

Object

We study some “good” properties of S-systems and T-systems

Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

Example

$$P = \{p_1, p_2, p_3\} \quad M = 2p_1 + 3p_2 \quad M(P) = 2 + 3 + 0 = 5$$

S-systems

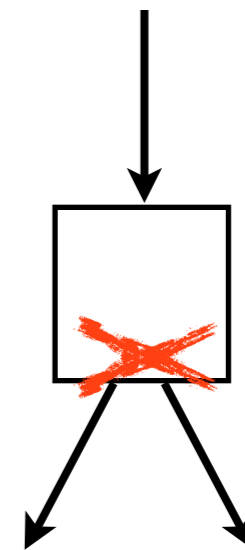
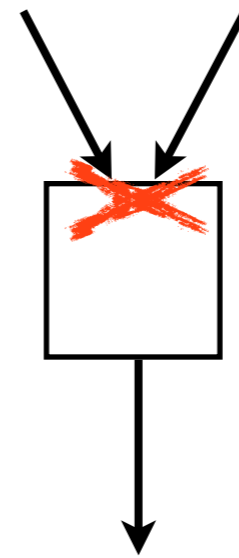
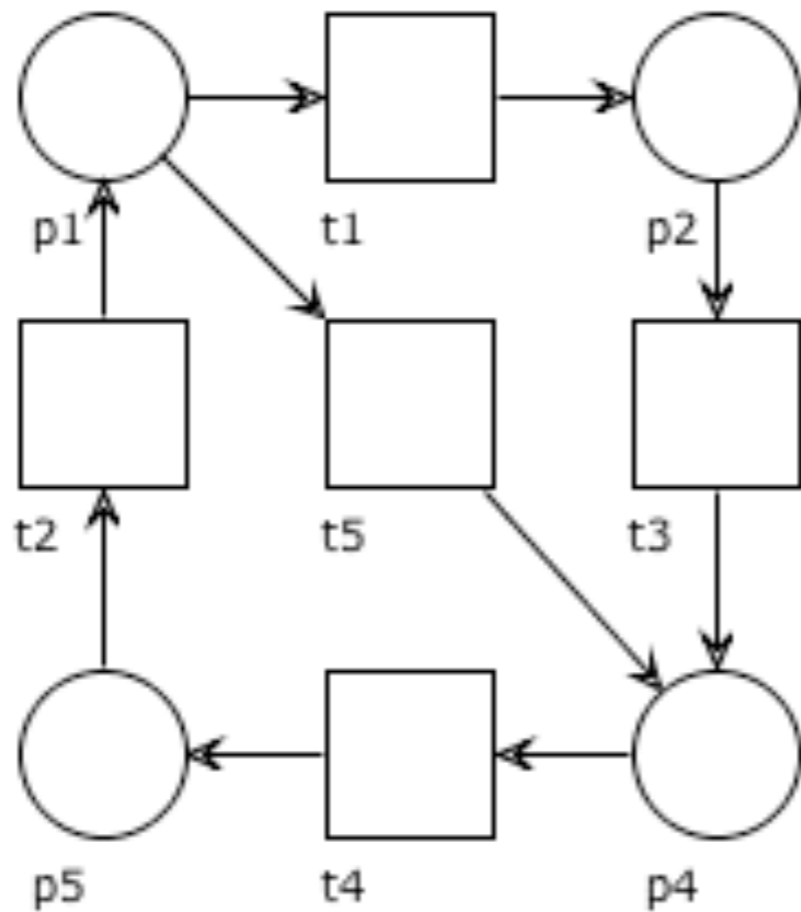
S-system

Definition: We recall that a net N is an **S-net** if each transition has exactly one input place and exactly one output place

$$\forall t \in T, \quad | \bullet t | = 1 = | t \bullet |$$

A system (N, M_0) is an **S-system** if N is an S-net

S-system: example



S-net N^*

Proposition: A workflow net N is an S-net
iff N^* is an S-net

N and N^* differ only for the reset transition,
that has exactly one incoming arc
and exactly one outgoing arc

Fundamental property of S-systems

Observation: each transition t that fires removes exactly one token from some place p and inserts exactly one token in some place p' (p and p' can also coincide)

Thus, **the overall number of tokens in the net is an invariant** under any firing.

Fundamental property of S-systems

Proposition: Let (P, T, F, M_0) be an S-system.
If M is a reachable marking, then $M(P) = M_0(P)$

We show that for any $M \xrightarrow{\sigma} M'$ we have $M'(P) = M(P)$

base ($\sigma = \epsilon$): trivial ($M' = M$)

induction ($\sigma = \sigma' t$ for some $\sigma' \in T^*$ and $t \in T$):

Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By inductive hypothesis: $M''(P) = M(P)$

By definition of S-system: $|\bullet t| = |t \bullet| = 1$

Thus, $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

A consequence of the fundamental property

Corollary: Any S-system is bounded

Let $M \in [M_0 \rangle$.

By the fundamental property of S-systems: $M(P) = M_0(P)$.

Then, for any $p \in P$ we have $M(p) \leq M(P) = M_0(P)$.

Thus the S-system is k -bounded for any $k \geq M_0(P)$.

$$M(P) = \sum_{p \in P} M(p)$$

S-invariants of S-nets

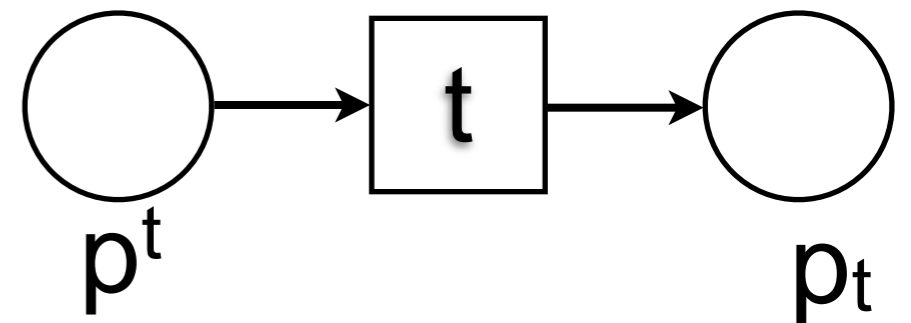
Proposition: Let $N=(P,T,F)$ be a connected S-net.
 I is a rational-valued S-invariant of N iff $I=[x \dots x]$
 for some rational value x

S-invariance $\forall t \in T, \sum_{p \in \bullet t} I(p) = \sum_{p \in t \bullet} I(p)$

S-nets $\forall t \in T, |\bullet t| = |t \bullet| = 1$

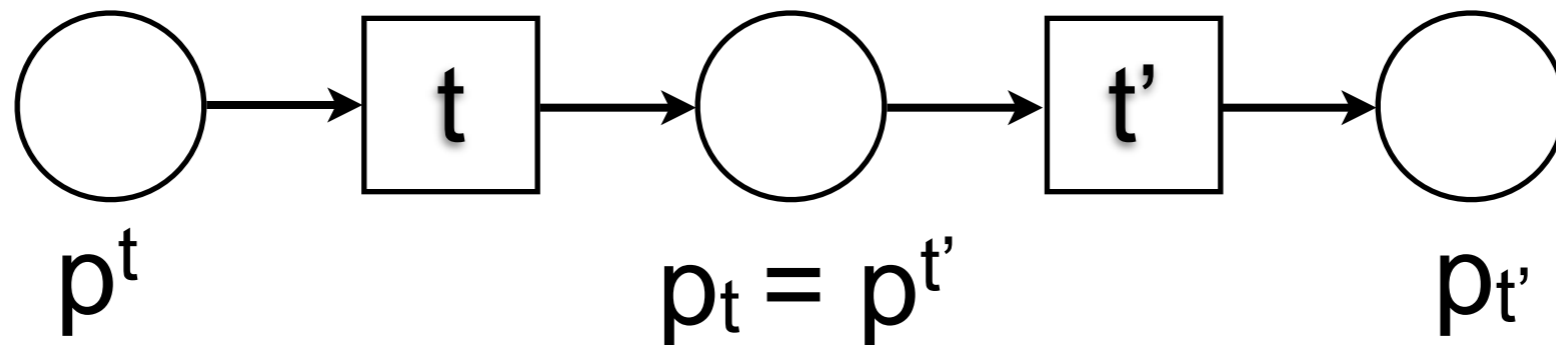
Let $\bullet t = \{p^t\}$ and $t \bullet = \{p_t\}$

$$\forall t \in T, I(p^t) = I(p_t)$$



S-invariants of S-nets

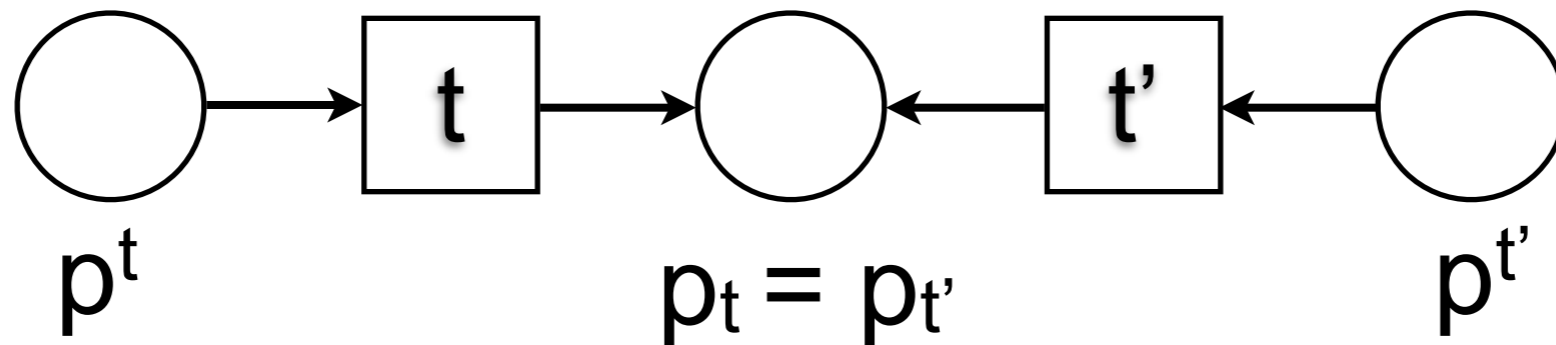
Proposition: Let $N=(P,T,F)$ be a connected S-net.
I is a rational-valued S-invariant of N iff $I=[x \dots x]$
for some rational value x



