Business Processes Modelling

MPB (6 cfu, 295AA)

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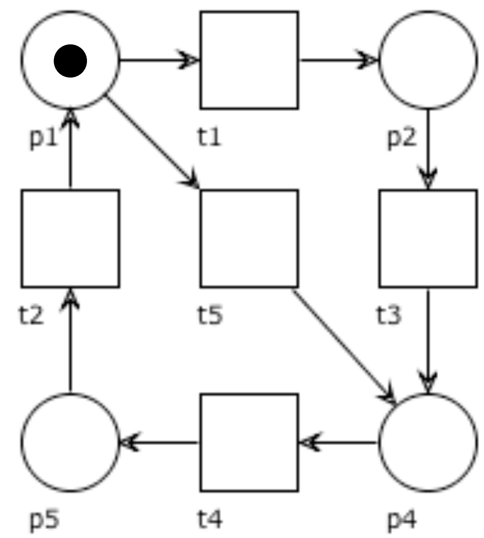
16 - S-systems

Object



We study some “good” properties of S-systems

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

S-systems

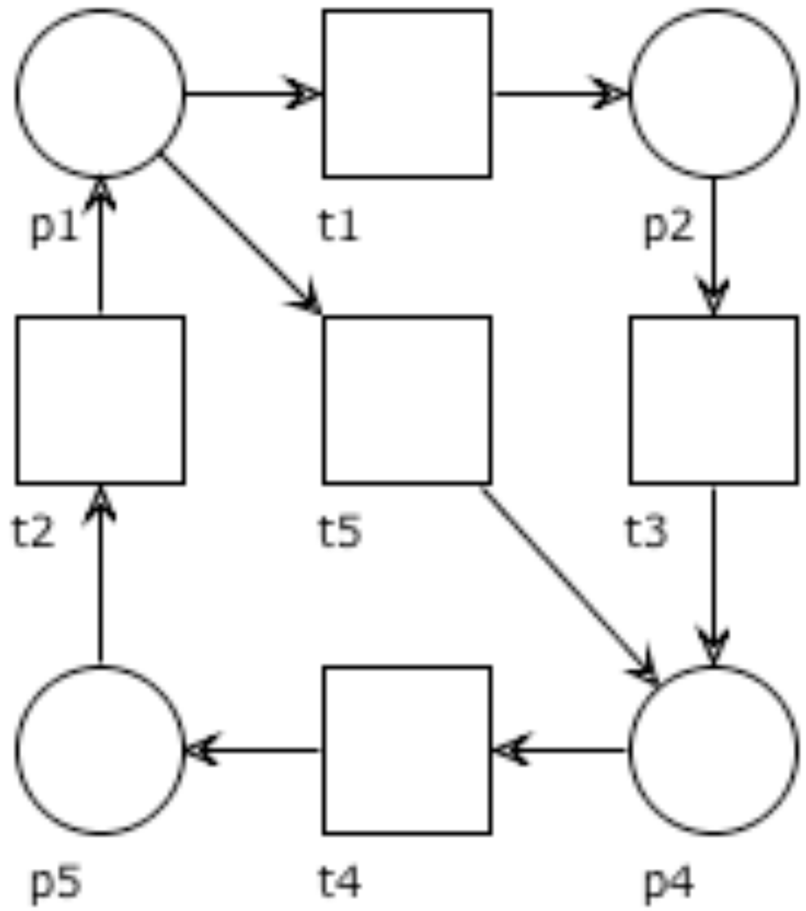
S-system

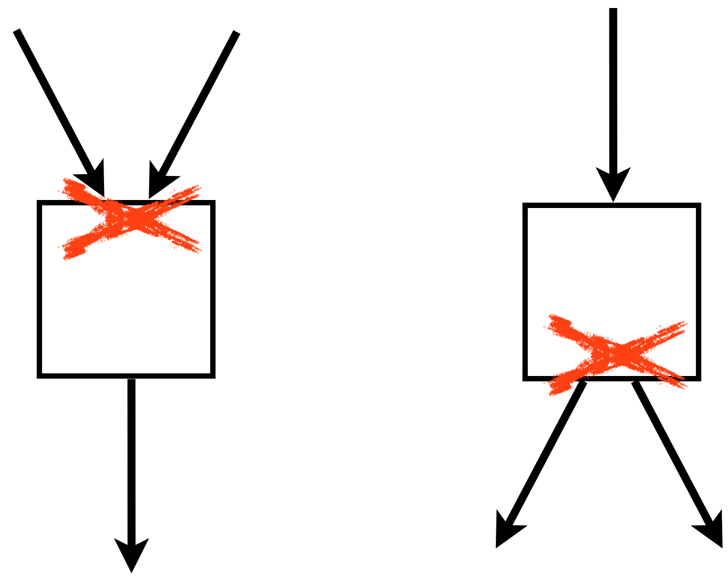
Definition: We recall that a net N is an **S-net** if each transition has exactly one input place and exactly one output place

