]

$$\frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p_3)} = \frac{2}{2} = 1$$

Example

How many tokens are at most in p_3?

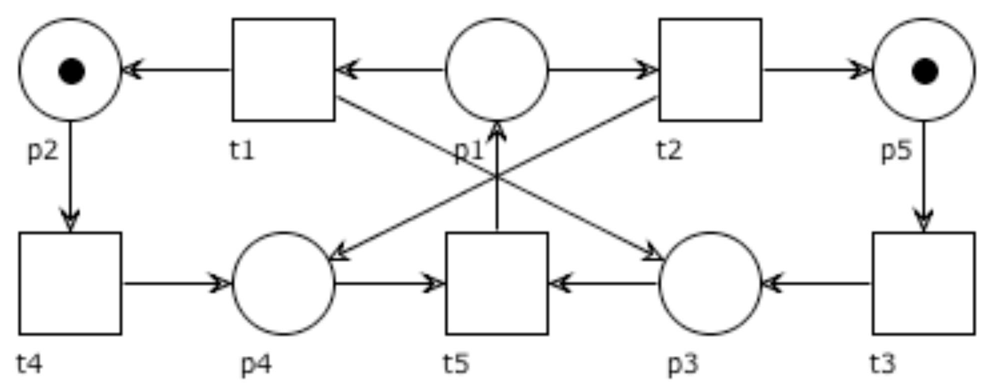


I = [1 0 1 1]

$$\frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p_3)} = \frac{1}{1} = 1$$

Question time

live, **deadlock-free**, **bounded**, **safe**, cyclic
Prove boundedness by exhibiting an S-invariant



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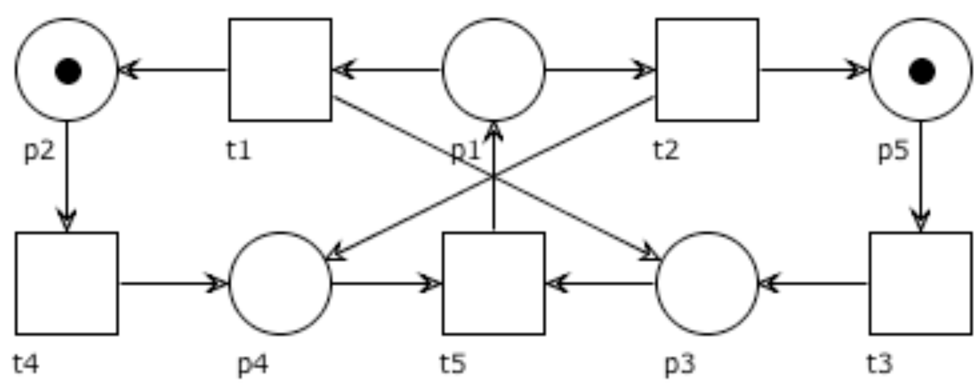
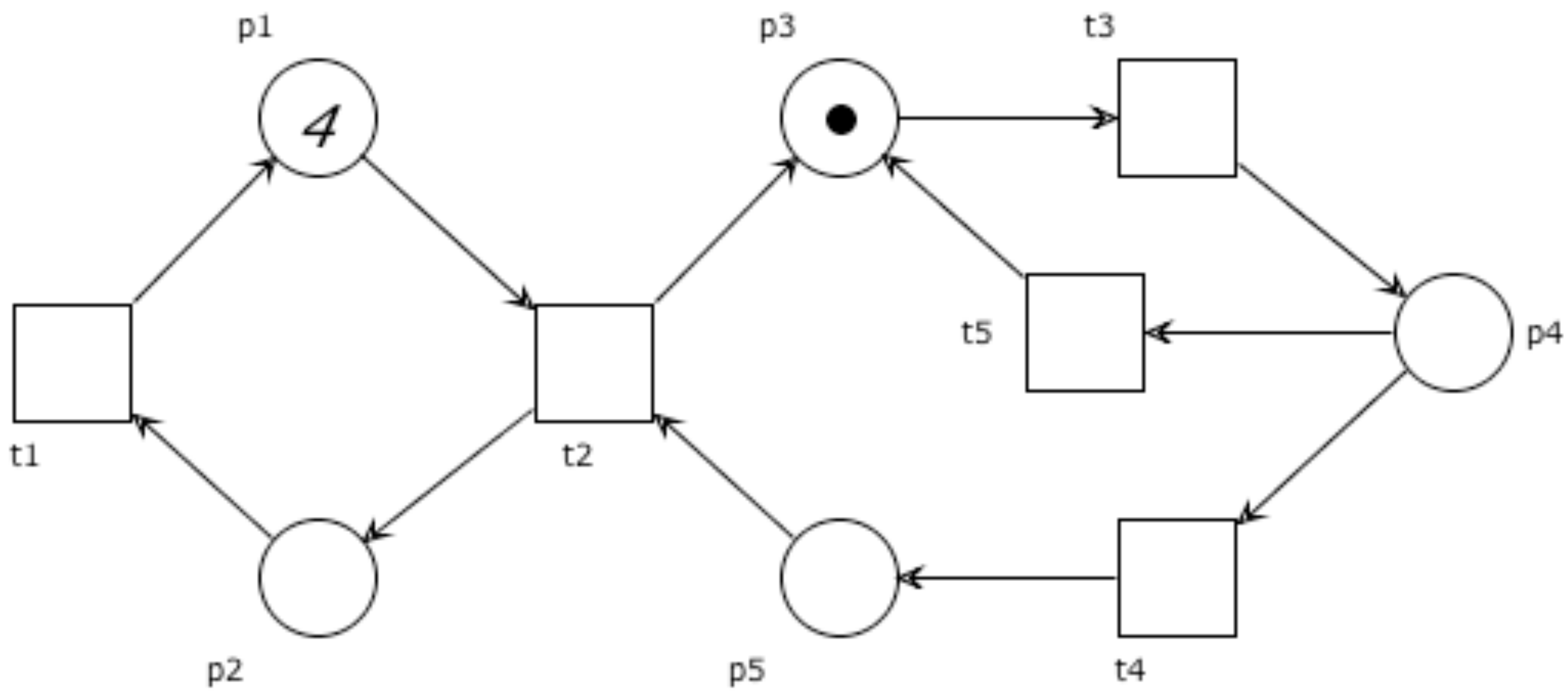
Prove boundedness by exhibiting an S-invariant



I = [2 1 1 1 1]

Exercises

Find a positive S-invariant for the net below



A necessary condition for liveness

Theorem:
If (P, T, F, M_0) is live then for every semi-positive invariant $\mathbf{I}$:

$$\mathbf{I} \cdot M_0 > 0$$

Let $p \in \langle \mathbf{I} \rangle$ and take any $t \in \bullet p \cup p \bullet$.

By liveness, there are $M, M' \in [\, M_0 \,\rangle$ with $M \xrightarrow{t} M'$

Then, $M(p) > 0$ (if $t \in p\bullet$) or $M'(p) > 0$ (if $t \in \bullet p$)

If $M(p) > 0$, then $\mathbf{I} \cdot M \geq \mathbf{I}(p)M(p) > 0$
If $M'(p) > 0$, then $\mathbf{I} \cdot M' \geq \mathbf{I}(p)M'(p) > 0$

In any case, $\mathbf{I} \cdot M_0 = \mathbf{I} \cdot M = \mathbf{I} \cdot M' > 0$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q) M(q)$$