$$I(p^t) = I(p_t) = I(p^{t'}) = I(p_{t'})$$

S-invariants of S-nets

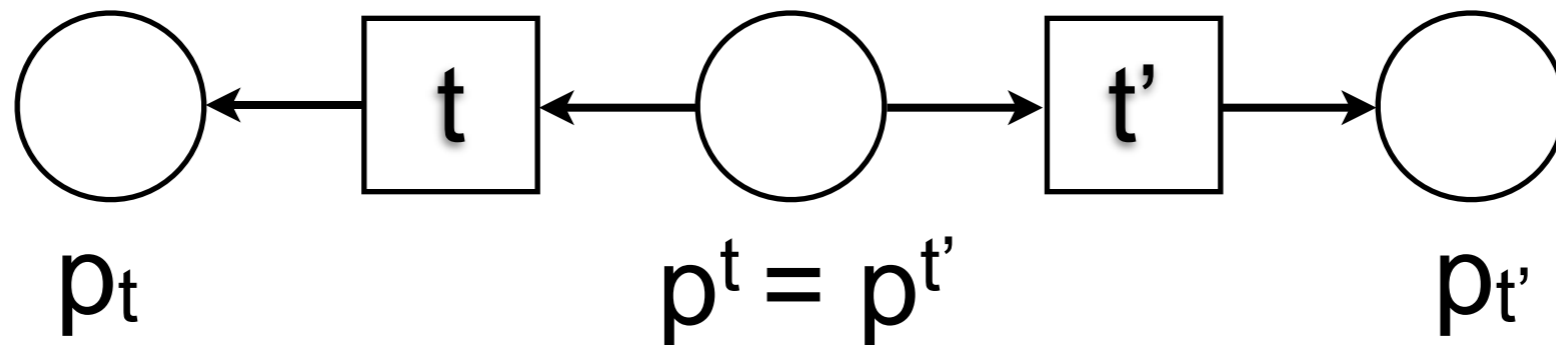
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$$I(p^t) = I(p_t) = I(p_{t'}) = I(p^{t'})$$

S-invariants of S-nets

Proposition: Let $N=(P,T,F)$ be a connected S-net.
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$$I(p_t) = I(p^t) = I(p^{t'}) = I(p_{t'})$$

S-invariants of S-nets

Proposition: Let $N=(P,T,F)$ be a connected S-net.
 I is a rational-valued S-invariant of N iff $I=[x \dots x]$
 for some rational value x

weak
 connectivity $\forall p_0, p_n \in P, \quad p_0 \ t_1 \ p_1 \ t_2 \ p_2 \ t_3 \ p_3 \ \dots \ t_n \ p_n$
 S-net $(\forall t_i, \text{ either } (p_i, t_i)(t_i, p_{i+1}) \text{ or } (t_i, p_i)(p_{i+1}, t_i))$

$$\forall p_0, p_n \in P, \mathbf{I}(p_0) = \mathbf{I}(p_n)$$

A note on S-invariants and S-nets

S-invariance

$$\forall M \in [M_0 \rangle, \quad \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$$

S-invariant
of S-nets

$$\mathbf{I} = [1 \ 1 \ \dots \ 1]$$

consequence

$$\forall M, \quad \mathbf{I} \cdot M = \sum_{p \in P} 1 \cdot M(p) = \sum_{p \in P} M(p) = M(P)$$

We recover the
Fundamental
property of S-nets

$$\forall M \in [M_0 \rangle, \quad M(P) = \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 = M_0(P)$$

Liveness theorem for S-systems

Theorem: An S-system (N, M_0) is live iff N is strongly connected and M_0 marks at least one place

\Rightarrow) (quite obvious)

(N, M_0) is live by hypothesis and bounded (because S-system).
By the strong connectedness theorem, N is strongly connected.

Since (N, M_0) is live, then $M_0 \xrightarrow{t}$ for some t .

Assume $\bullet t = \{p\}$. Thus, $M_0(p) \geq 1$.

Liveness theorem for S-systems

Theorem: An S-system (N, M_0) is live iff N is strongly connected and M_0 marks at least one place

\Leftarrow) (more interesting)

Take any $M \in [M_0 \rangle$ and $t \in T$.

We want to find $M' \in [M \rangle$ such that $M' \xrightarrow{t}$.

Take $p_1 \in P$ such that $M(p_1) \geq 1$ (it exists, because $M(P) = M_0(P) \geq 1$).

By strong connectedness: there is a path from p_1 to $t_n = t$

$(p_1, t_1)(t_1, p_2)(p_2, t_2) \dots (p_n, t_n)$

By definition of S-system: $\bullet t_i = \{p_i\}$ and $t_i \bullet = \{p_{i+1}\}$.

Thus, $M \xrightarrow{\sigma} M' \xrightarrow{t}$ for $\sigma = t_1 t_2 \dots t_{n-1}$.

Reachability lemma for S-nets

Lemma: Let (P, T, F) be a strongly connected S-net.
 If $M(P) = M'(P)$, then M' is reachable from M

We proceed by induction on $M(P)$

base ($M(P) = M'(P) = 0$): trivial ($M' = M$)

induction ($M(P) = M'(P) > 0$):

Let $p, p' \in P$ be such that $M(p) > 0$ and $M'(p') > 0$.

Let $K = M - p$ and $K' = M' - p'$.

Clearly $K'(P) = K(P) < M(P) = M'(P)$.

By inductive hypothesis: $\exists \sigma, K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from $p_0 = p$ to $p_n = p'$

$(p_0, t_1)(t_1, p_1)(p_1, t_2) \dots (t_n, p_n)$

By definition of S-system: $\bullet t_i = \{p_{i-1}\}$ and $t_i \bullet = \{p_i\}$.

Thus, $p = p_0 \xrightarrow{\sigma'} p_n = p'$ for $\sigma' = t_1 t_2 \dots t_n$.

By the monotonicity lemma: $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

Reachability Theorem for S-systems

Theorem: Let (P, T, F, M_0) be a live S-system.
A marking M is reachable **iff** $M(P) = M_0(P)$

\Rightarrow) Follows from the fundamental property of S-systems

\Leftarrow) By the previous liveness theorem, the S-net is strongly connected.

We conclude by applying the reachability lemma for S-systems.

S-systems: recap

S-system \Rightarrow bounded

S-system: str. conn. + $M_0(P) > 0 \Leftrightarrow$ live

S-system + M reachable $\Rightarrow M(P) = M_0(P)$

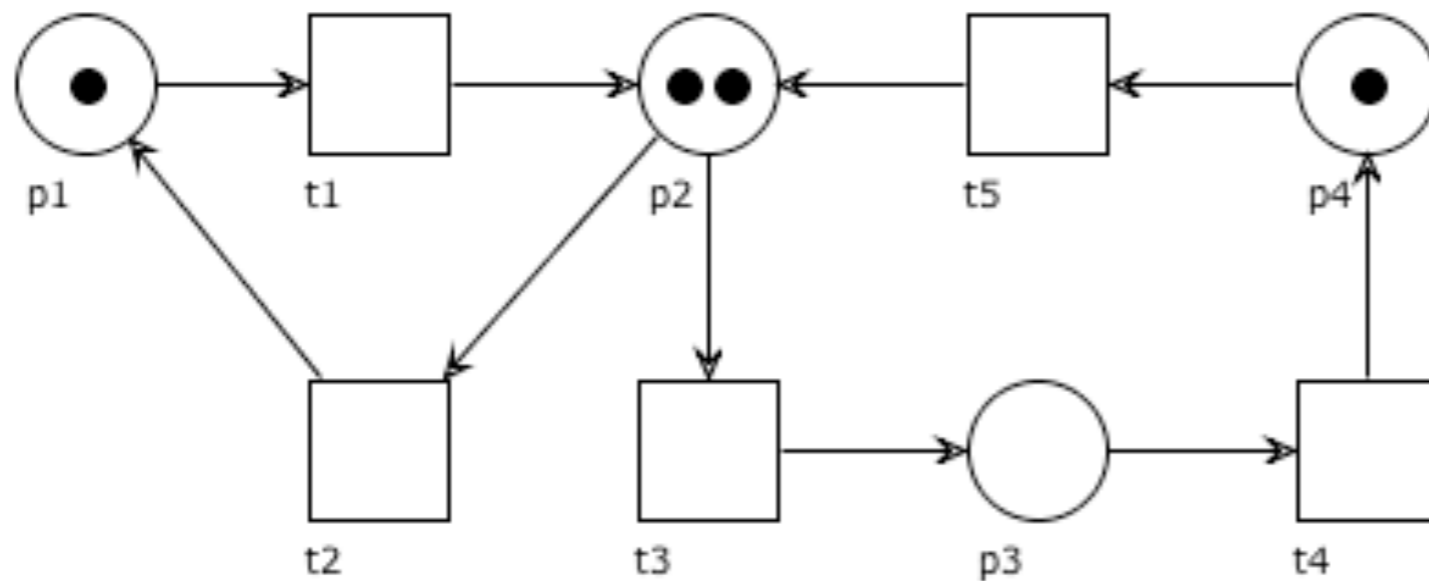
S-system + str. conn.: $M(P) = M_0(P) \Leftrightarrow$ M reachable

S-system + live: $M(P) = M_0(P) \Leftrightarrow$ M reachable

S-invariant $I \Rightarrow I = [x \ x \ \dots \ x]$

Question time

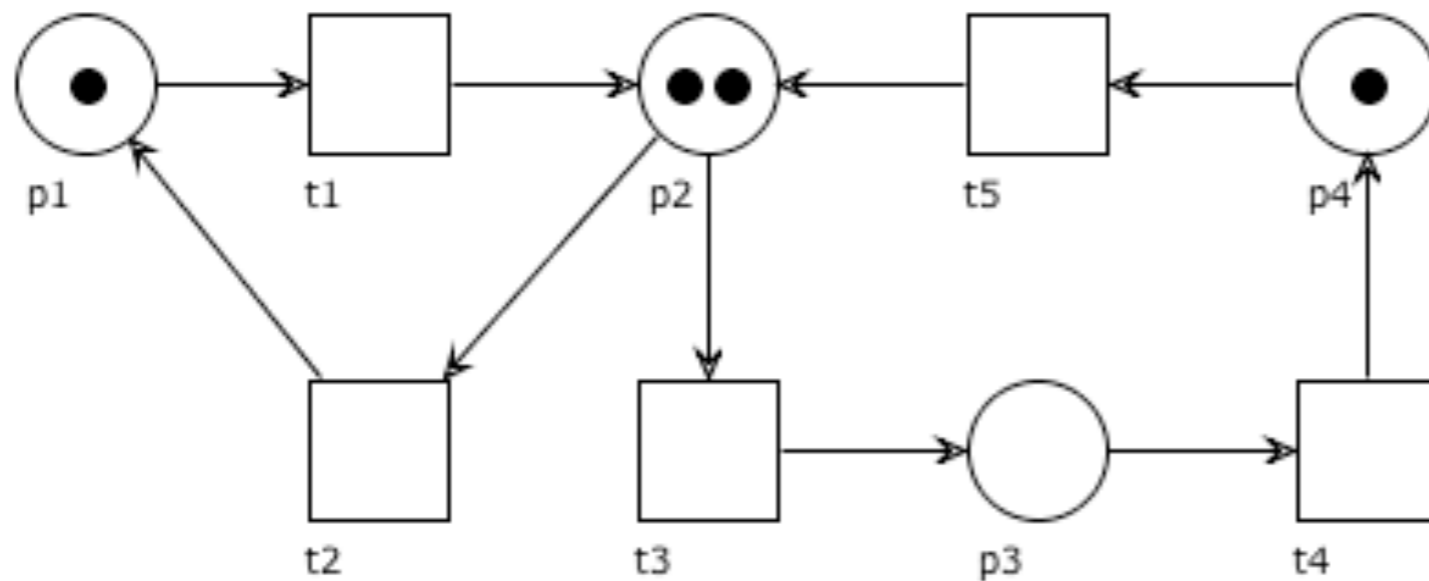
Which of the following markings are reachable? (why?)



