

# Business Processes Modelling

## MPB (6 cfu, 295AA)

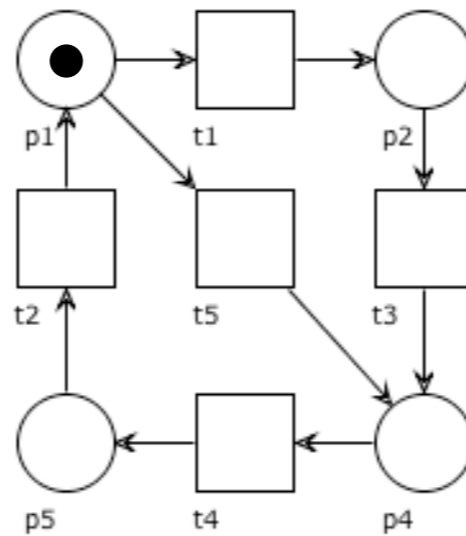
Roberto Bruni

<http://www.di.unipi.it/~bruni>

16 - S-systems



# Object



We study some “good” properties of S-systems

Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

S-systems

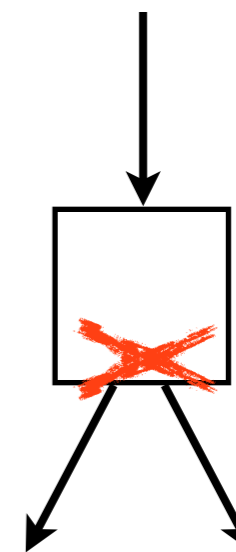
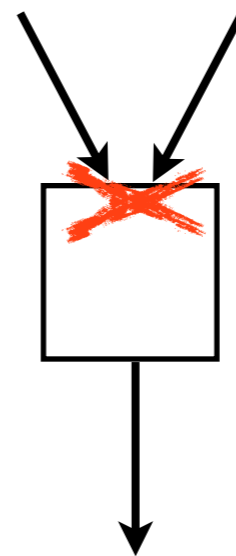
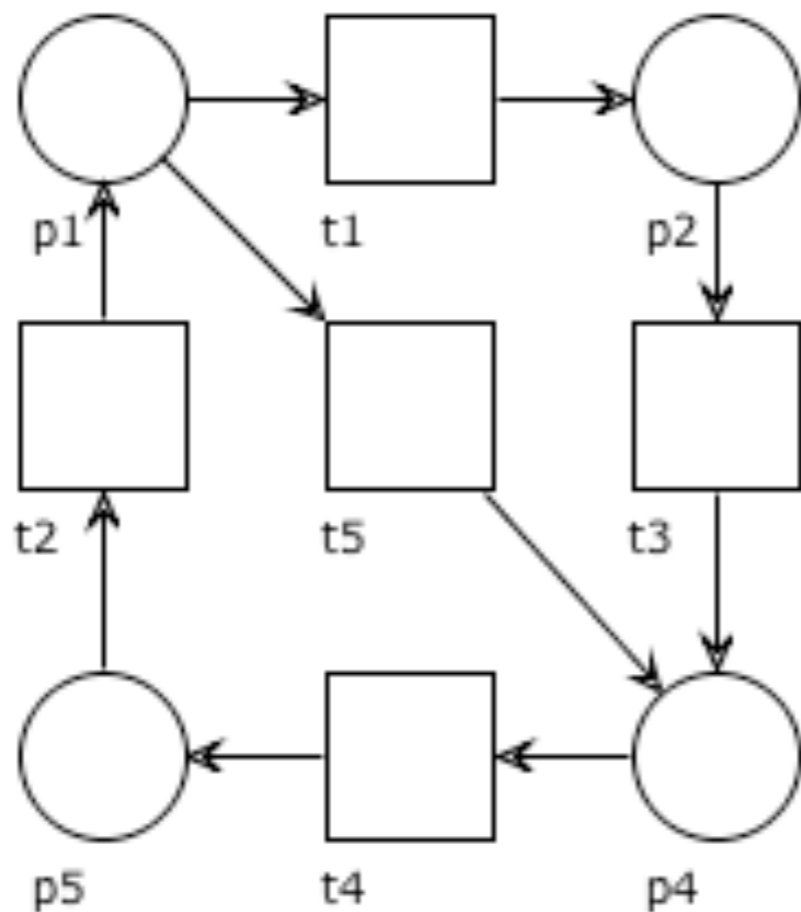
# S-system

**Definition:** We recall that a net  $N$  is an **S-net** if each transition has exactly one input place and exactly one output place

$$\forall t \in T, \quad | \bullet t | = 1 = | t \bullet |$$

A system  $(N, M_0)$  is an **S-system** if  $N$  is an S-net

# S-net: example



# S-net $N^*$

**Proposition:** A workflow net  $N$  is an S-net  
iff  $N^*$  is an S-net

$N$  and  $N^*$  differ only for the reset transition,  
that has exactly one incoming arc  
and exactly one outgoing arc

# S-systems: an observation

**Observation:** each transition  $t$  that fires removes exactly one token from some place  $p$  and inserts exactly one token in some place  $q$  ( $p$  and  $q$  can also coincide)

# Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

Example

$$P = \{p_1, p_2, p_3\} \quad M = 2p_1 + 3p_2 \quad M(P) = 2 + 3 + 0 = 5$$



# Fundamental property of S-systems

**The overall number of tokens in the net is an invariant under any firing.**

# Fundamental property of S-systems

**Proposition:** Let  $(P, T, F, M_0)$  be an S-system.  
If  $M$  is a reachable marking, then  $M(P) = M_0(P)$

We show that for any  $M \xrightarrow{\sigma} M'$  we have  $M'(P) = M(P)$

base ( $\sigma = \epsilon$ ): trivial ( $M' = M$ )

induction ( $\sigma = \sigma' t$  for some  $\sigma' \in T^*$  and  $t \in T$ ):

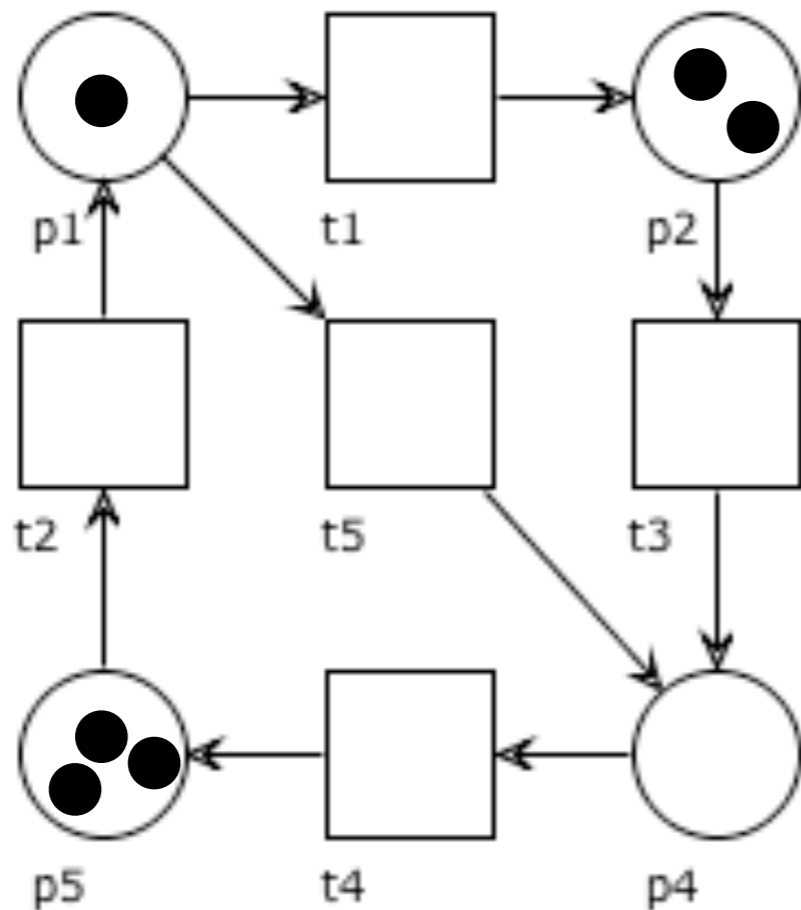
Let  $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$ .

By inductive hypothesis:  $M''(P) = M(P)$

By definition of S-system:  $|\bullet t| = |t \bullet| = 1$

Thus,  $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

# Question time



Is the marking  
 $M = p_2 + 4p_4 + 2p_5$   
reachable?

# A consequence of the fundamental property

**Corollary:** Any S-system is bounded

Let  $M \in [M_0 \rangle$ .

By the fundamental property of S-systems:  $M(P) = M_0(P)$ .

Then, for any  $p \in P$  we have  $M(p) \leq M(P) = M_0(P)$ .

Thus the S-system is  $k$ -bounded for any  $k \geq M_0(P)$ .

$$M(P) = \sum_{p \in P} M(p)$$

# S-invariants of S-nets

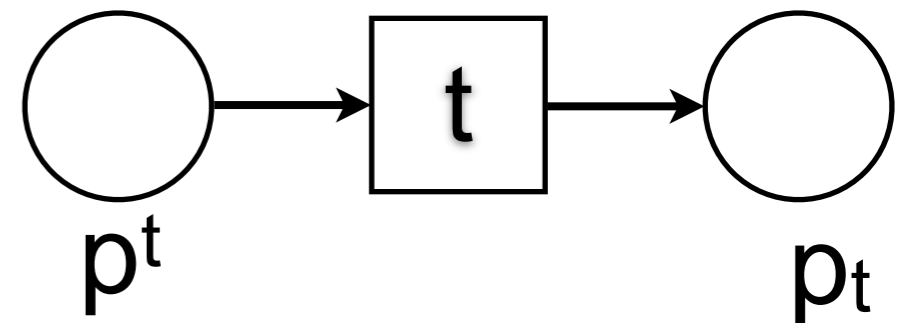
**Proposition:** Let  $N=(P,T,F)$  be a (connected) S-net.  
 $I$  is an S-invariant of  $N$  iff  $I=[k \dots k]$  for some value  $k$