Consequence of previous theorem

If we find a semi-positive invariant such that

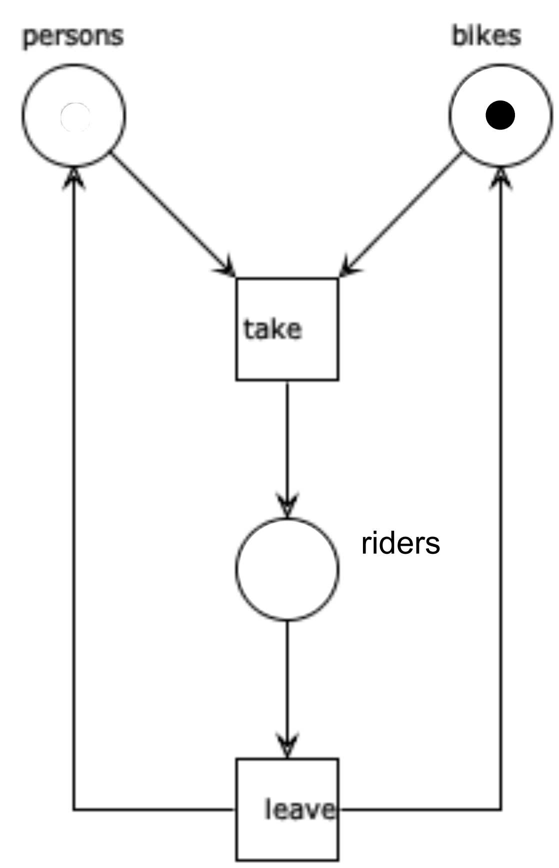
$$\mathbf{I} \cdot M_0 = 0$$

Then we can conclude that the system **is not live**

Example

the system is not live

It is immediate to check the counter-example



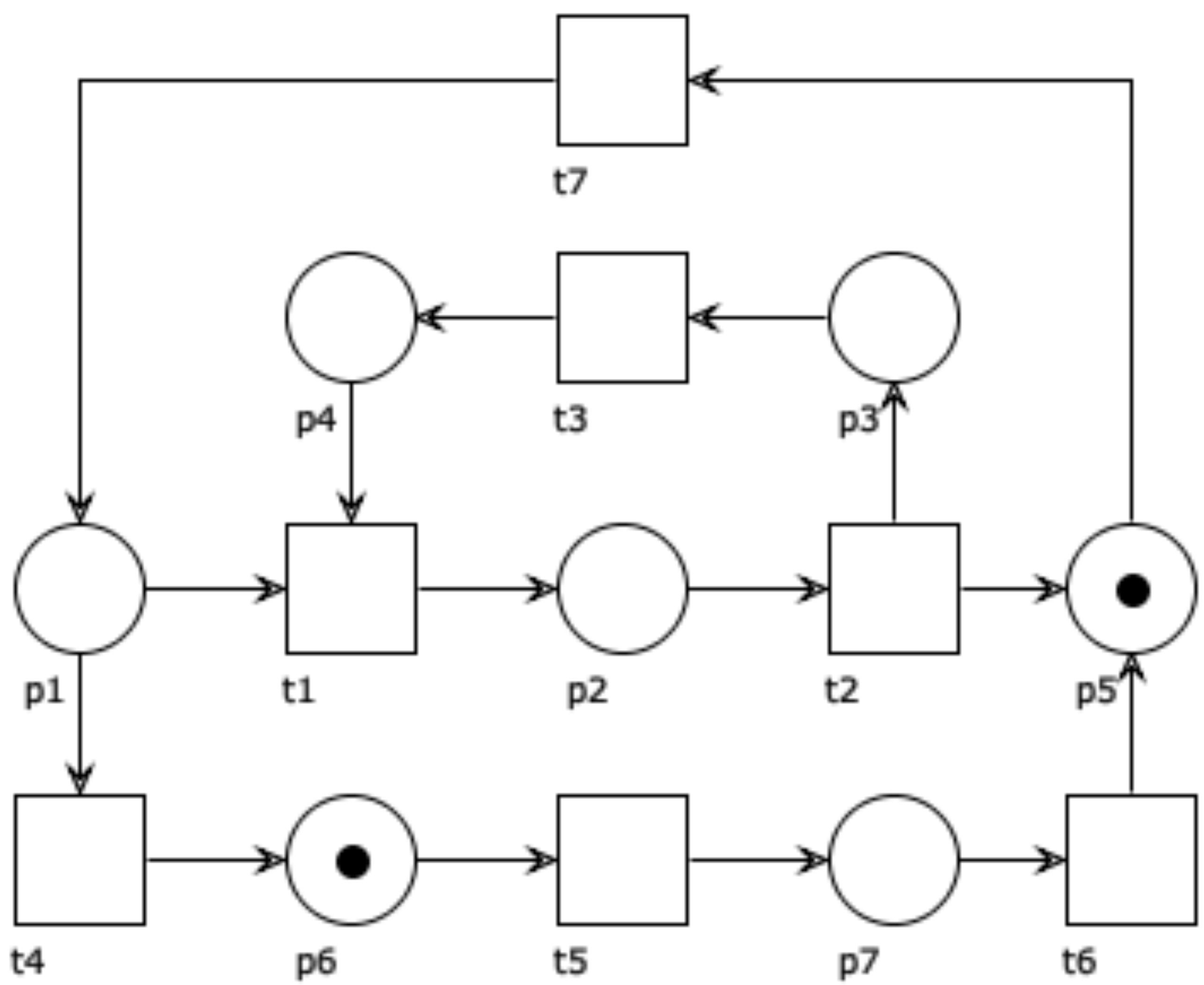
p1 - persons
p2 - bikes
p3 - riders

$$\mathbf{I} = [\,1 \;\; 0 \;\; 1\,]$$

$$\underbrace{[\,1\;0\;1\,]}_{\mathbf{I}}\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{M_0} = 0$$

Exercises

Find an S-invariant that proves the net non-live



Markings that agree on all S-invariant

Definition: M and M' **agree on all S-invariants** if for every S-invariant **I** we have **I**$\cdot M$ = **I**$\cdot M'$

Note: by properties of linear algebra,
this corresponds to require that the equation on **y**
N$\cdot$**y** = M'-M has some rational-valued solution

Remark: In general, there can exist M and M' that agree on all S-invariants but such that none of them is reachable from the other

A necessary condition for reachability

reachability problem: is M reachable from M_0? $M \overset{?}{\in} [M_0\rangle$

decidable, but computationally expensive (EXPSPACE-hard)

S-invariants provide a preliminary check that can be computed more efficiently

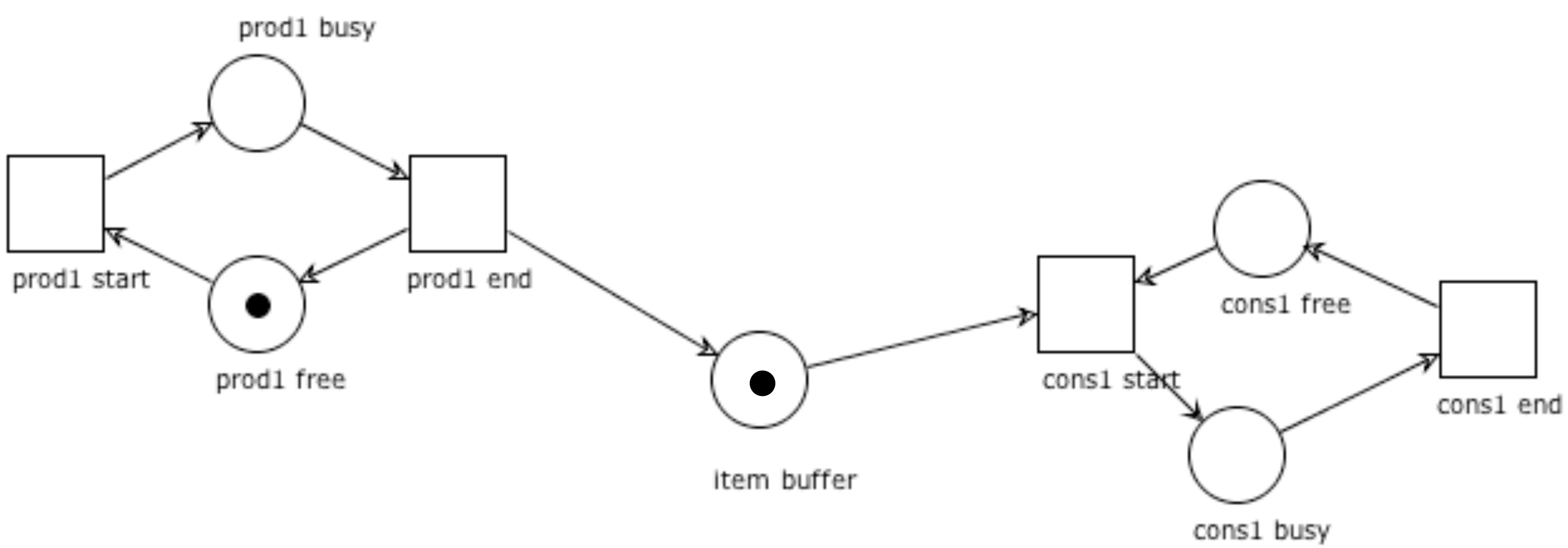
Let (P, T, F, M_0) be a system.