[1	1	1	1]
[2	0	2	0]
[1	2	1	2]
[4	0	0	0]
[0	4	0	4]
[0	3	2	1]
[0	0	4	0]
[0	3	0	0]
[0	3	0	1]

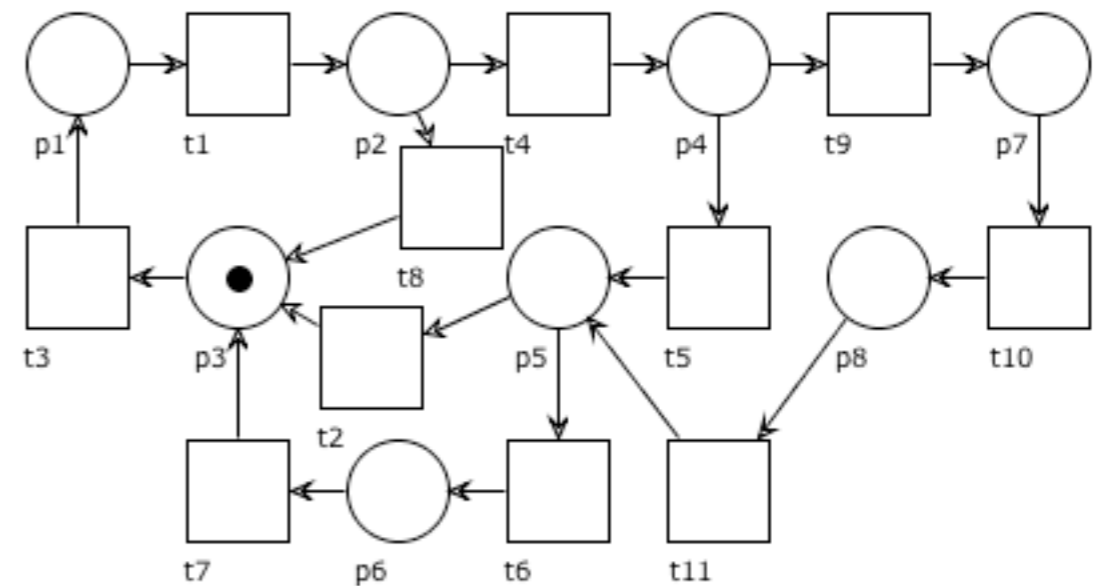
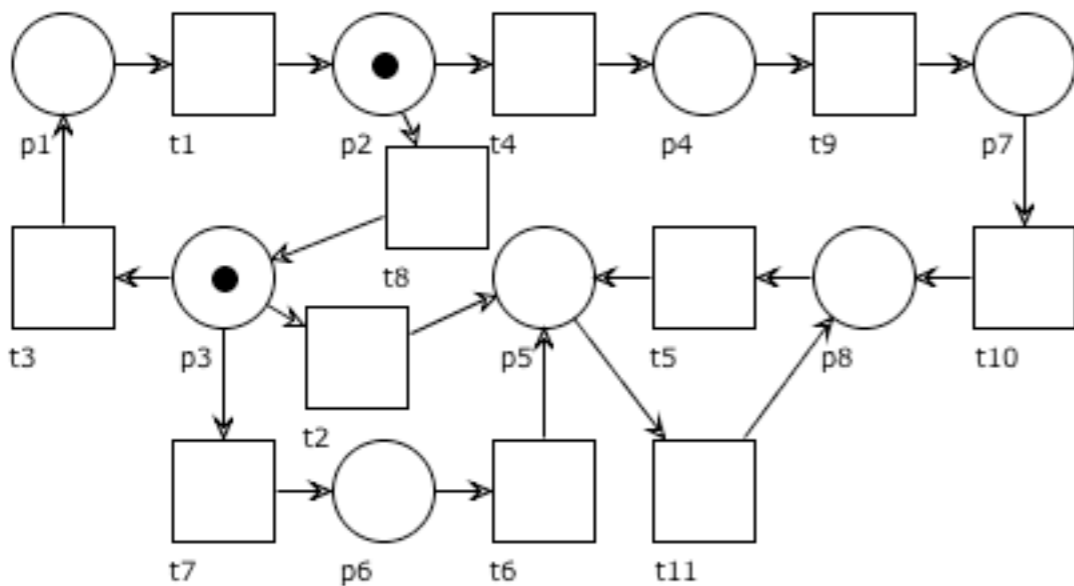
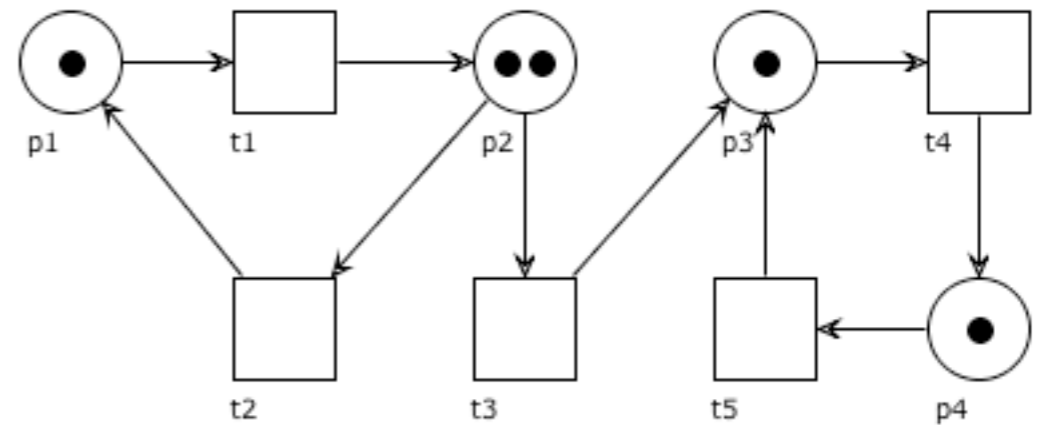
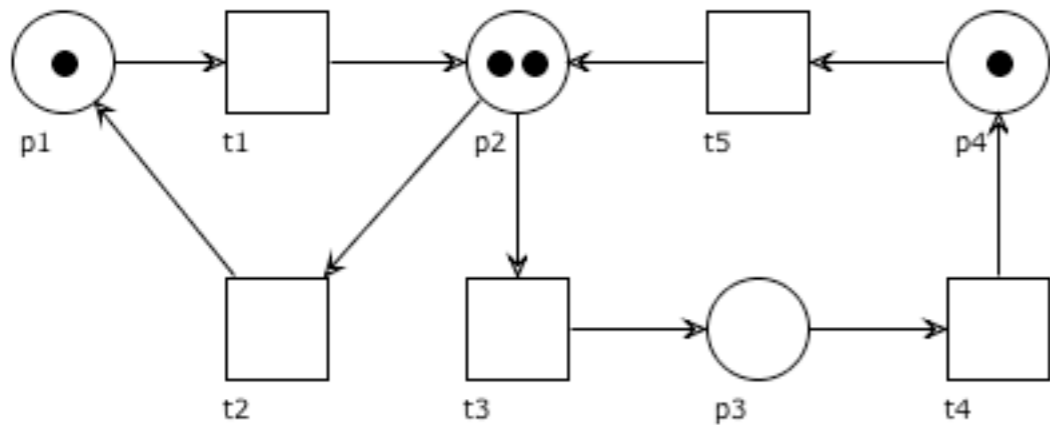
Question time

Which of the following are S-invariants? (why?)


$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

Exercises

Which of the following S-systems are live? (why?)



Boundedness Theorem for S-systems

Theorem:

A live S-system (P, T, F, M_0) is k -bounded iff $M_0(P) \leq k$

Exercise

Prove the boundedness theorem for live S-systems

T-systems

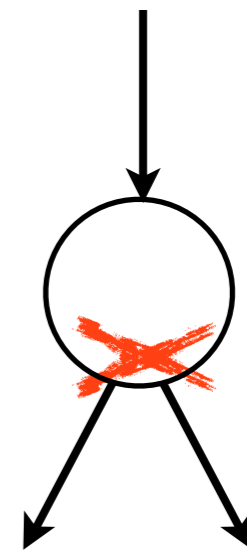
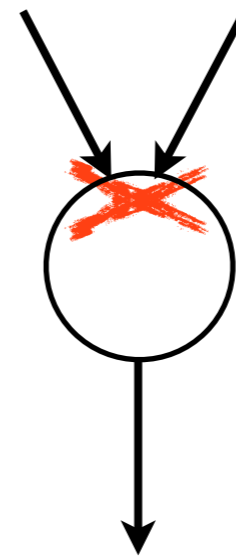
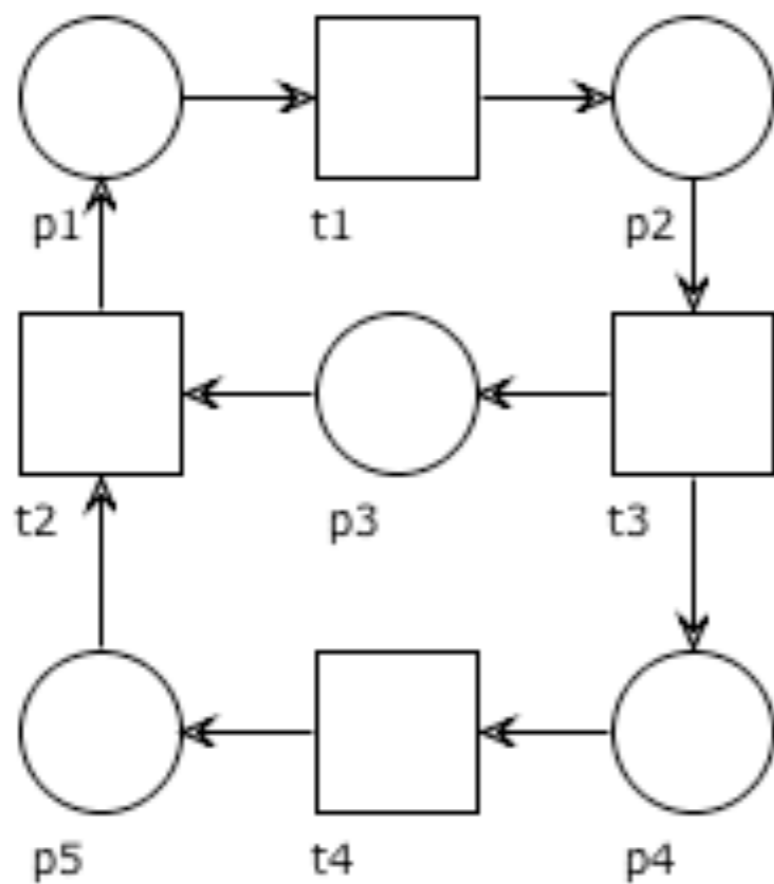
T-system

Definition: We recall that a net N is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \quad |\bullet p| = 1 = |p \bullet|$$

A system (N, M_0) is a **T-system** if N is a T-net

T-system: example



T-systems: an observation

Notably, computation in T-systems is concurrent,
but essentially deterministic:

the firing of a transition t in M cannot disable
another transition t' enabled at M

T-net N^*

Is it true that: A workflow net N is a T-net
iff N^* is a T-net ?