S-invariance  $\forall t \in T, \sum_{p \in \bullet t} I(p) = \sum_{p \in t \bullet} I(p)$

S-nets  $\forall t \in T, |\bullet t| = |t \bullet| = 1$

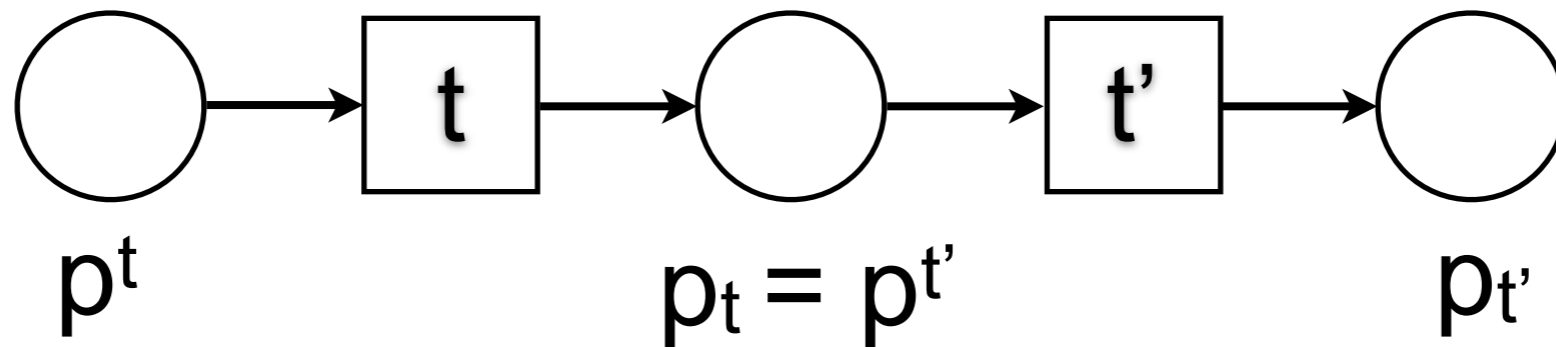
Let  $\bullet t = \{p^t\}$  and  $t \bullet = \{p_t\}$

$$\forall t \in T, I(p^t) = I(p_t)$$



# S-invariants of S-nets

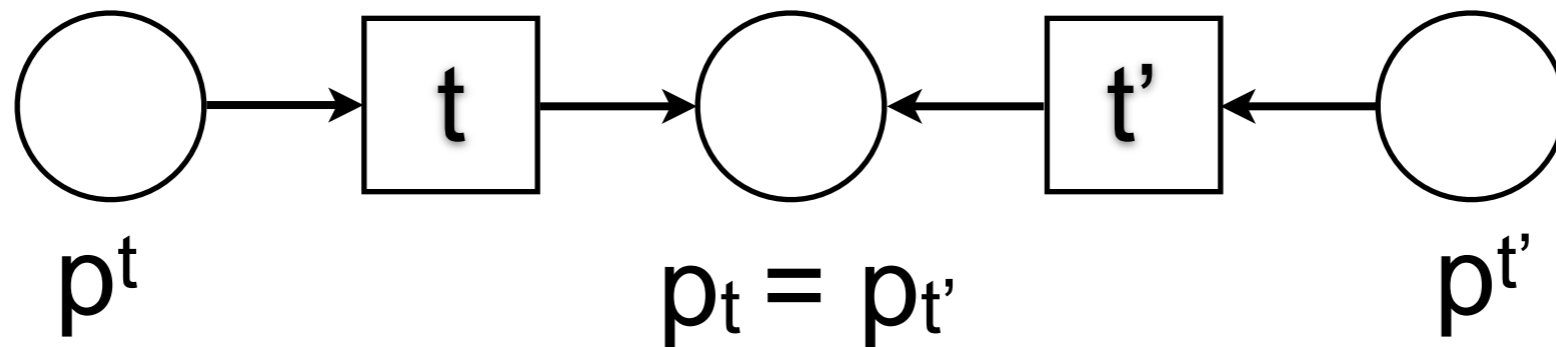
**Proposition:** Let  $N=(P,T,F)$  be a (connected) S-net.  
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$$I(p^t) = I(p_t) = I(p^{t'}) = I(p_{t'})$$