If there is an S-invariant $\mathbf{I}$ s.t. $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$ then $M \notin [\, M_0 \,\rangle$

If the equation $\mathbf{N} \cdot \mathsf{y} = M - M_0$ has no rational-valued solution, then $M \notin [\, M_0 \,\rangle$

Example

Prove that the marking
M = prod1free + cons1busy
is not reachable



$$\mathbf{I} = [\ 0\ 0\ 0\ 1\ 1\ ]$$

$$\mathbf{I} \cdot M_0 = 0 \neq 1 = \mathbf{I} \cdot M$$

S-invariants: recap

Positive S-invariant => boundedness
Unboundedness => no positive S-invariant

Semi-positive S-invariant **I** and liveness => $\mathbf{I} \cdot M_0 > 0$
Semi-positive S-invariant **I** and $\mathbf{I} \cdot M_0 = 0$ => non-live

S-invariant **I** and M reachable => $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$
S-invariant **I** and $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$ => M not reachable

S-invariants: pay attention to implication

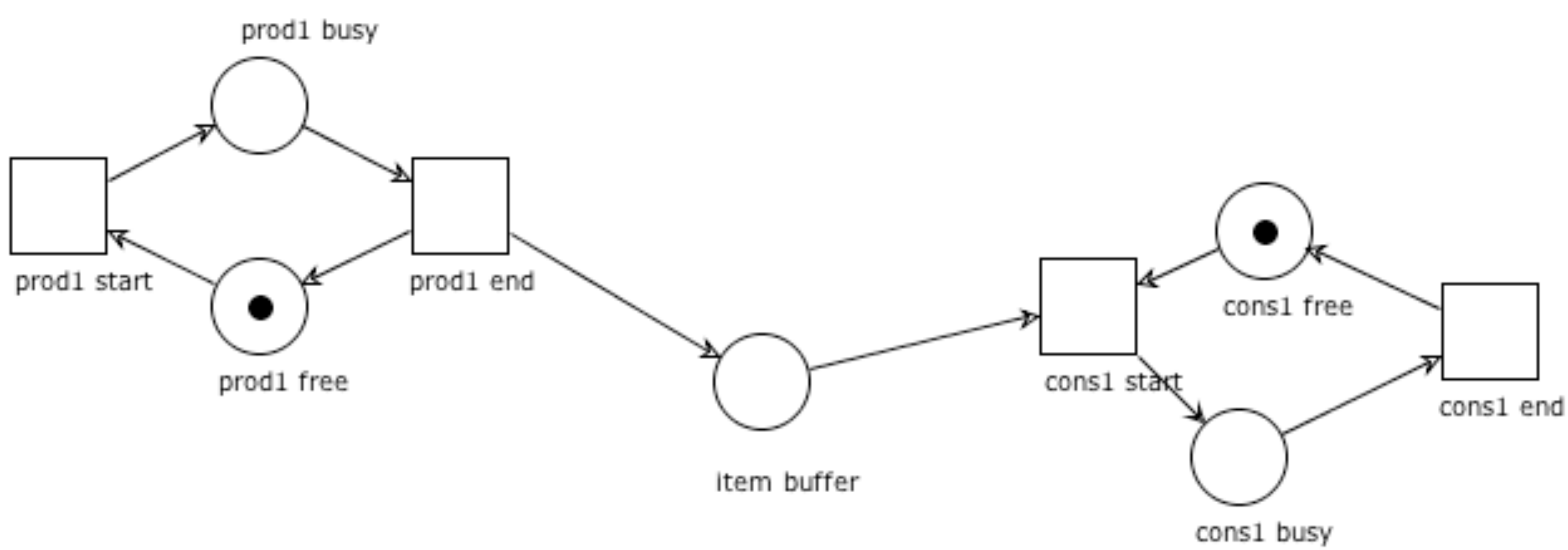
No positive S-invariant => **maybe** unbounded

Semi-positive S-invariant **I** and $\mathbf{I} \cdot M_0 > 0$ => **maybe** live

S-invariant **I** and $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$ => **maybe** M reachable

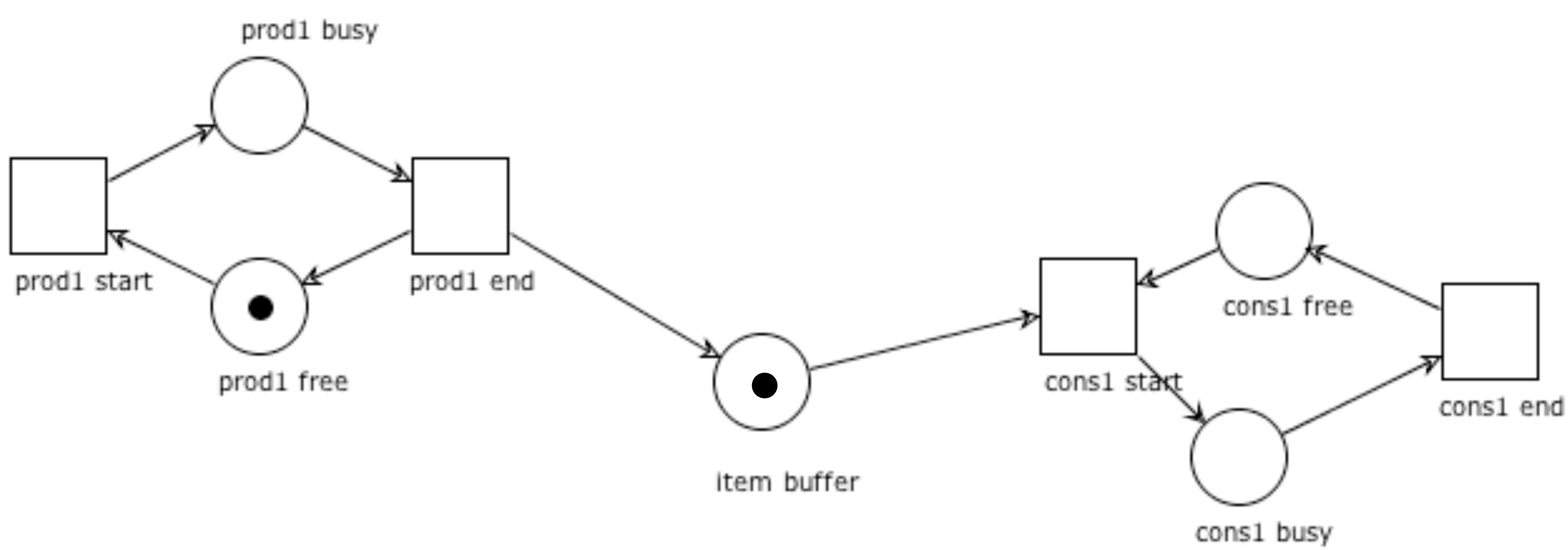
Exercises

Can you find a positive S-invariant?



Exercises

Prove that the system is not live by exhibiting a suitable S-invariant



T-invariants

Dual reasoning

The S-invariants of a net N are vectors satisfying the equation

$$\mathbf{x} \cdot \mathbf{N} = \mathbf{0}$$

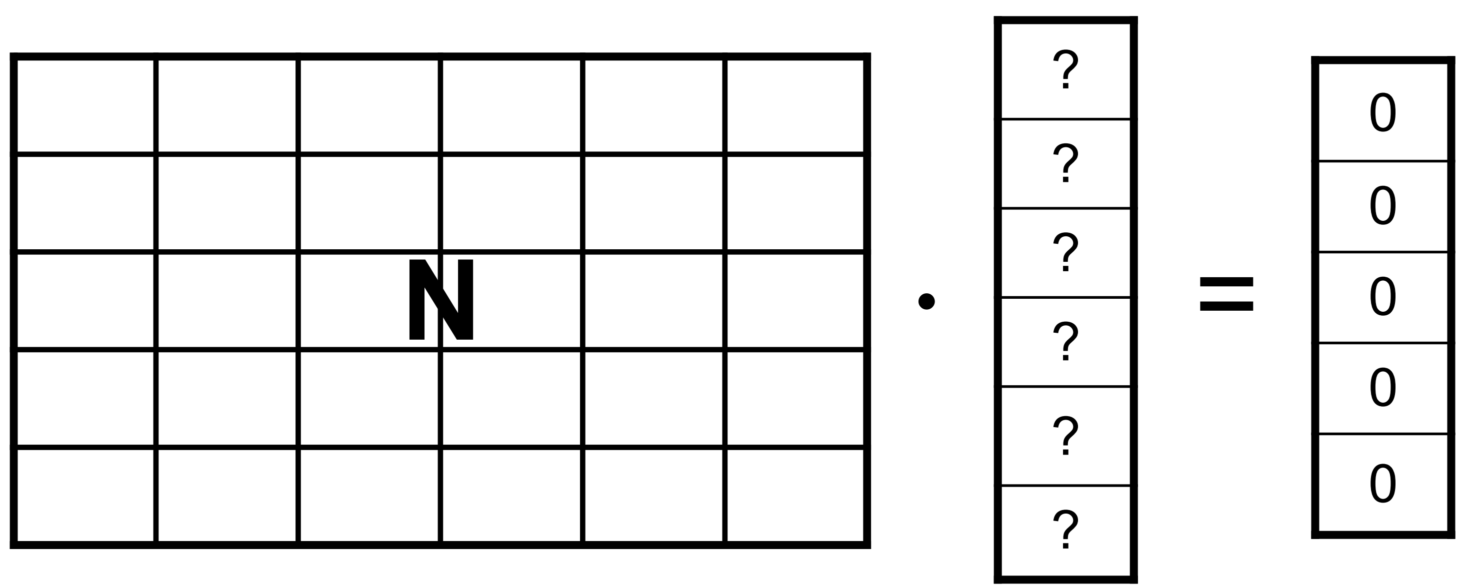
It seems natural to ask if we can find some interesting properties also for the vectors satisfying the equation

$$\mathbf{N} \cdot \mathbf{y} = \mathbf{0}$$

T-invariant
(aka transition-invariant)

Definition: A **T-invariant** of a net N=(P,T,F) is a rational-valued solution **y** of the equation

$$\mathbf{N} \cdot \mathbf{y} = \mathbf{0}$$



Fundamental property of T-invariants

Proposition: Let $M \xrightarrow{\sigma} M'$.