T-systems: another observation

Determination of control:

the transitions responsible for enabling t are
one for each input place of t

Notation: token count of a circuit

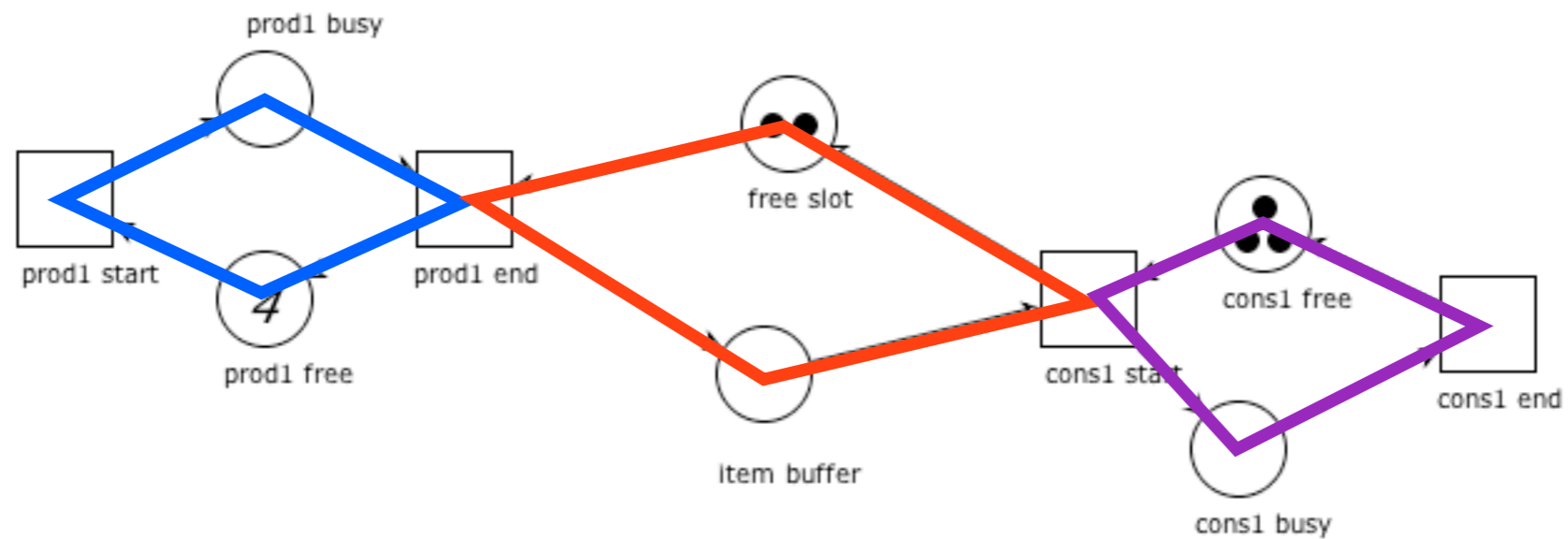
Let $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2) \dots (x_n, y_n)$ be a circuit.

Let $P_{|\gamma} \subseteq P$ be the set of places in γ .

$$M(\gamma) = M(P_{|\gamma}) = \sum_{p \in P_{|\gamma}} M(p)$$

We say that γ is **marked at** M if $M(\gamma) > 0$

Example



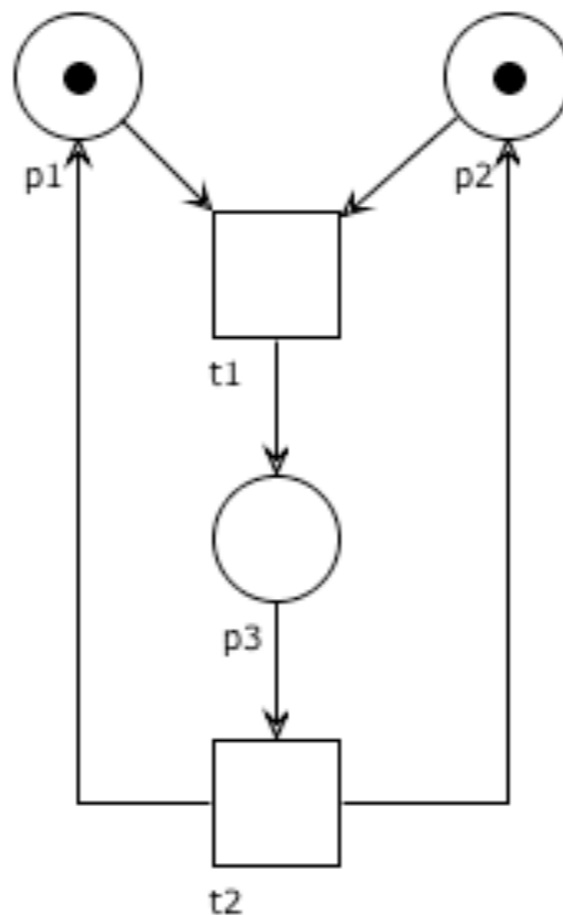
$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

Question time

Trace two circuits over the T-system below



Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

Fundamental property of T-systems

Proposition: Let γ be a circuit of a T-system (P, T, F, M_0) .
If M is a reachable marking, then $M(\gamma) = M_0(\gamma)$

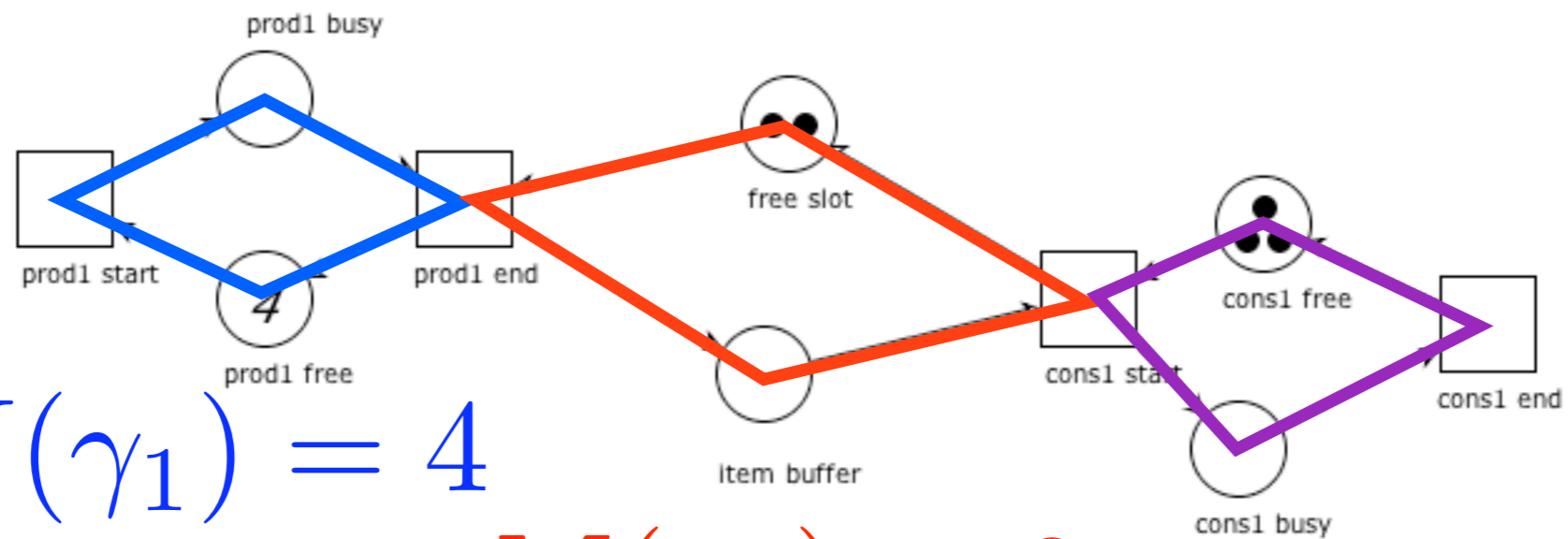
Take any $t \in T$: either $t \notin \gamma$ or $t \in \gamma$.

If $t \notin \gamma$, then no place in $\bullet t \cup t \bullet$ is in γ
(otherwise, by definition of T-nets, t would be in γ).

Then, an occurrence of t does not change the token count of γ .

If $t \in \gamma$, then exactly one place in $\bullet t$ and one place in $t \bullet$ are in γ .
Then, an occurrence of t does not change the token count of γ .

Example



$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

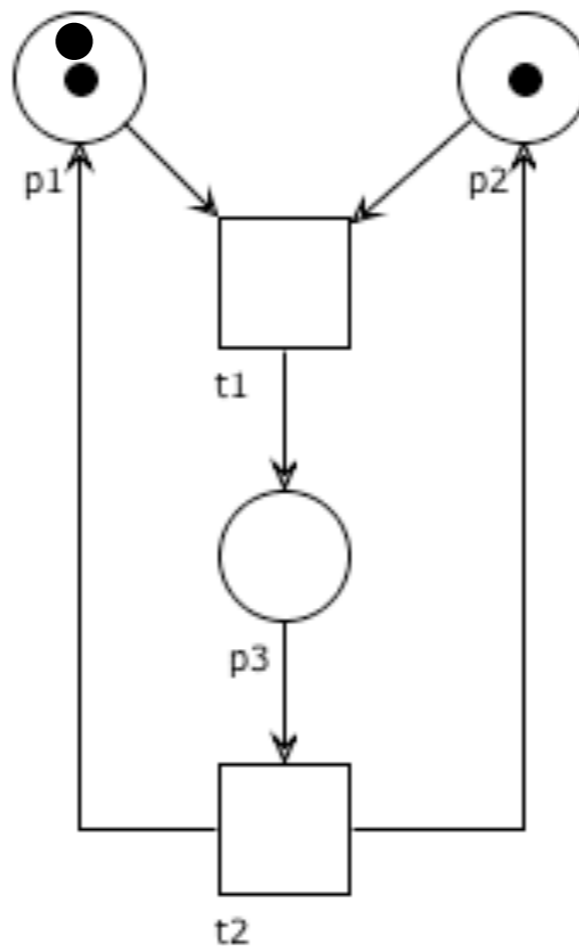
$$M_0 = [0 \quad 4 \quad 2 \quad 0 \quad 3 \quad 0]$$

$$M = [2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 1]$$

$$M' = [2 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2]$$

Question time

Is the marking $p_1 + 2p_2$ reachable? (why?)