# S-invariants of S-nets

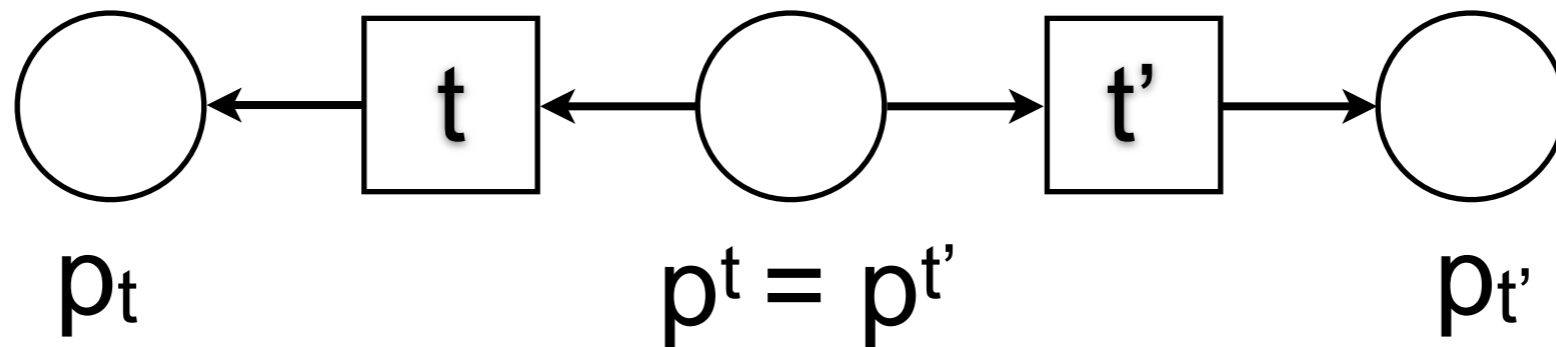
**Proposition:** Let  $N=(P,T,F)$  be a (connected) S-net.  
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# S-invariants of S-nets

**Proposition:** Let  $N=(P,T,F)$  be a (connected) S-net.  
 $I$  is an S-invariant of  $N$  **iff**  $I=[k \dots k]$  for some value  $k$

weak  
connectivity  $\forall p_0, p_n \in P, \quad p_0 \ t_1 \ p_1 \ t_2 \ p_2 \ t_3 \ p_3 \ \dots \ t_n \ p_n$   
S-net  $(\forall t_i, \text{ either } (p_i, t_i)(t_i, p_{i+1}) \text{ or } (t_i, p_i)(p_{i+1}, t_i))$

$$\forall p_0, p_n \in P, \mathbf{I}(p_0) = \mathbf{I}(p_n)$$

# A note on S-invariants and S-nets

S-invariance  $\forall M \in [M_0 \rangle, \quad \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

S-invariant  
of S-nets  $\mathbf{I} = [1 \ 1 \ \dots \ 1]$

consequence  $\forall M, \quad \mathbf{I} \cdot M = \sum_{p \in P} 1 \cdot M(p) = \sum_{p \in P} M(p) = M(P)$

We recover the  
Fundamental  
property of S-nets  $\forall M \in [M_0 \rangle, \quad M(P) = \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 = M_0(P)$

# Liveness theorem for S-systems

**Theorem:** An S-system  $(N, M_0)$  is live iff  
 $N$  is strongly connected and  $M_0$  marks at least one place

$\Rightarrow$ ) (quite obvious)

$(N, M_0)$  is live by hypothesis and bounded (because S-system).  
By the strong connectedness theorem,  $N$  is strongly connected.

Since  $(N, M_0)$  is live, then  $M_0 \xrightarrow{t}$  for some  $t$ .

Assume  $\bullet t = \{p\}$ . Thus,  $M_0(p) \geq 1$ .

# Liveness theorem for S-systems

**Theorem:** An S-system  $(N, M_0)$  is live iff  $N$  is strongly connected and  $M_0$  marks at least one place

$\Leftarrow$ ) (more interesting)

Take any  $M \in [M_0 \rangle$  and  $t \in T$ .

We want to find  $M' \in [M \rangle$  such that  $M' \xrightarrow{t}$ .

Take  $p_1 \in P$  such that  $M(p_1) \geq 1$  (it exists, because  $M(P) = M_0(P) \geq 1$ ).

By strong connectedness: there is a path from  $p_1$  to  $t_n = t$

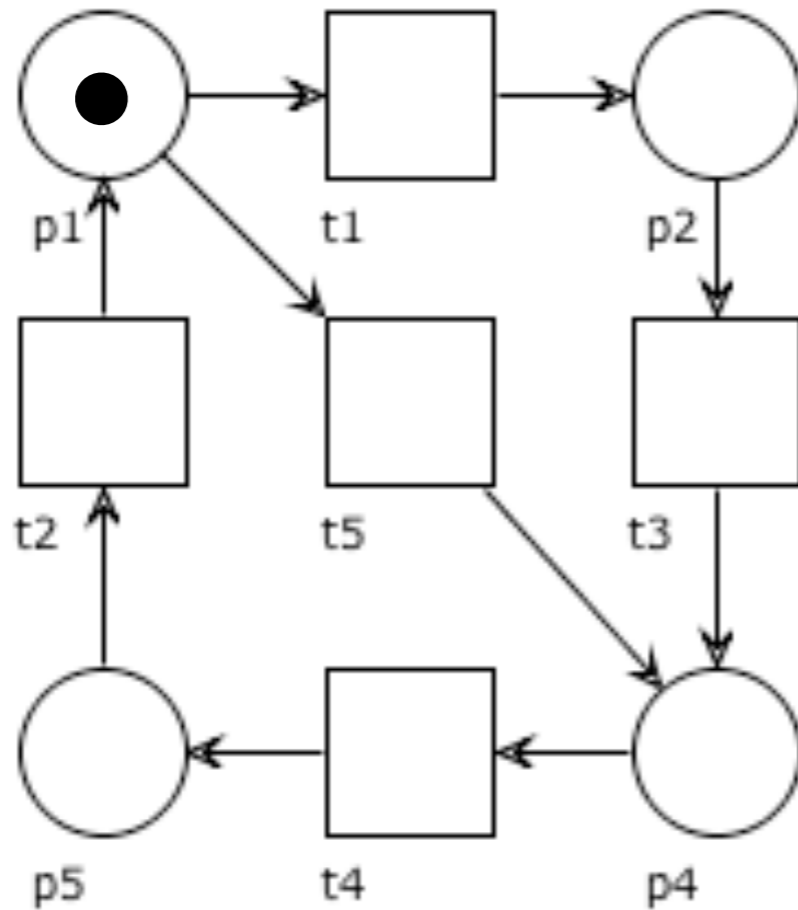
$(p_1, t_1)(t_1, p_2)(p_2, t_2) \dots (p_n, t_n)$