The Parikh vector $\vec{\sigma}$ is a T-invariant iff $M' = M$

$\Rightarrow$) By the marking equation lemma $M' = M + \mathbf{N} \cdot \vec{\sigma}$
Since $\vec{\sigma}$ is a T-invariant $\mathbf{N} \cdot \vec{\sigma} = \mathbf{0}$, thus $M' = M$.

$\Leftarrow$) If $M \xrightarrow{\sigma} M$, by the marking equation lemma $M = M + \mathbf{N} \cdot \vec{\sigma}$
Thus $\mathbf{N} \cdot \vec{\sigma} = M - M = \mathbf{0}$ and $\vec{\sigma}$ is a T-invariant

Transition-invariant, intuitively

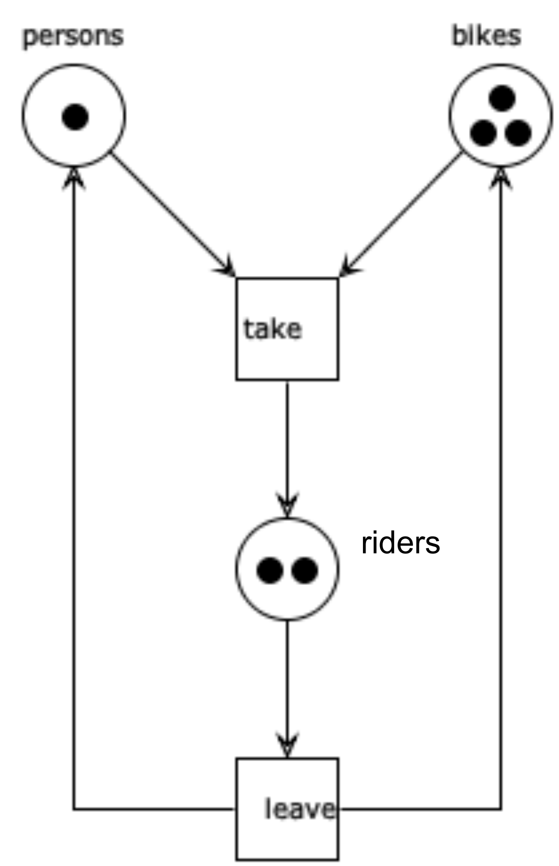
A transition-invariant assigns a **number of occurrences to each transition** such that any occurrence sequence comprising exactly those transitions leads to the same marking where it started (independently from the order of execution)

Example

An easy-to-be-found T-invariant

p1 - persons
p2 - bikes
p3 - riders

t1 - take
t2 - leave



$$\begin{array}{cc} t & l \\ \end{array}$$
$$[\ 1 \quad 1\ ]$$

Alternative definition of T-invariant

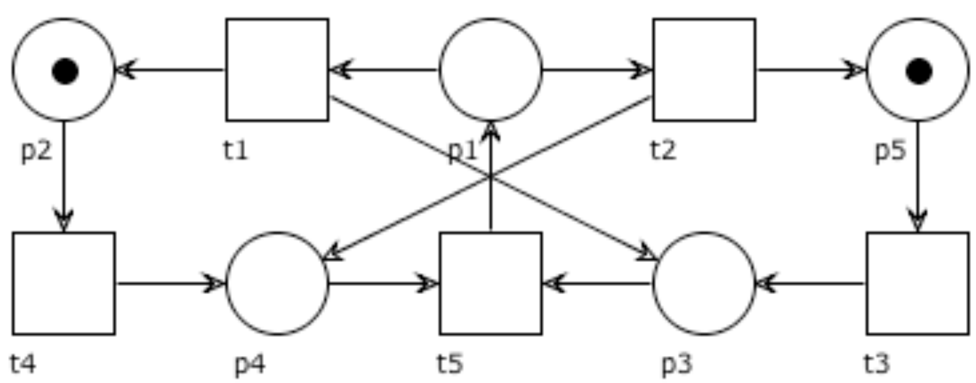
Proposition:

A mapping $\mathbf{J} : T \to \mathbb{Q}$ is a T-invariant of N iff for any $p \in P$:

$$\sum_{t \in \bullet p} \mathbf{J}(t) = \sum_{t \in p \bullet} \mathbf{J}(t)$$

Question time

Which of the following are T-invariants?



t_1	t_2	t_3	t_4	t_5
[1	0	0	1	1]
[1	1	2	1	2]
[1	1	1	0	2]
[1	1	1	1	2]
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$$\forall p \in P, \ \sum_{t \in \bullet p} \mathbf{J}(t) \stackrel{?}{=} \sum_{t \in p \bullet} \mathbf{J}(t)$$

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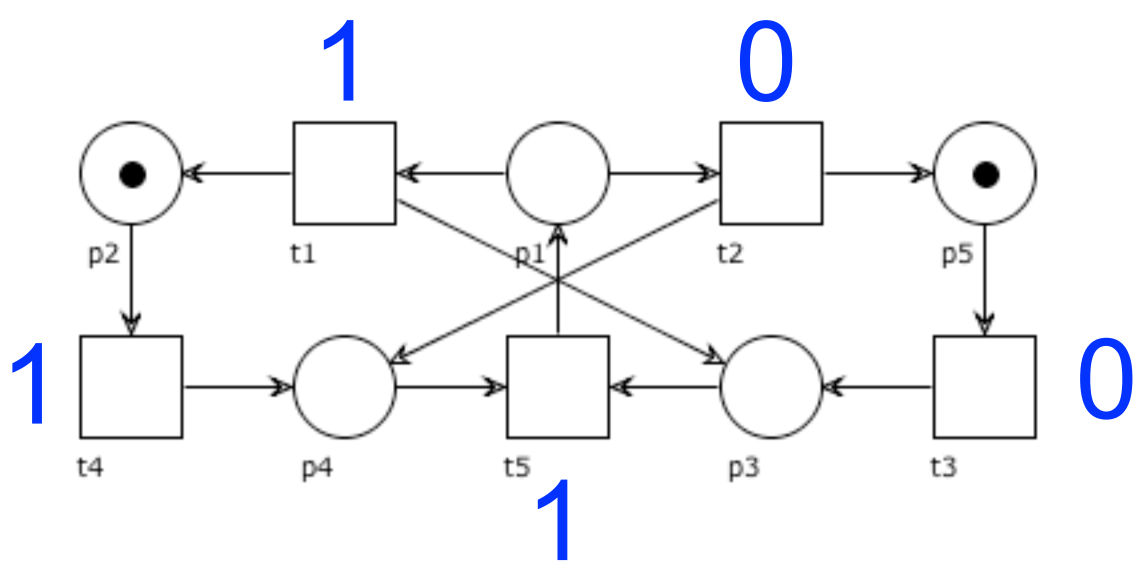

$$
\begin{array}{ccccc}
t_1 & t_2 & t_3 & t_4 & t_5 \\
\end{array}
$$

$$
\begin{array}{l}
[\,1 \quad 0 \quad 0 \quad 1 \quad 1\,] \\
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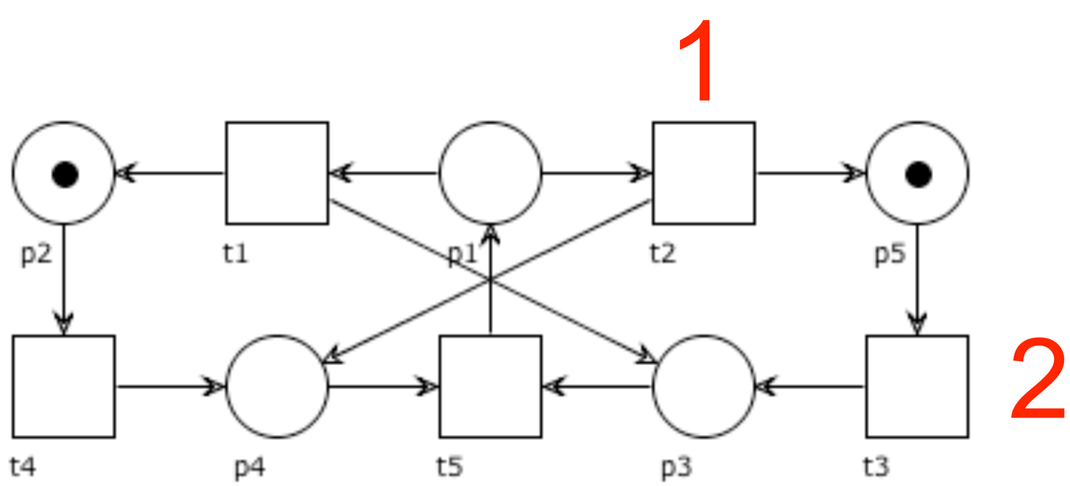