T-invariants of T-nets

Proposition: Let $N=(P,T,F)$ be a connected T-net.
 J is a rational-valued T-invariant of N iff $J=[x \dots x]$
for some rational value x

(the proof is dual to the analogous proposition for
S-invariants of S-nets)

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
iff every circuit of N is marked at M_0

\Rightarrow) (quite obvious)

By contradiction, let γ be a circuit with $M_0(\gamma) = 0$.

By the fundamental property of T-systems: $\forall M \in [M_0 \rangle, M(\gamma) = 0$.

Take any $t \in T|_\gamma$ and $p \in P|_\gamma \cap \bullet t$.

For any $M \in [M_0 \rangle$, we have $M(p) = 0$.

Hence t is never enabled and the T-system is not live.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
iff every circuit of N is marked at M_0

\Leftarrow) (more involved)

Take any $t \in T$ and $M \in [M_0 \rangle$.

We need to show that some marking M' reachable from M enables t .

The key idea is to collect the places that control the firing of t :

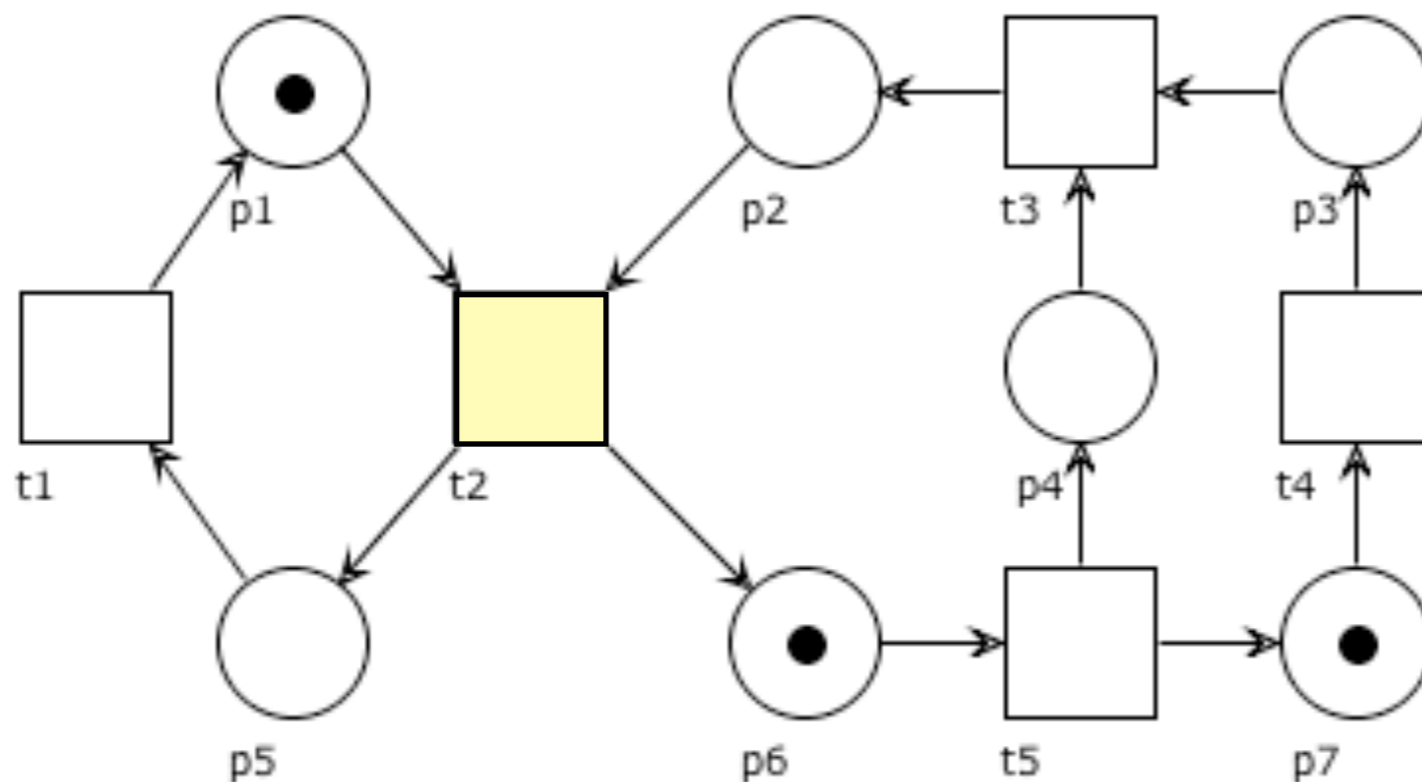
$p \in P_{M,t}$ if there is a path from p to t through places unmarked at M .

We then proceed by induction on the size of $P_{M,t}$.

We just sketch the key idea of the proof over a T-system.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

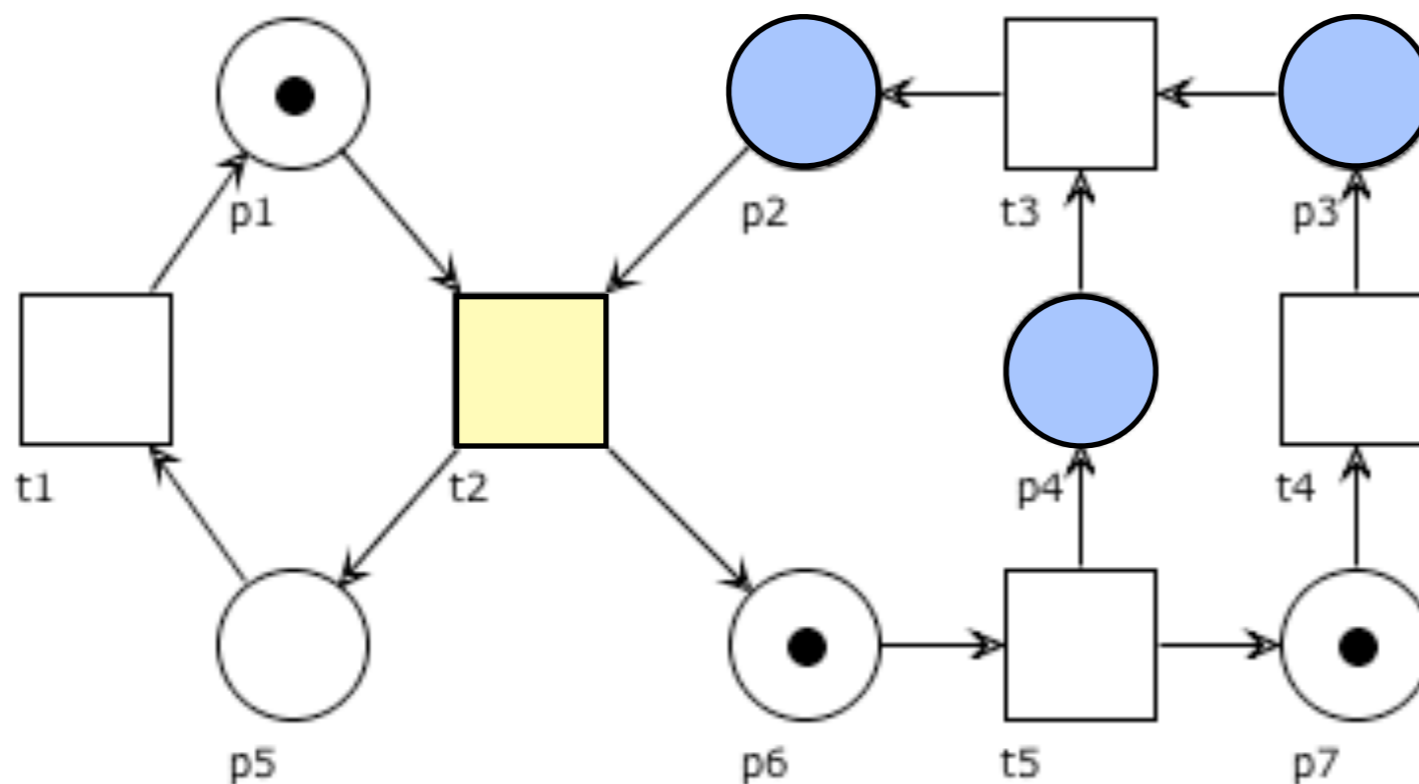


$$M = p_1 + p_6 + p_7$$

M' enabling t_2 ?

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0



$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (continued proof sketch)

Base case: $|P_{M,t}| = 0$.

Every place in $\bullet t$ is already marked at M .

Hence t is enabled at M .

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (continued proof sketch)

Inductive case: $|P_{M,t}| > 0$.

Therefore t is not enabled at M .

We look for a path π of maximal length necessary for firing t .