By definition of S-system:  $\bullet t_i = \{p_i\}$  and  $t_i \bullet = \{p_{i+1}\}$ .

Thus,  $M \xrightarrow{\sigma} M' \xrightarrow{t}$  for  $\sigma = t_1 t_2 \dots t_{n-1}$ .

# Question time

Is this system live?



# Reachability lemma for S-nets

**Lemma:** Let  $(P, T, F)$  be a strongly connected S-net.  
 If  $M(P) = M'(P)$ , then  $M'$  is reachable from  $M$

We proceed by induction on  $M(P)$

**base** ( $M(P) = M'(P) = 0$ ): trivial ( $M' = M$ )

**induction** ( $M(P) = M'(P) > 0$ ):

Let  $p, p' \in P$  be such that  $M(p) > 0$  and  $M'(p') > 0$ .

Let  $K = M - p$  and  $K' = M' - p'$ .

Clearly  $K'(P) = K(P) < M(P) = M'(P)$ .

By inductive hypothesis:  $\exists \sigma, K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from  $p_0 = p$  to  $p_n = p'$

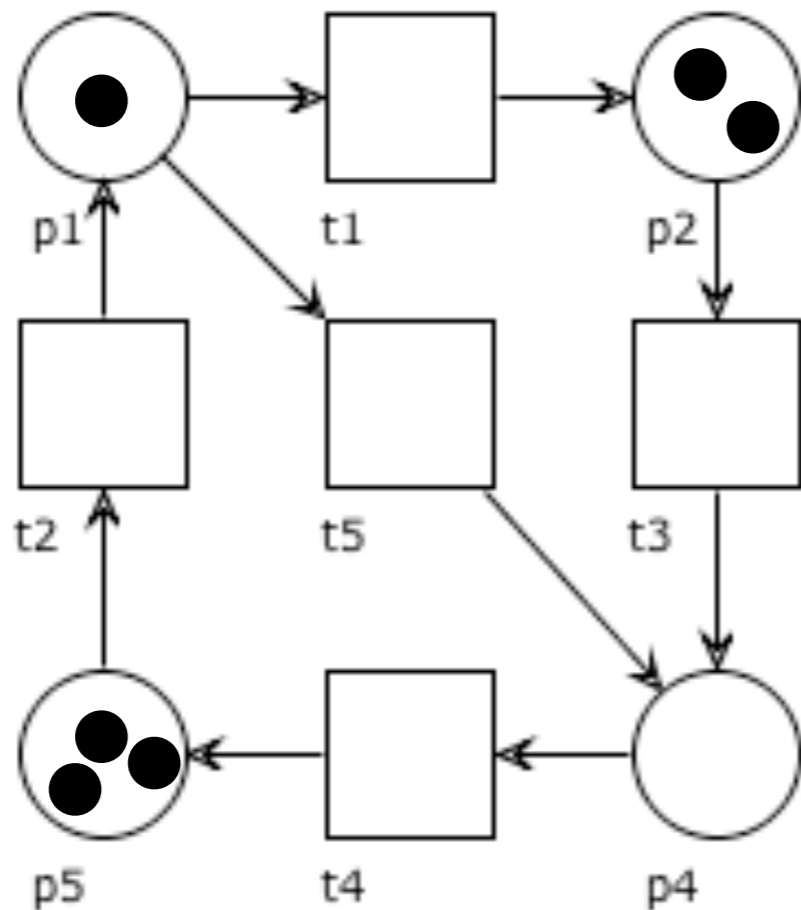
$(p_0, t_1)(t_1, p_1)(p_1, t_2) \dots (t_n, p_n)$

By definition of S-system:  $\bullet t_i = \{p_{i-1}\}$  and  $t_i \bullet = \{p_i\}$ .

Thus,  $p = p_0 \xrightarrow{\sigma'} p_n = p'$  for  $\sigma' = t_1 t_2 \dots t_n$ .