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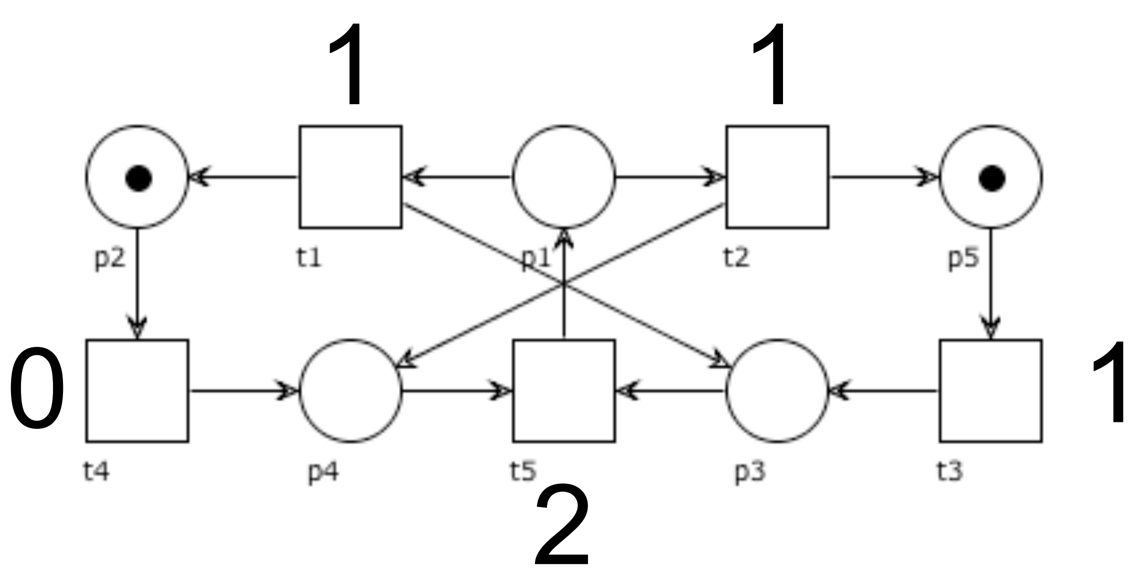

$$\begin{array}{c} & t_1 & t_2 & t_3 & t_4 & t_5 \\ & [\,1 & 0 & 0 & 1 & 1\,] \\ & [\,1 & 1 & 2 & 1 & 2\,] \\ \rightarrow & [\,1 & 1 & 1 & 0 & 2\,] \\ & [\,1 & 1 & 1 & 1 & 2\,] \\ & [\,0 & 1 & 1 & 0 & 1\,] \end{array}$$

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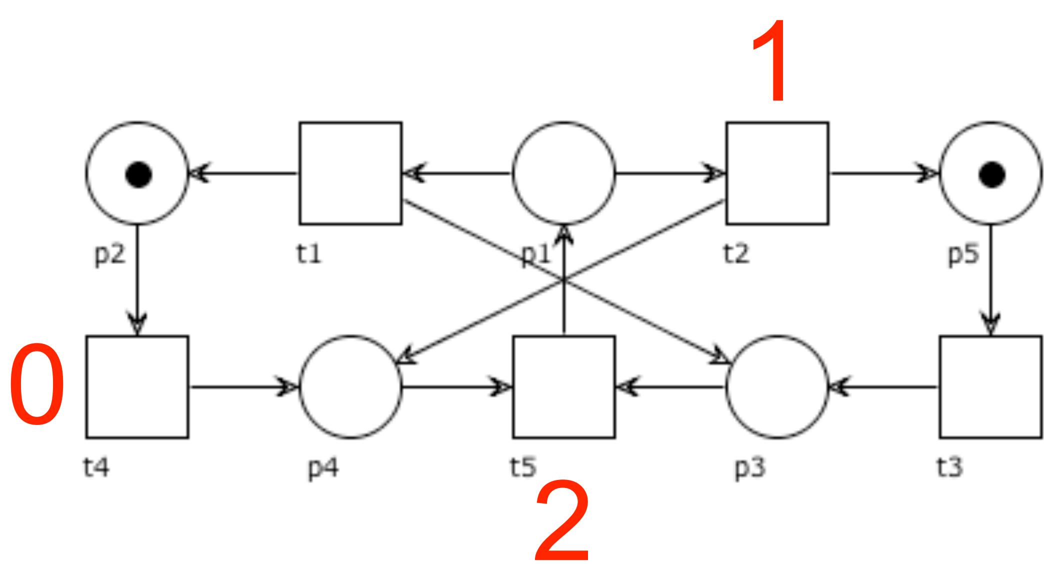

$$\begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array}$$

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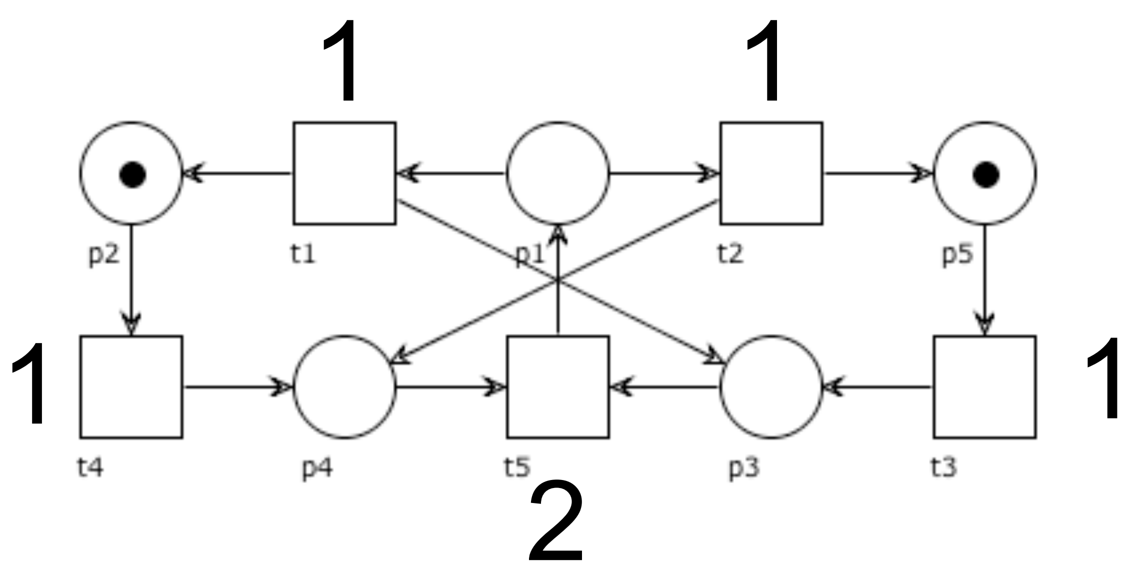

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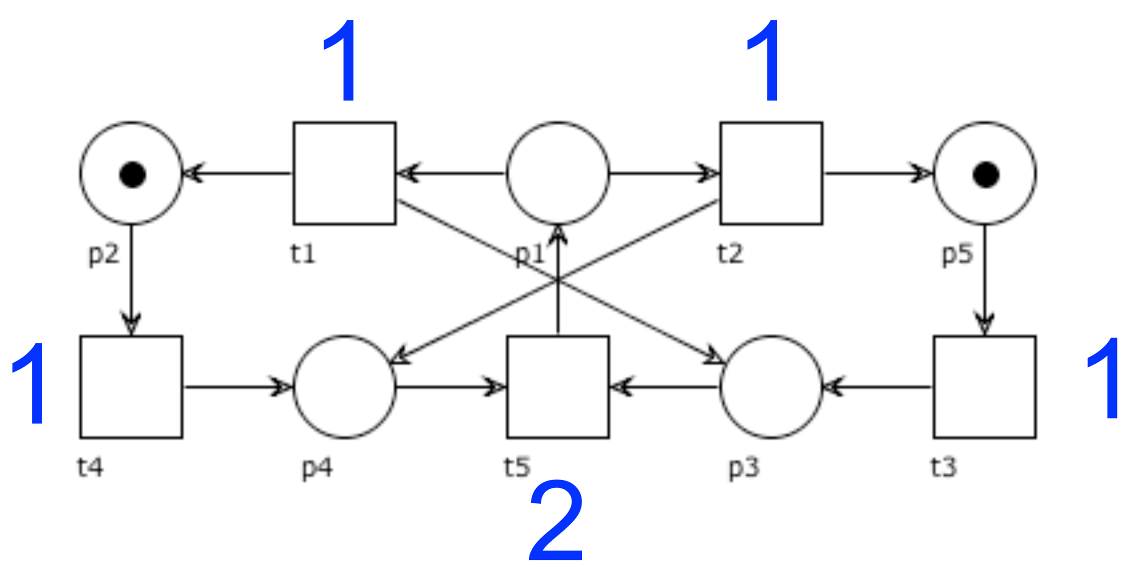