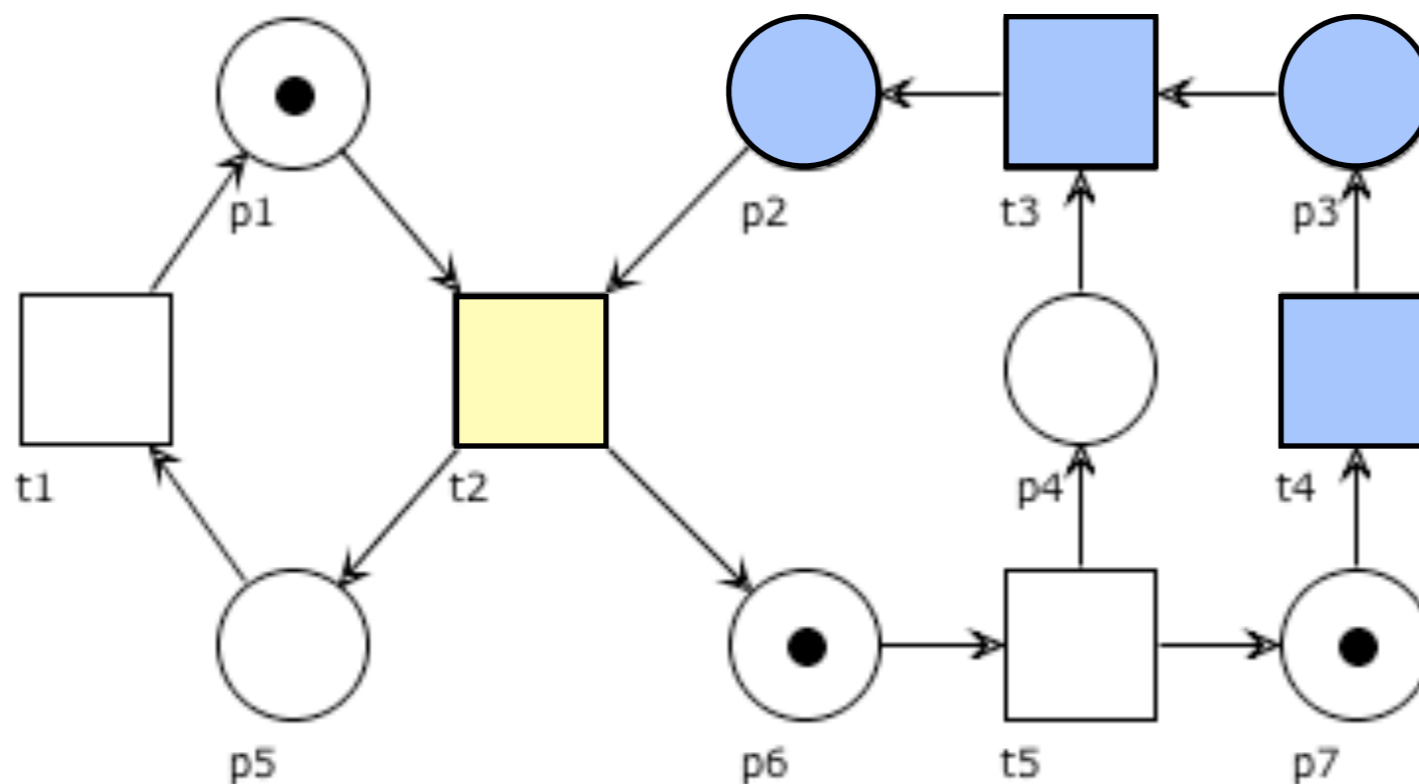
π must contain only places unmarked at M .

By the fundamental property of T-systems: all circuits are marked at M .

π is not necessarily unique, but exists (no cycle in it).

Liveness theorem for T-systems

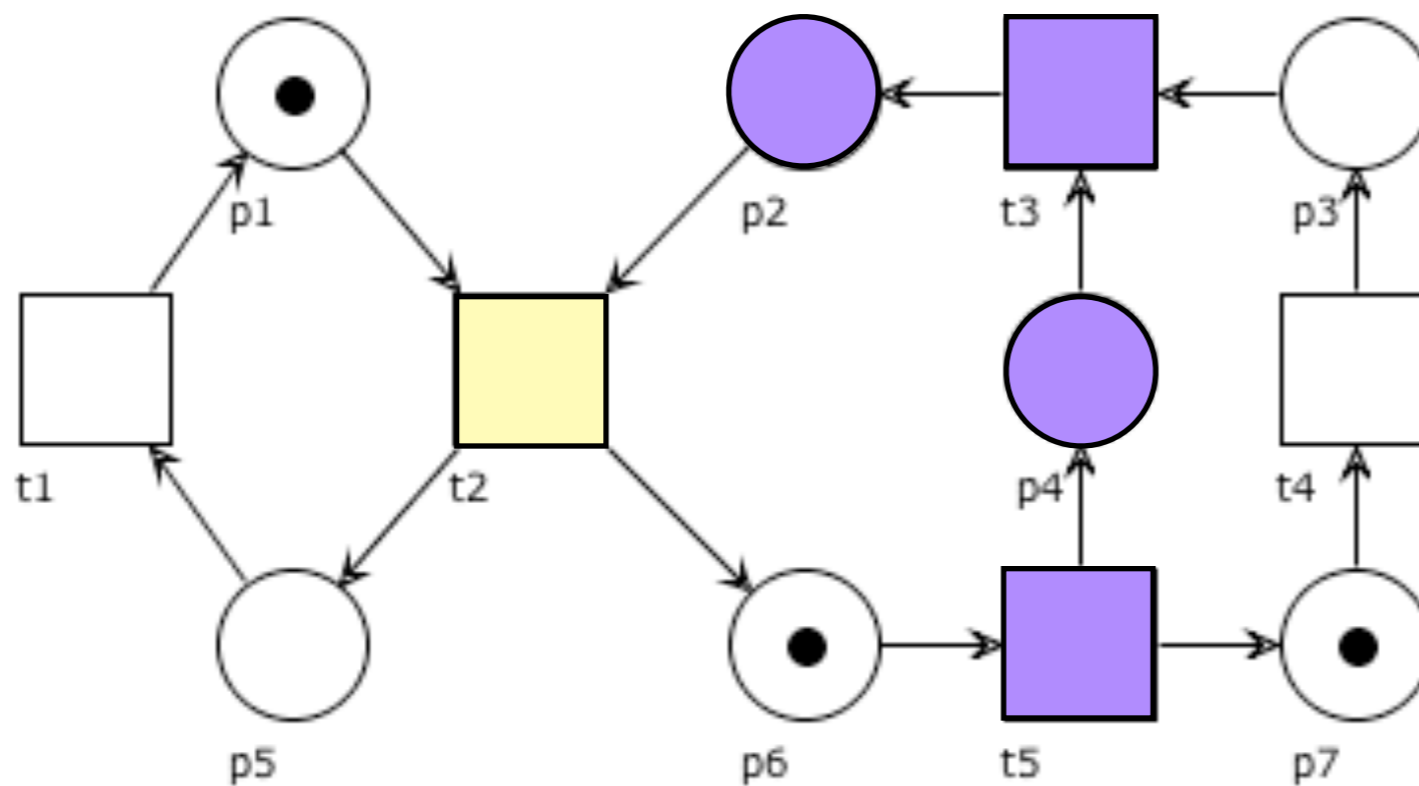
Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0



$$\pi = t_4 p_3 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0



$$\pi = t_5 p_4 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (Inductive case: $|P_{M,t}| > 0$, continued proof sketch)

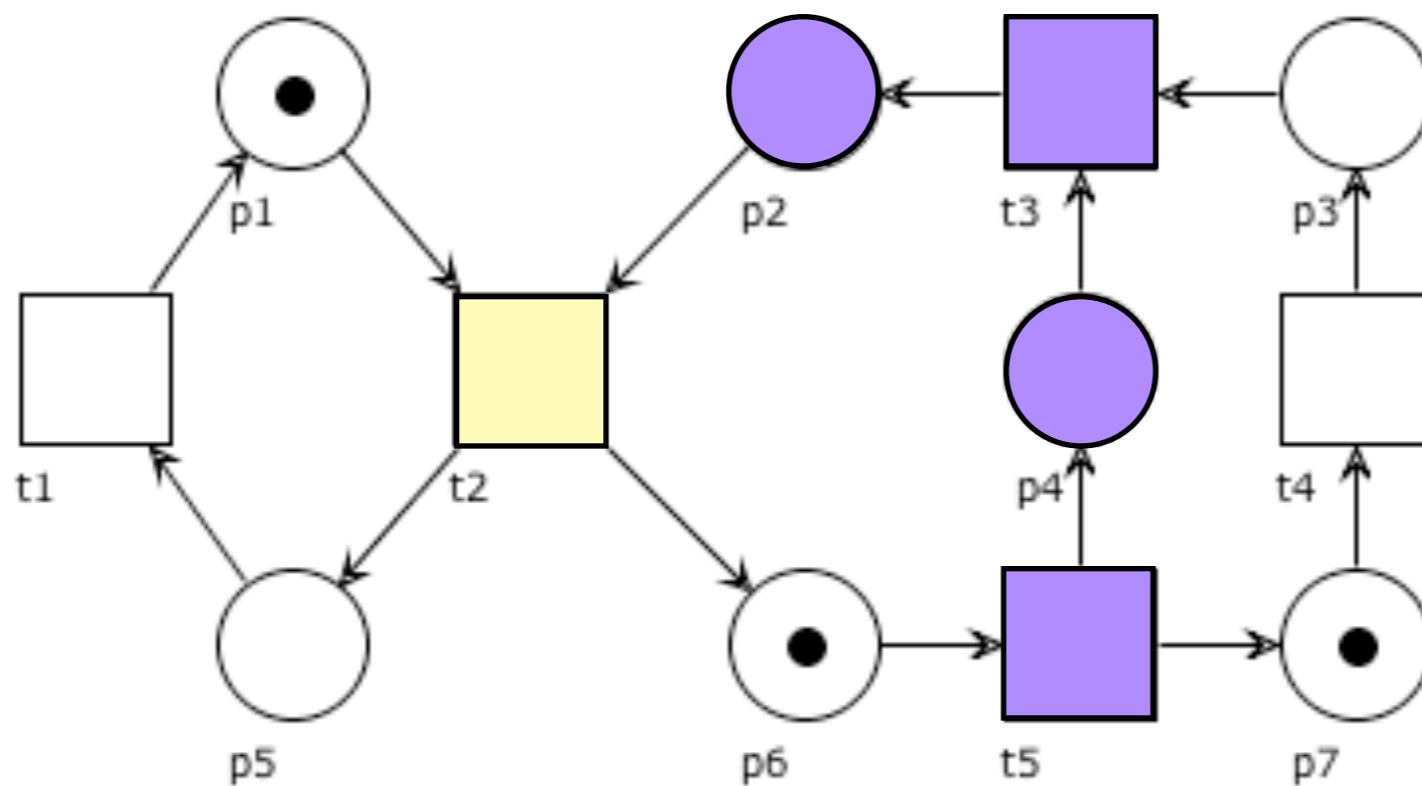
π begins with a transition t' enabled at M .
(otherwise a longer path could be found).

By firing t' we reach a marking M'' such that $P_{M'',t} \subset P_{M,t}$.