$$\forall t \in T, \qquad |\bullet t| = 1 = |t \bullet|$$

A system (N,M_0) is an **S-system** if N is an S-net

S-net: example





S-net N*

Proposition: A workflow net N is an S-net
iff N* is an S-net

N and N* differ only for the reset transition,
that has exactly one incoming arc
and exactly one outgoing arc

S-systems: an observation

Observation: each transition t that fires removes exactly one token from some place p and inserts exactly one token in some place q (p and q can also coincide)

Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

Example

$$P = \{p_1, p_2, p_3\} \qquad M = 2p_1 + 3p_2 \qquad M(P) = 2 + 3 + 0 = 5$$

Fundamental property of S-systems

The overall number of tokens in the net is an invariant under any firing.

Fundamental property of S-systems

Proposition: Let (P,T,F,M_0) be an S-system.
If M is a reachable marking, then M(P) = M_0(P)

We show that for any $M \xrightarrow{\sigma} M'$ we have $M'(P) = M(P)$

$\mathbf{base}$ $(\sigma = \epsilon)$**:** trivial $(M' = M)$

induction $(\sigma = \sigma' t$ **for some** $\sigma' \in T^*$ **and** $t \in T)$**:**

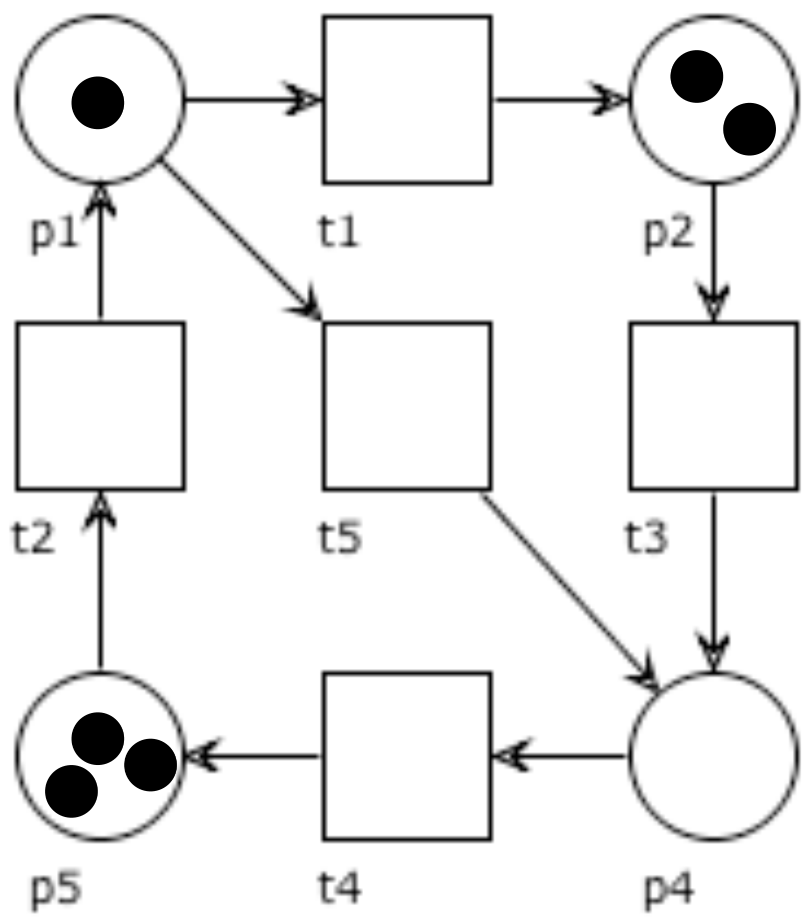
Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By inductive hypothesis: $M''(P) = M(P)$

By definition of S-system: $|\bullet t| = |t \bullet| = 1$

Thus, $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

Question time



Is the marking
$M = p_2 + 4p_4 + 2p_5$
reachable?