By the monotonicity lemma:  $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

# Question time



Is the marking  
 $M = p_2 + 4p_4 + p_5$   
reachable?

# Reachability Theorem for S-systems

**Theorem:** Let  $(P, T, F, M_0)$  be a live S-system.  
A marking  $M$  is reachable **iff**  $M(P) = M_0(P)$

$\Rightarrow$ ) Follows from the fundamental property of S-systems

$\Leftarrow$ ) By the previous liveness theorem, the S-net is strongly connected.

We conclude by applying the reachability lemma for S-systems.



# S-systems: recap

S-system  $\Rightarrow$  bounded

S-system: strong conn. +  $M_0(P) > 0 \Leftrightarrow$  live

S-system + M reachable  $\Rightarrow M(P) = M_0(P)$

S-system + str. conn.:  $M(P) = M_0(P) \Leftrightarrow$  M reachable

S-system + live:  $M(P) = M_0(P) \Leftrightarrow$  M reachable

S-system: S-invariant  $I \Leftrightarrow I = [k \ k \ \dots \ k]$

# Workflow S-nets

**Theorem:** If a workflow net  $N$  is an S-system  
then it is safe and sound

$N$  is S-system  $\Leftrightarrow N^*$  is S-system

$N$  and  $N^*$  S-systems  $\Rightarrow N$  and  $N^*$  bounded

$M_0(P)=1$  (initially one token in place  $i$ )

$N$  and  $N^*$  S-systems +  $M_0(P)=1 \Rightarrow N$  and  $N^*$  safe

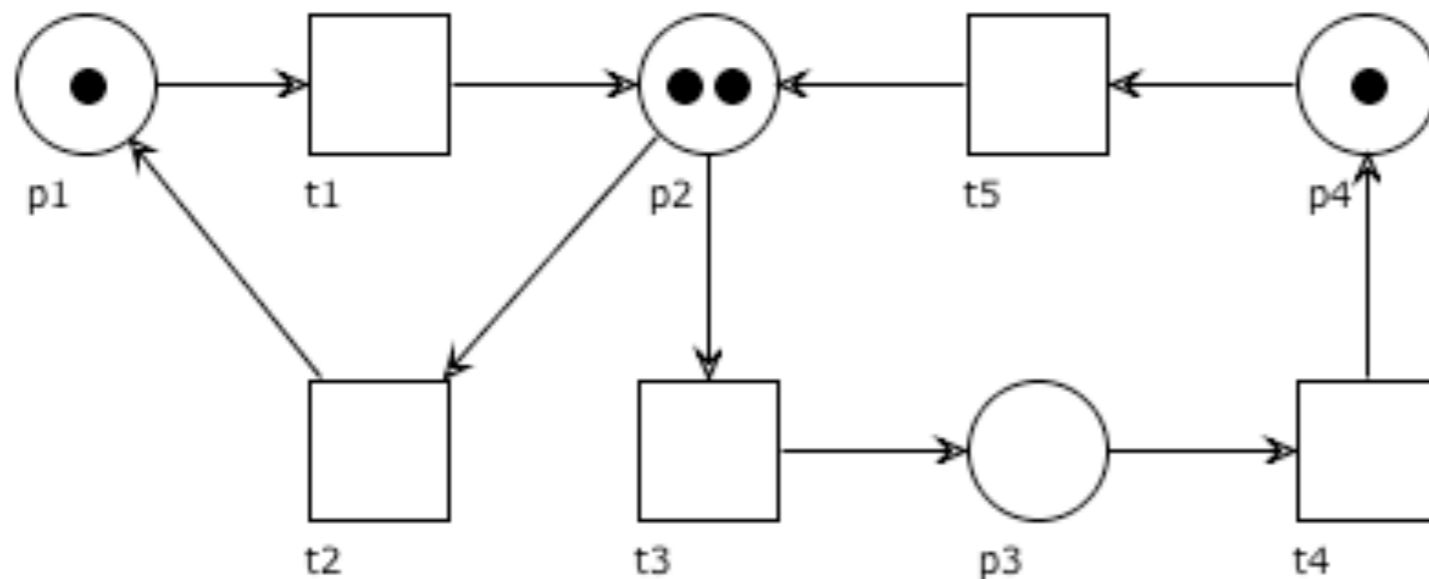
$N$  workflow net  $\Rightarrow N^*$  strong connected