Hence $|P_{M'',t}| < |P_{M,t}|$ and we conclude by inductive hypothesis.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

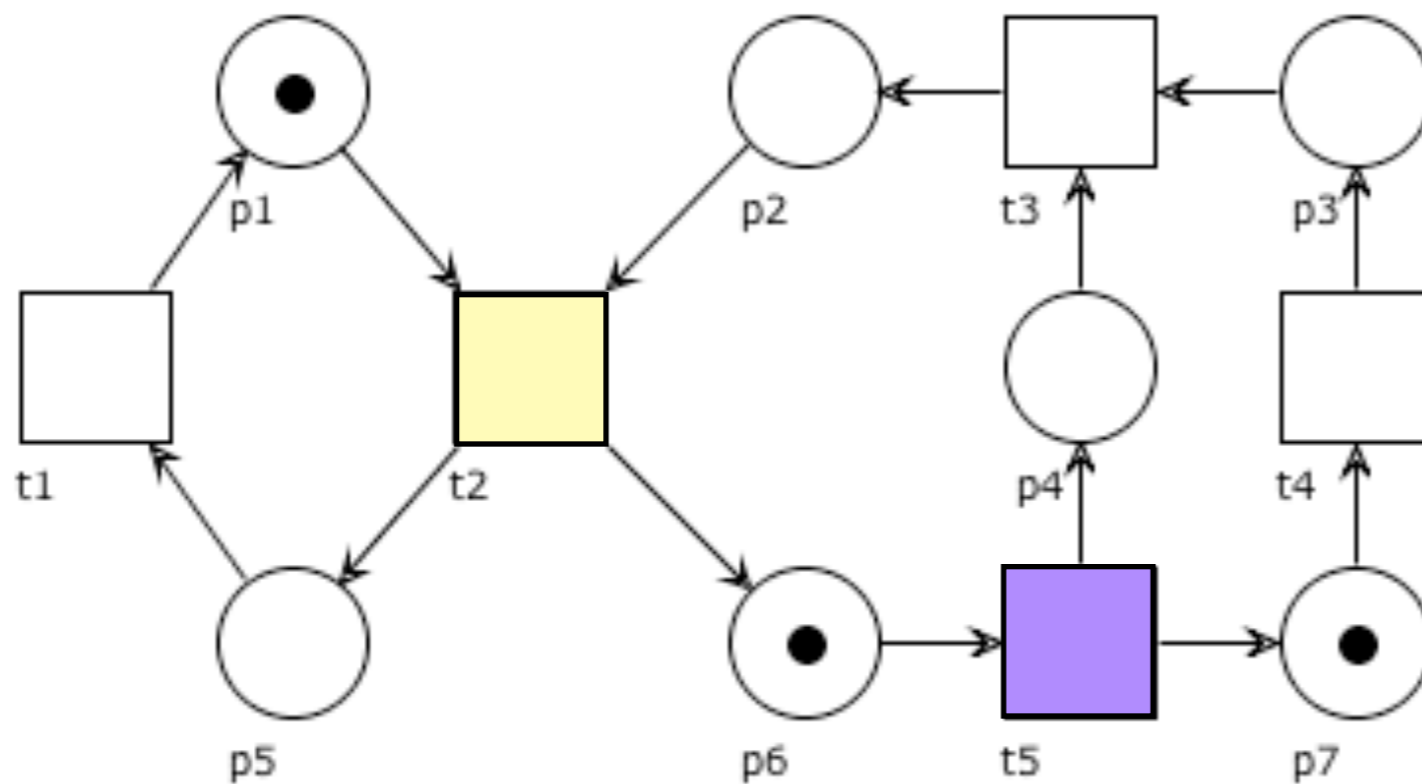


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$\pi = t_5 p_4 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
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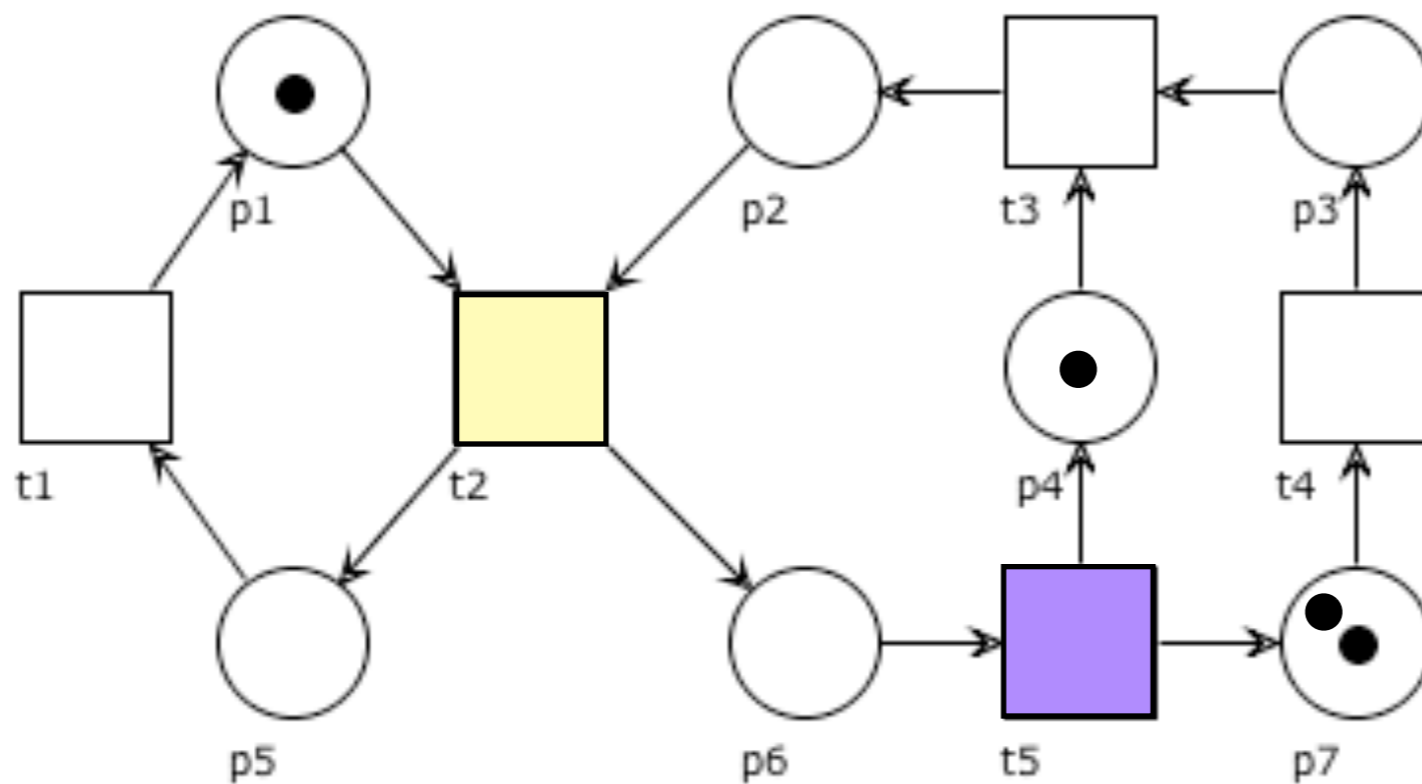


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Liveness theorem for T-systems

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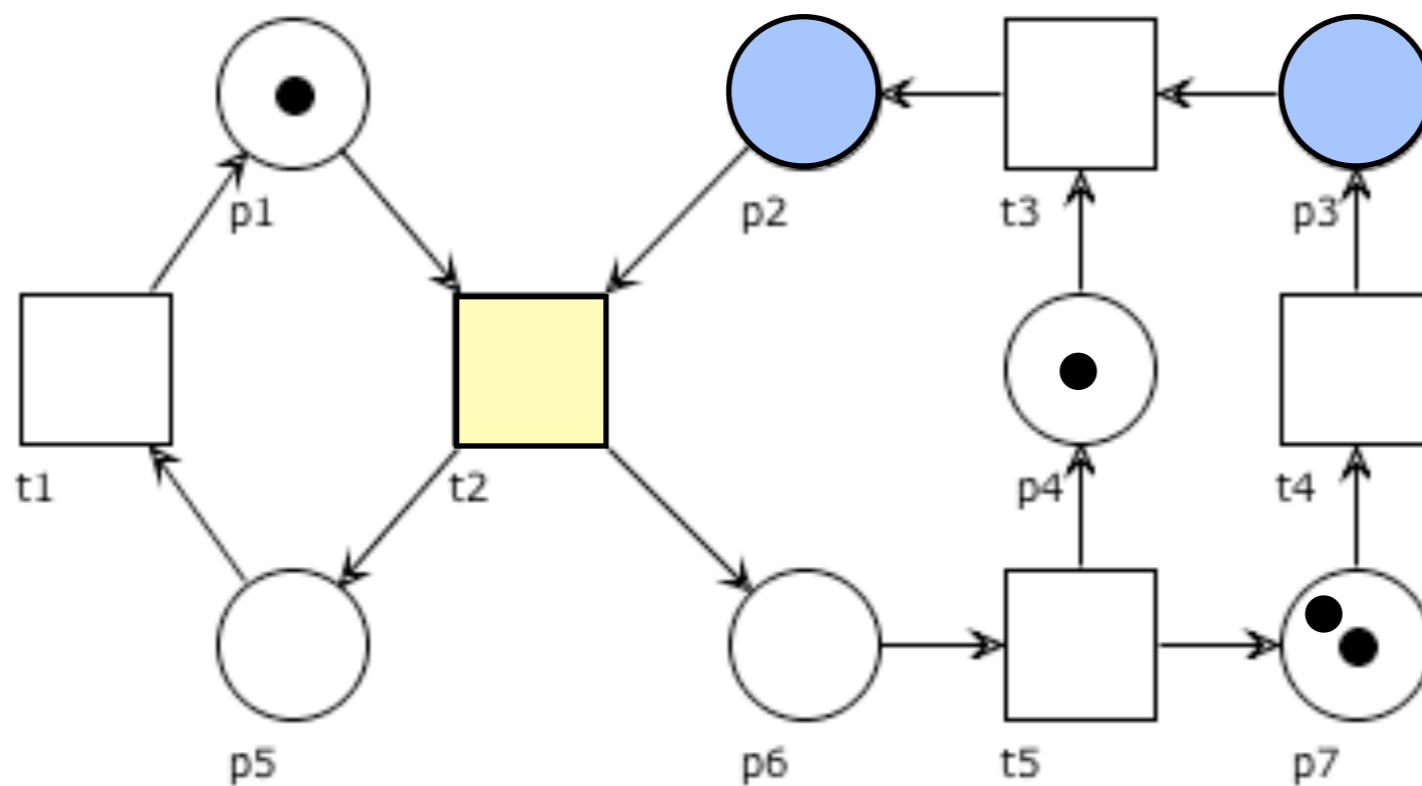


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

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Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live \Leftrightarrow every circuit of N is marked at M_0