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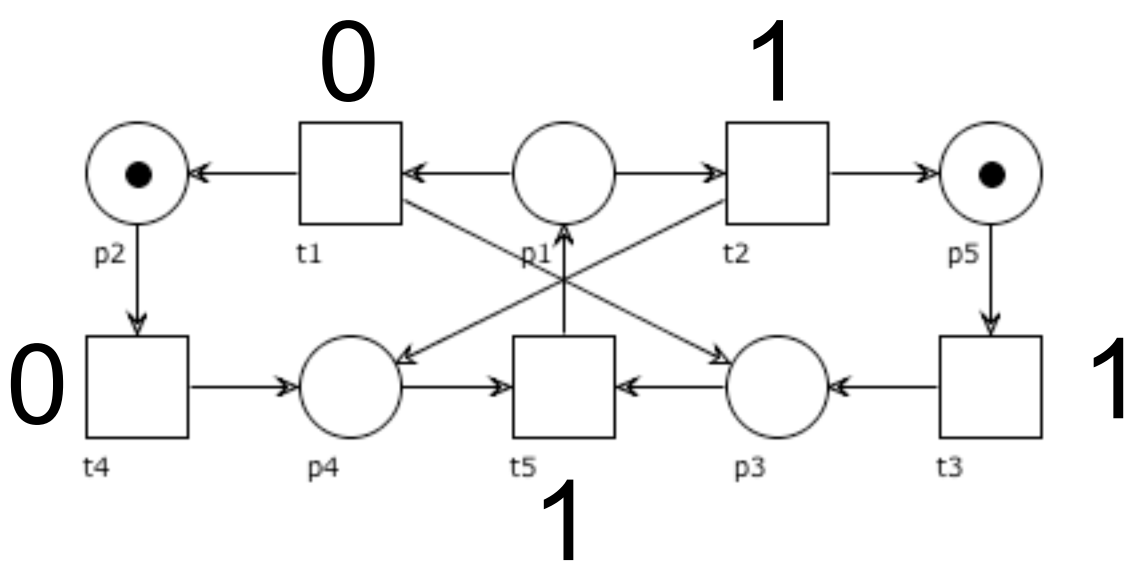

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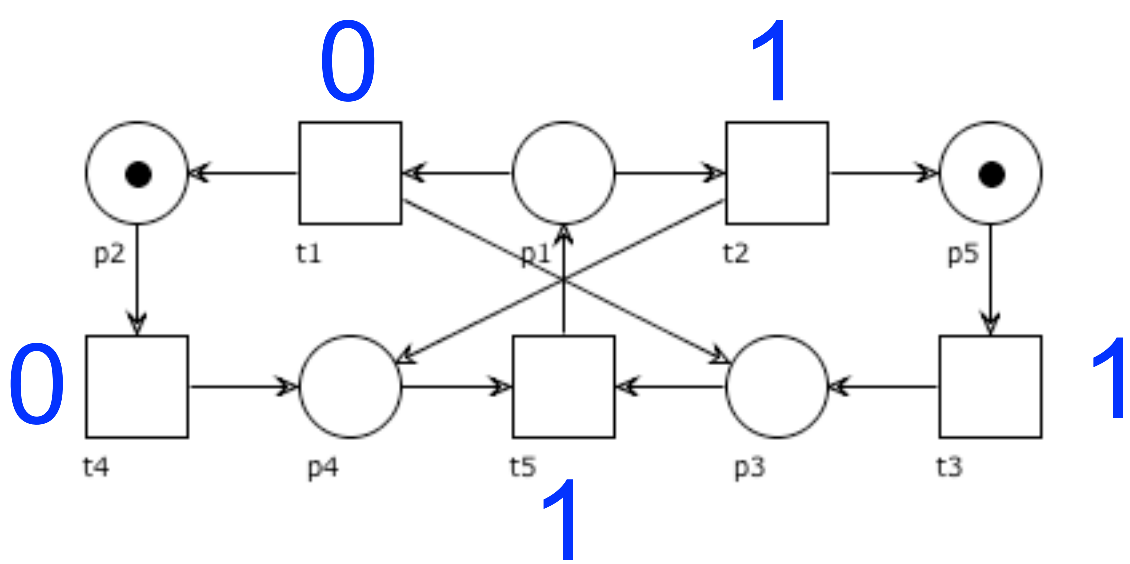

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T-invariants and system properties

Pigeonhole principle

If n items are put into m slots, with n > m, then at least one slot must contain more than one item

Reproduction lemma

Lemma: Let (P, T, F, M_0) be a bounded system.
If $M_0 \xrightarrow{\sigma}$ for some infinite sequence σ, then
there is a semi-positive T-invariant $\mathbf{J}$ such that $\langle\, \mathbf{J} \,\rangle \subseteq \{\, t \mid t \in \sigma \,\}$.

Assume $\sigma = t_1\, t_2\, t_3 \ldots$ and $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$

By boundedness: $[\, M_0 \,\rangle$ is finite.

By the pigeonhole principle, there are $0 \leq i < j$ s.t. $M_i = M_j$
Let $\sigma' = t_{i+1} \ldots t_j$. Then $M_i \xrightarrow{\sigma'} M_j = M_i$

By the marking equation lemma: $\vec{\sigma'}$ is a T-invariant. (fund. prop. of T-inv.)
It is semi-positive, because σ' is not empty ($i < j$).
Clearly, $\langle\, \mathbf{J} \,\rangle$ only includes transitions in σ.

Boundedness, liveness and positive T-invariant

Theorem: If a bounded system is live, then it has a positive T-invariant

By boundedness: $[\, M_0 \,\rangle$ is finite and we let $k = |[\, M_0 \,\rangle|$.

By liveness: $M_0 \xrightarrow{\sigma_1} M_1$ with $\vec{\sigma_1}(t) > 0$ for any $t \in T$
Similarly: $M_1 \xrightarrow{\sigma_2} M_2$ with $\vec{\sigma_2}(t) > 0$ for any $t \in T$
Similarly: $M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 ... \xrightarrow{\sigma_k} M_k$

By the pigeonhole principle, there are $0 \leq i < j \leq k$ s.t. $M_i = M_j$
Let $\sigma = \sigma_{i+1} ... \sigma_j$. Then $M_i \xrightarrow{\sigma} M_j = M_i$

By the marking equation lemma: $\vec{\sigma}$ is a T-invariant. (fund. prop. of T-inv.)
It is positive, because $\vec{\sigma}(t) \geq \vec{\sigma_j}(t) > 0$ for any $t \in T$.

Corollary of previous theorem

Every live and bounded system has:

a reachable marking M and
an occurrence sequence $M \xrightarrow{\sigma} M$

such that <u>all transitions of N occur in σ.</u>

T-invariants: recap

Boundedness + liveness => positive T-invariant

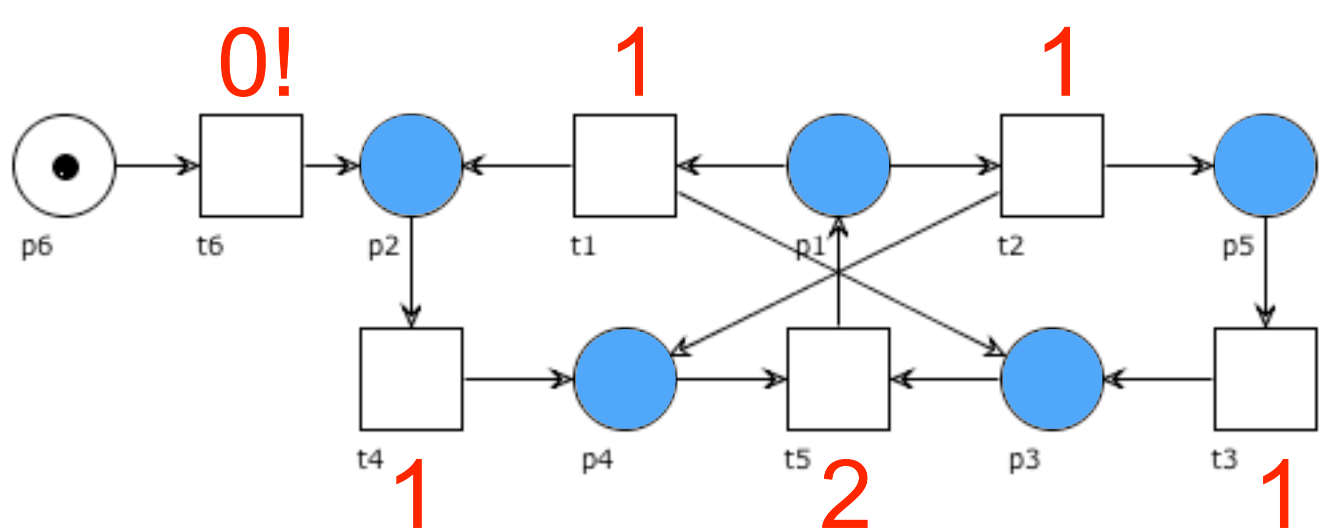
No positive T-invariant => non (live + bounded)
No positive T-invariant => non-live OR unbounded
No positive T-invariant + liveness => unbounded
No positive T-invariant + boundedness => non-live
No positive T-inv. + positive S-inv. => non-live