$N^*$  strong connected +  $M_0(P) = 1 \Leftrightarrow N^*$  live

$N^*$  bounded and live  $\Leftrightarrow N$  sound

# Question time

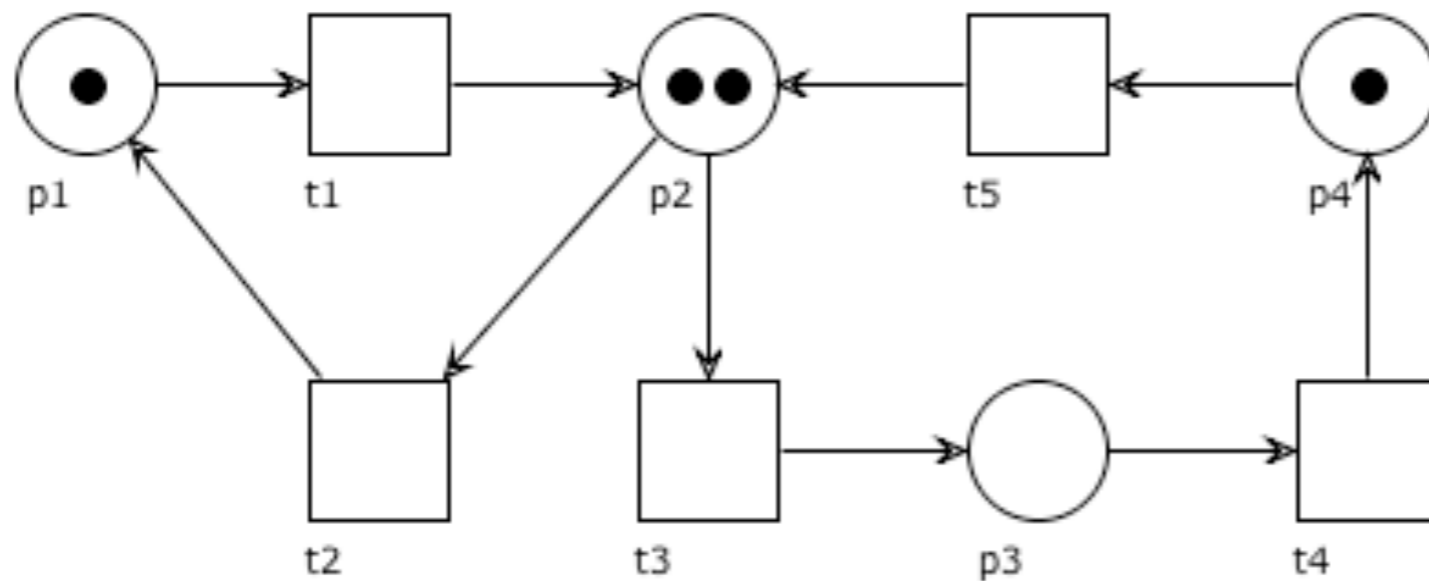
Which of the following markings are reachable? (why?)



[	1	1	1	]
[	2	0	2	]
[	1	2	1	]
[	4	0	0	]
[	0	4	0	]
[	0	3	2	]
[	0	0	4	]
[	0	3	0	]
[	0	3	0	]