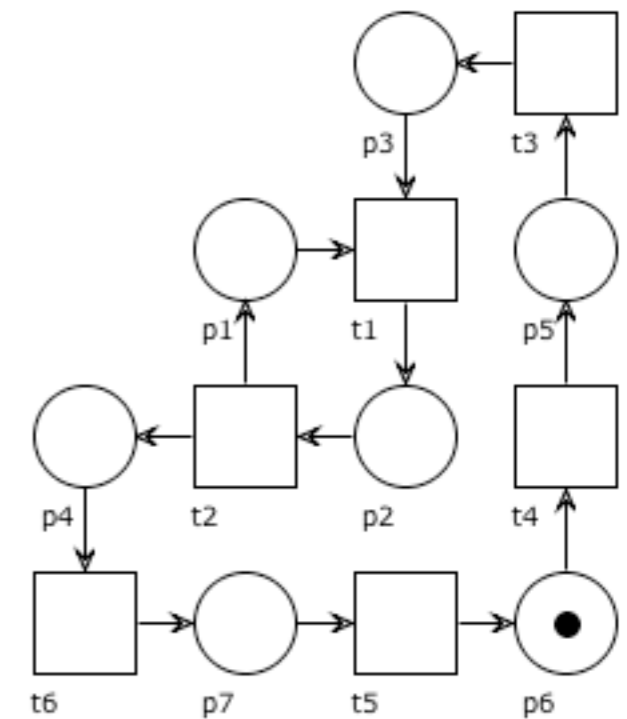
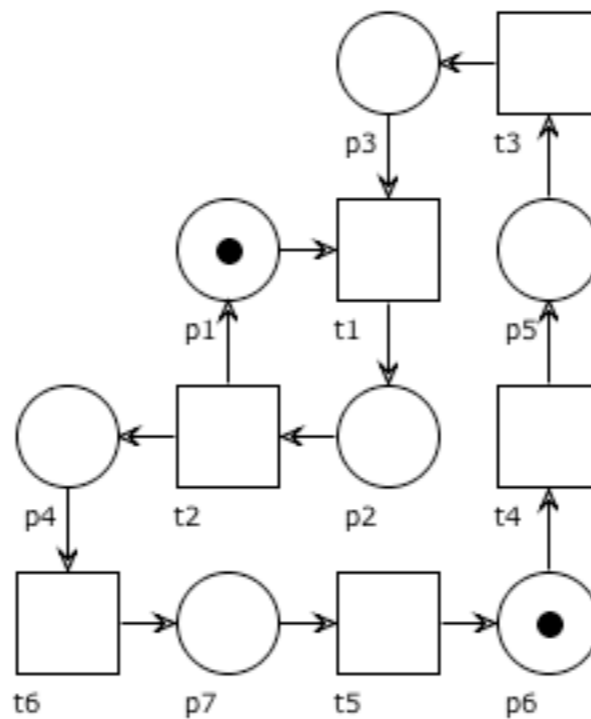
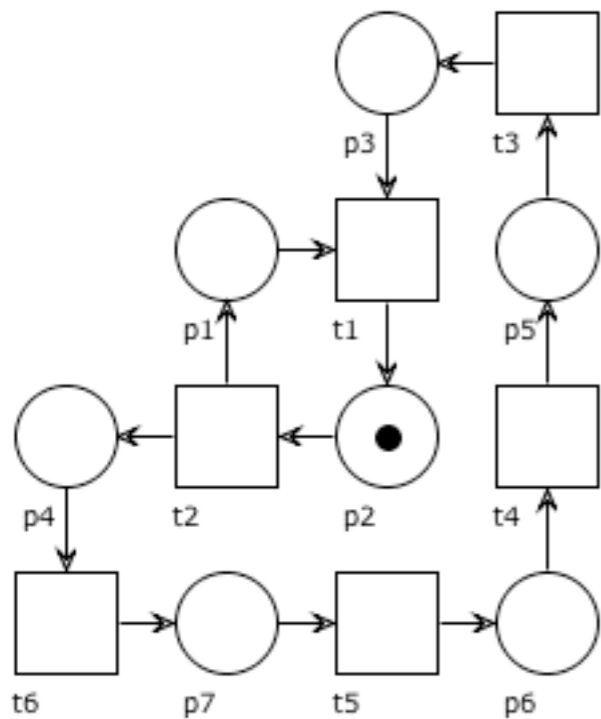


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$P_{M'', t_2} = \{ p_2, p_3 \}$$

Question time

Which of the T-systems below is live? (why?)



Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k -bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

\Leftarrow) Let $M \in [M_0 \rangle$ and take any $p \in P$.

By the fundamental property of T-systems:

$$M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$$

Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k -bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

\Rightarrow) Let $k_p \leq k$ be the bound of p .
Take $M \in [M_0]$ with $M(p) = k_p$.

Define $L = M - k_p p$ and note that the T-system (N, L) is not live.
(otherwise $L \xrightarrow{\sigma} L'$ with $L'(p) > 0$ for enabling $t \in p\bullet$. But then:
 $M = L + k_p p \xrightarrow{\sigma} L' + k_p p = M'$ with $M'(p) = L'(p) + k_p > k_p!$)

By the liveness theorem: some circuit γ is not marked at L .
Since (N, M) is live, the circuit γ is marked at $M \supset L$.
Since $M - L = k_p p$, the circuit γ contains p and
 $M_0(\gamma) = M(\gamma) = M(p) = k_p \leq k$.

Boundedness in strongly connected T-systems

Lemma: If a T-system (N, M_0) is strongly connected, then it is bounded

Let Γ be the set of the circuits of N and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$.

Since N is strongly connected, every place p belongs to some circuit γ_p .

By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M , we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is k -bounded.

Liveness in strongly connected T-systems

Lemma: If a T-system (N, M_0) is strongly connected, then
it is live iff it is deadlock-free iff it has an infinite run
 $\implies \implies$

It is obvious that (for any net):

Liveness implies deadlock freedom.

Deadlock freedom implies the existence of an infinite run.

We show that (for strongly connected T-systems):

The existence of an infinite run implies liveness.

Liveness in strongly connected T-systems

Lemma: Let (N, M_0) be a strongly connected T-system.
If it has an infinite run σ , then it is live

Since the T-system is strongly connected then it is bounded.

By the Reproduction lemma (holding for any bounded net):

There is a semi-positive T-invariant \mathbf{J} .

The support of \mathbf{J} is included in the set of transitions of the infinite run σ .

By T-invariance in T-systems: $\langle \mathbf{J} \rangle = T$

(σ is an infinite run that contains all transitions).

Hence every transition can occur from M_0 .

Hence every place can become marked.

Hence every circuit can become marked.

By the fundamental property of T-systems: every circuit is marked at M_0 .

By the liveness theorem, (N, M_0) is live.

Place bounds in live T-systems

Let (P, T, F, M_0) be a live T-system.

We can draw some easy consequences of the above results:

1) If $p \in P$ is bounded, then it belongs to some circuit.
(see part \Rightarrow of the proof of the boundedness theorem)

2) If $p \in P$ belongs to some circuit, then it is bounded.
(by the fundamental property of T-systems)

3) If (N, M_0) is bounded, then it is strongly connected.
(by strong connectedness theorem, holding for any system)

4) If N is strongly connected, then (N, M_0) is bounded.
(by 1, since any $p \in P$ belongs to a circuit by strong connectedness)

Place bounds in live T-systems

Let (P, T, F, M_0) be a live T-system.

We can draw some easy consequences of the above results:

1+2) $p \in P$ is bounded iff it belongs to some circuit.

3+4) (N, M_0) is bounded iff it is strongly connected.

T-systems: recap

T-system + M reachable + γ circuit $\Rightarrow M(\gamma) = M_0(\gamma)$

T-system + $\gamma_1 \dots \gamma_n$ circuits: $\exists i. p \in \gamma_i \Leftrightarrow p$ bounded

T-system: $M_0(\gamma) > 0$ for all circuits $\gamma \Leftrightarrow$ live

T-system: strongly connected \Rightarrow bounded

T-system + live: strongly connected \Leftrightarrow bounded

T-system + str. conn.: deadlock-free \Leftrightarrow live

T-system + str. conn.: infinite run \Leftrightarrow live

T-invariant **J** $\Rightarrow \mathbf{J} = [x \ x \ \dots \ x]$

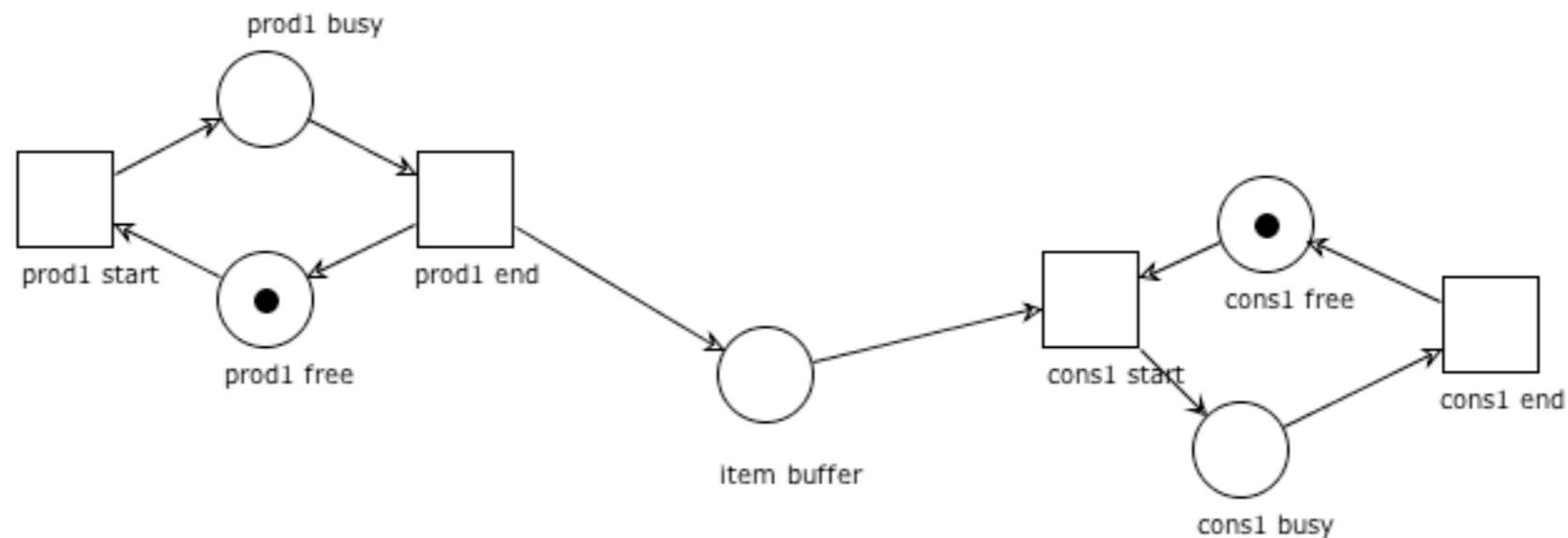
Exercises

Which are the circuits of the T-system below?

Is the T-system below live? (why?)

Which places are bounded? (why?)

Assign a bound to each bounded place.



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Which are the circuits of the T-systems below?

Are the T-systems below live? (why?)

Which places are bounded? (why?)

Assign a bound to each bounded place.