T-invariants: pay attention to implication

No positive T-invariant => **maybe** non live

Example

The system below has a positive S-invariant
but no positive T-invariant:
thus it is bounded but not live

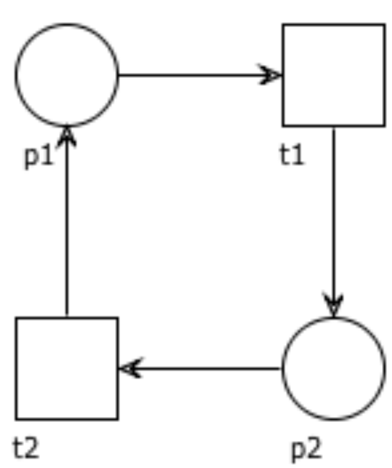


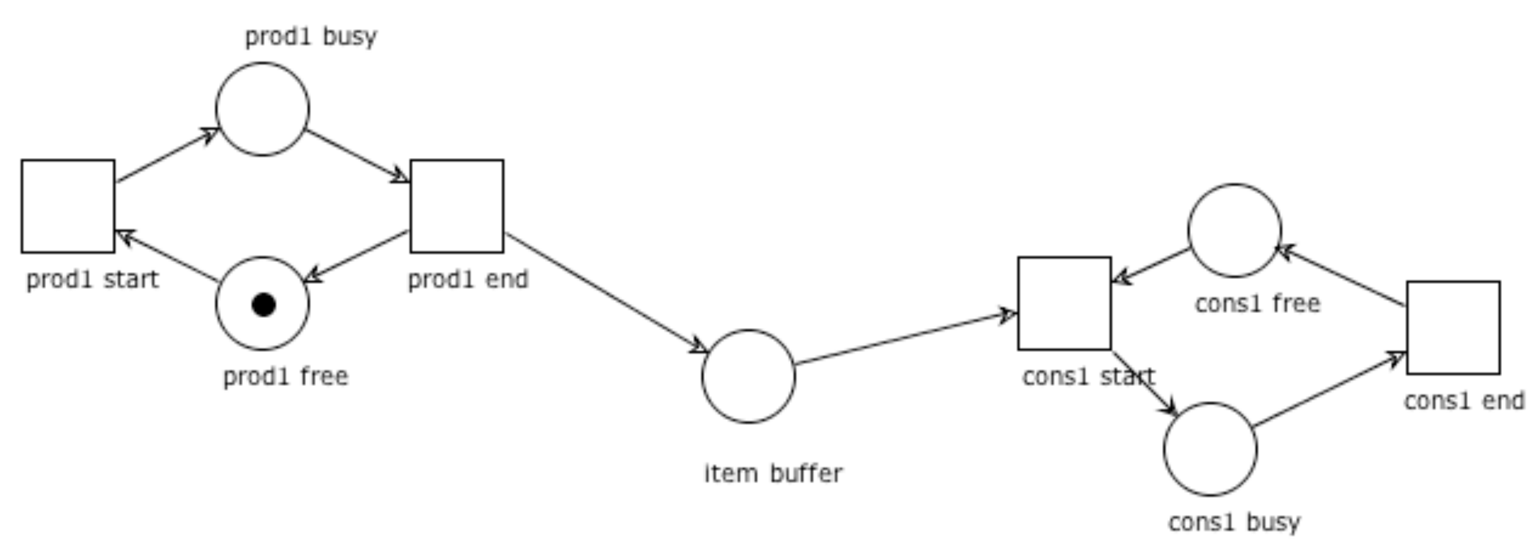
I = [2 1 1 1 1 1]

J = [??]

Exercises

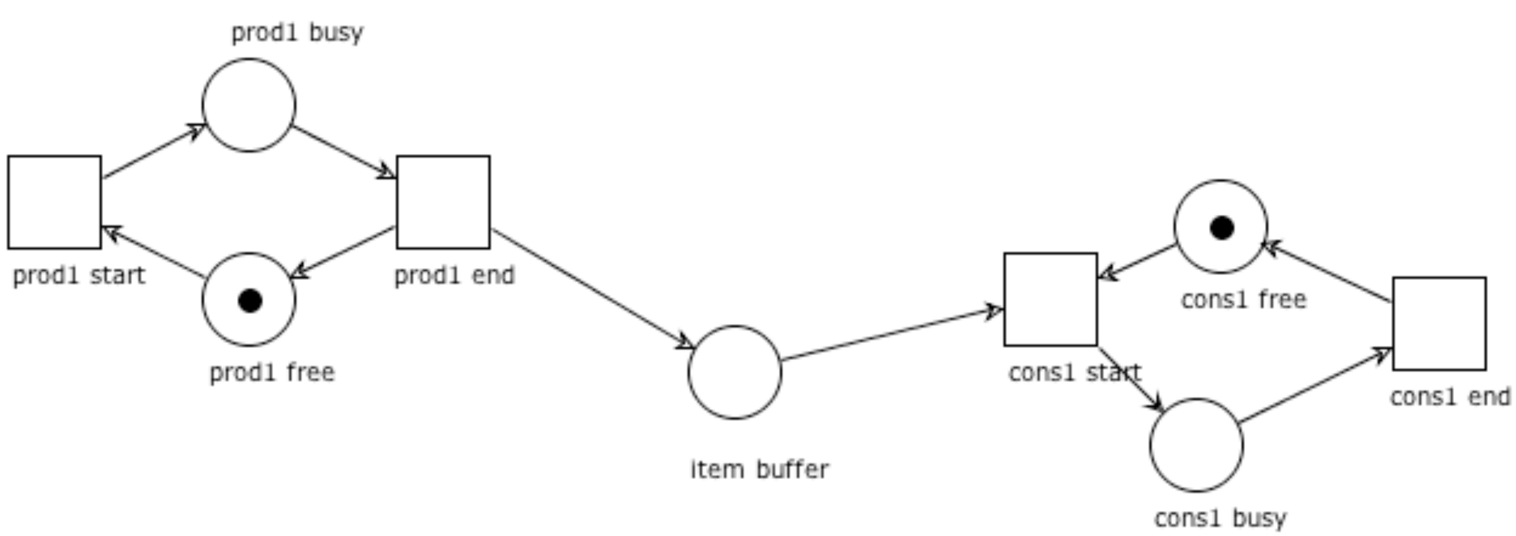
Which system has a positive T-invariant
but is not
live and bounded?

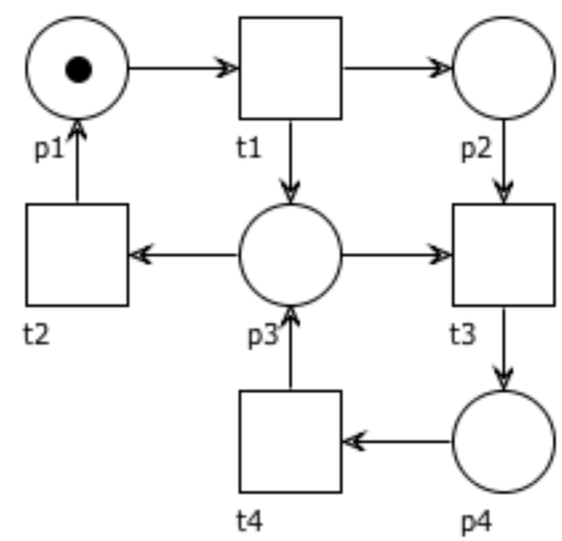




Exercises

Which live system has a positive T-invariant
but is not bounded?