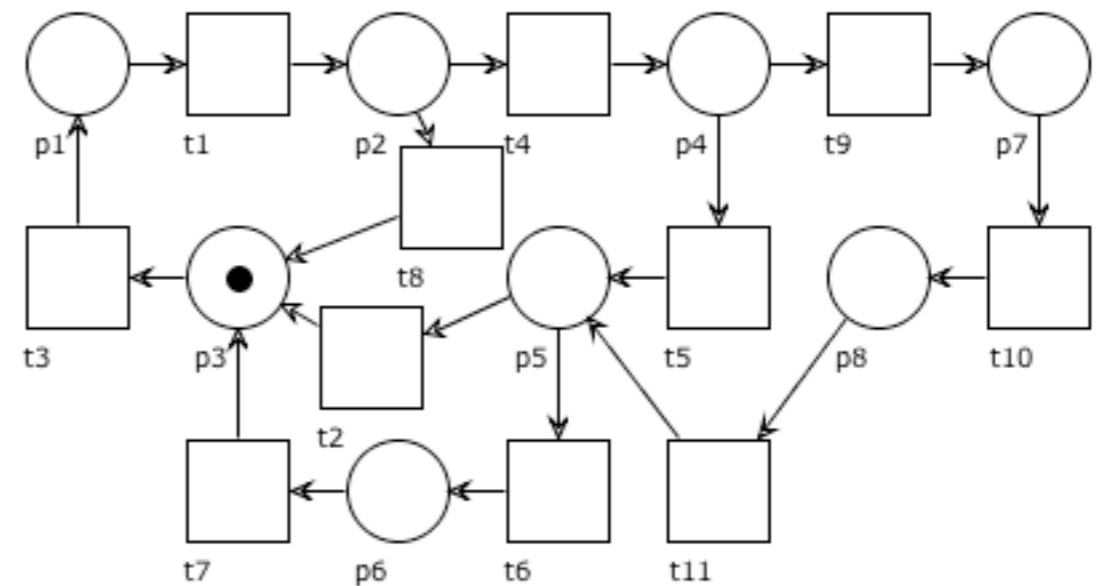
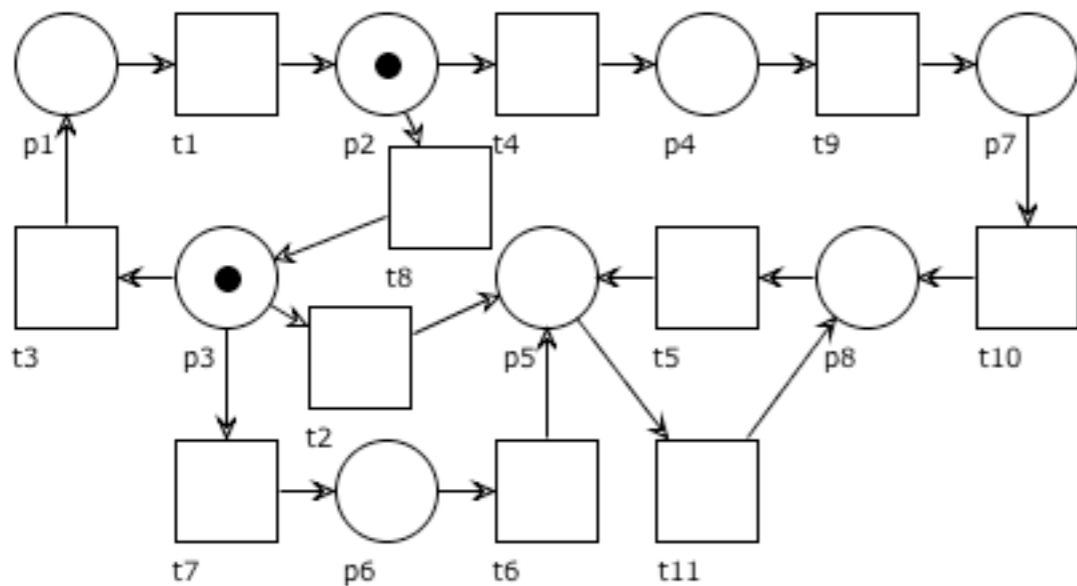
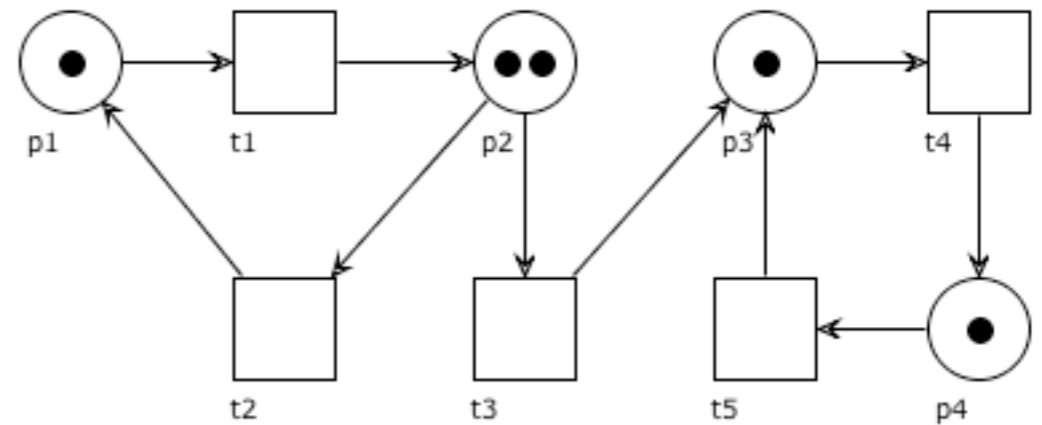
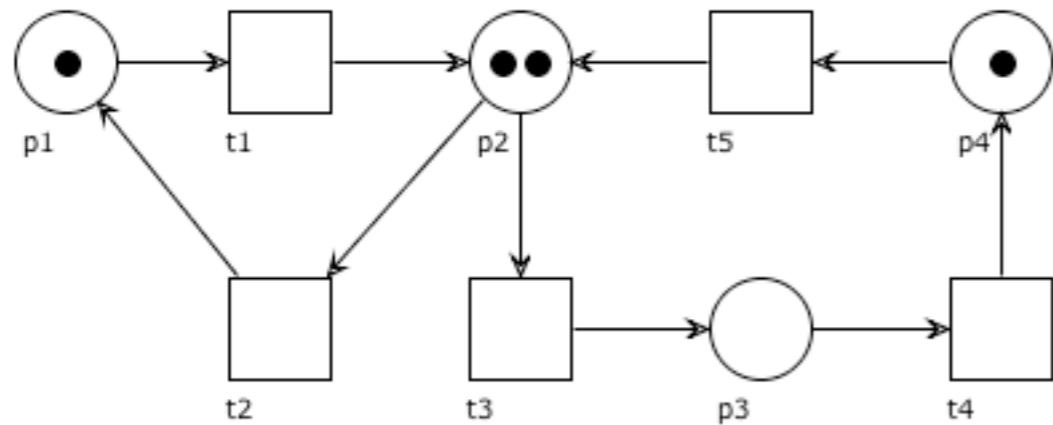
# Question time

Which of the following are S-invariants? (why?)


$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

# Exercises

Which of the following S-systems are live? (why?)



# Boundedness Theorem for S-systems

## Theorem:

A live S-system  $(P, T, F, M_0)$  is  $k$ -bounded iff  $M_0(P) \leq k$

# Exercise

Prove the boundedness theorem for live S-systems

# Exercise

Is the net below a workflow net?  
Is it sound?