Two theorems on strong connectedness (whose proofs we omit)

Strong connectedness theorem

Theorem: If a weakly connected system is live and bounded then it is strongly connected

Consequences

If a (weakly-connected) net is not strongly connected

then

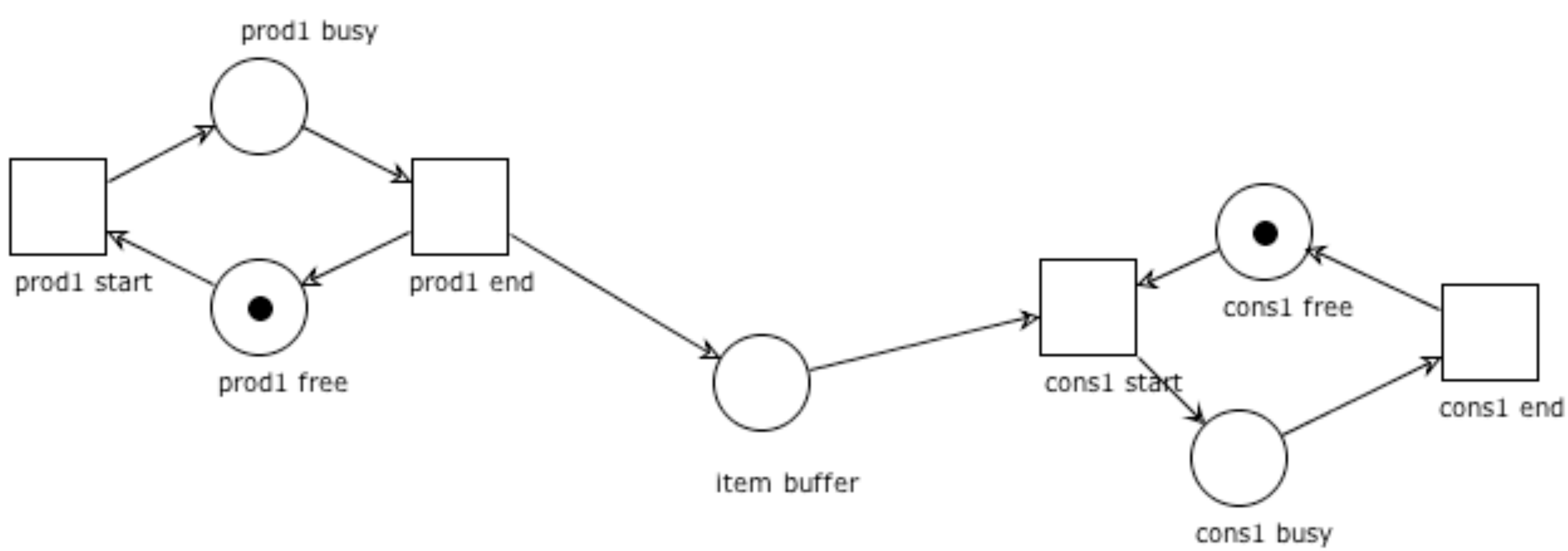
It is not “live and bounded”

If it is live, it is not bounded

If it is bounded, it is not live

Example

It is now immediate to see that this system (weakly connected, not strongly connected) cannot be live and bounded (it is live but not bounded)



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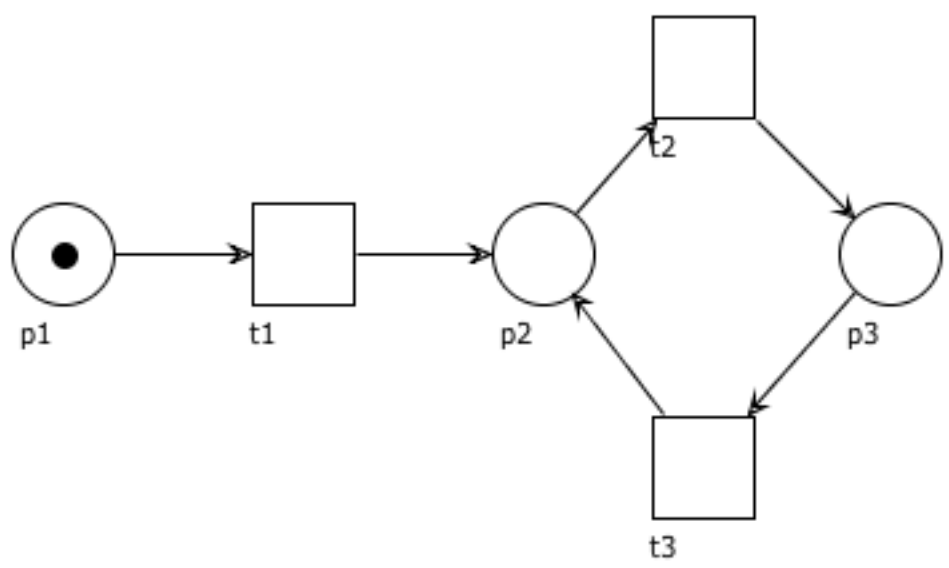
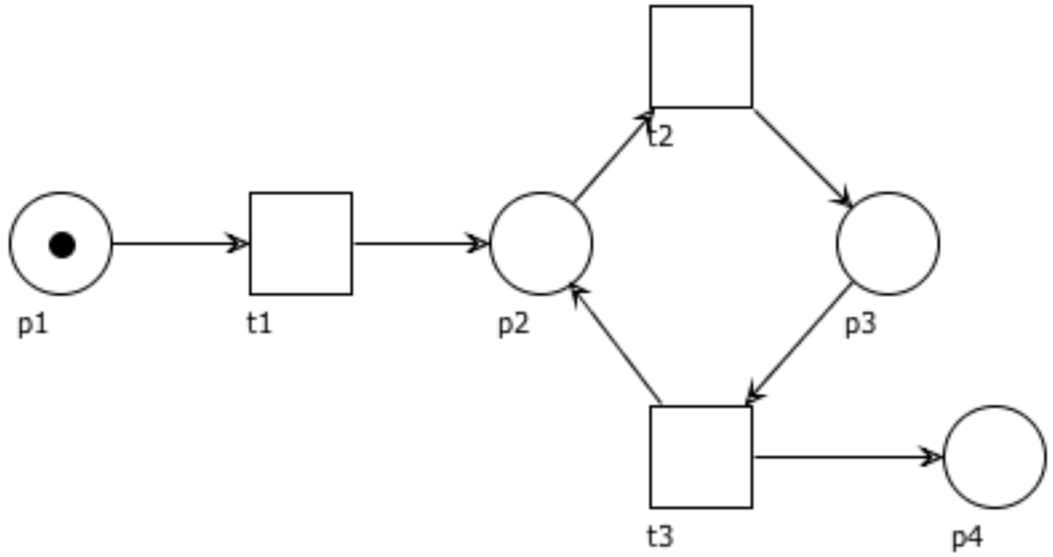


Example

It is now immediate to see that this system (weakly connected, not strongly connected) cannot be live and bounded (it is neither bounded nor live)



Strong connectedness via invariants

Theorem: If a weakly connected net has
a positive S-invariant **I** and a positive T-invariant **J**
then it is strongly connected

Consequences

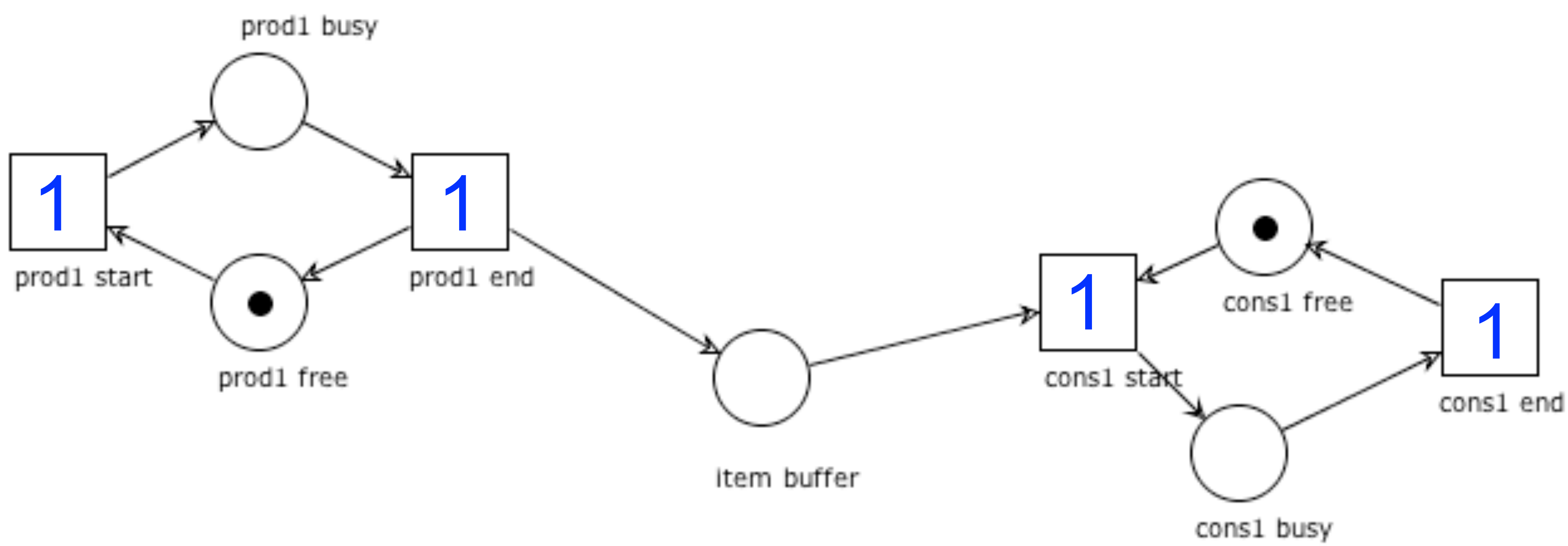
If a (weakly-connected) net is not strongly connected

then

we cannot find (two) positive S- and T-invariants

Example

It is now immediate to check that this system (weakly connected, not strongly connected) has a positive T-invariant, but not a positive S-Invariant



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