A consequence of the fundamental property

Corollary: Any S-system is bounded

Let $M \in [\, M_0 \,\rangle$.

By the fundamental property of S-systems: $M(P) = M_0(P)$.

Then, for any $p \in P$ we have $M(p) \leq M(P) = M_0(P)$.

Thus the S-system is k-bounded for any $k \geq M_0(P)$.

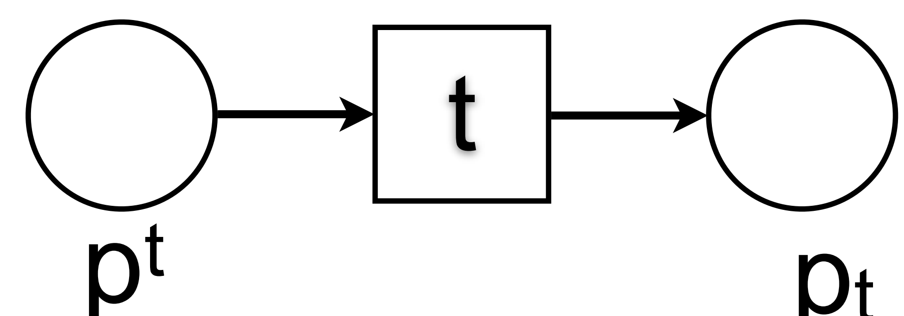
$$M(P) = \sum_{p \in P} M(p)$$

S-invariants of S-nets

Proposition: Let N=(P,T,F) be a (connected) S-net.
I is an S-invariant of N **iff I**=[k ... k] for some value k

S-invariance $$\forall t \in T,\ \sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$

S-nets $$\forall t \in T,\ |\bullet t| = |t \bullet| = 1$$



Let $\bullet t = \{p^t\}$ and $t\bullet = \{p_t\}$

$$\forall t \in T,\ \mathbf{I}(p^t) = \mathbf{I}(p_t)$$

S-invariants of S-nets

Proposition: Let N=(P,T,F) be a (connected) S-net.
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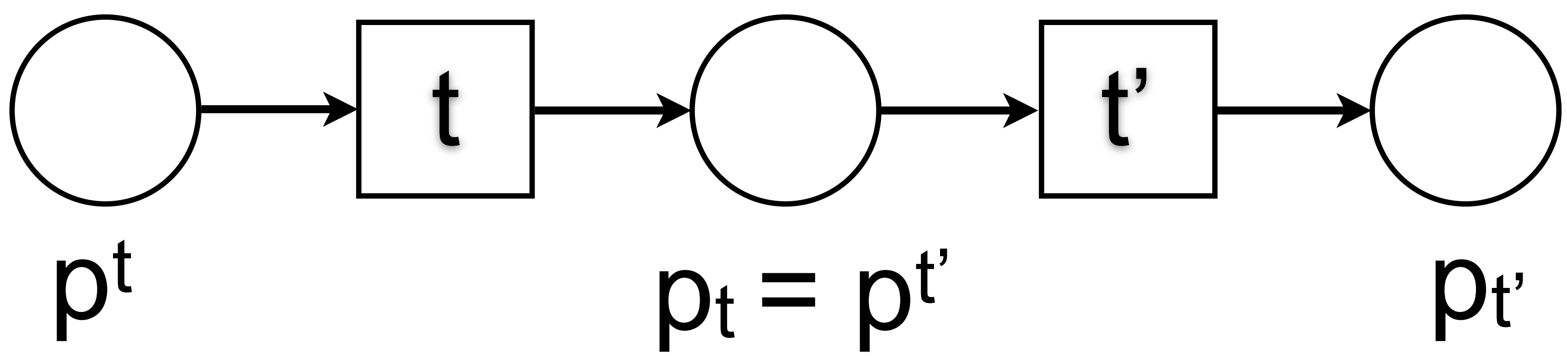

$$\mathbf{I}(p^t) = \mathbf{I}(p_t) = \mathbf{I}(p^{t'}) = \mathbf{I}(p_{t'})$$

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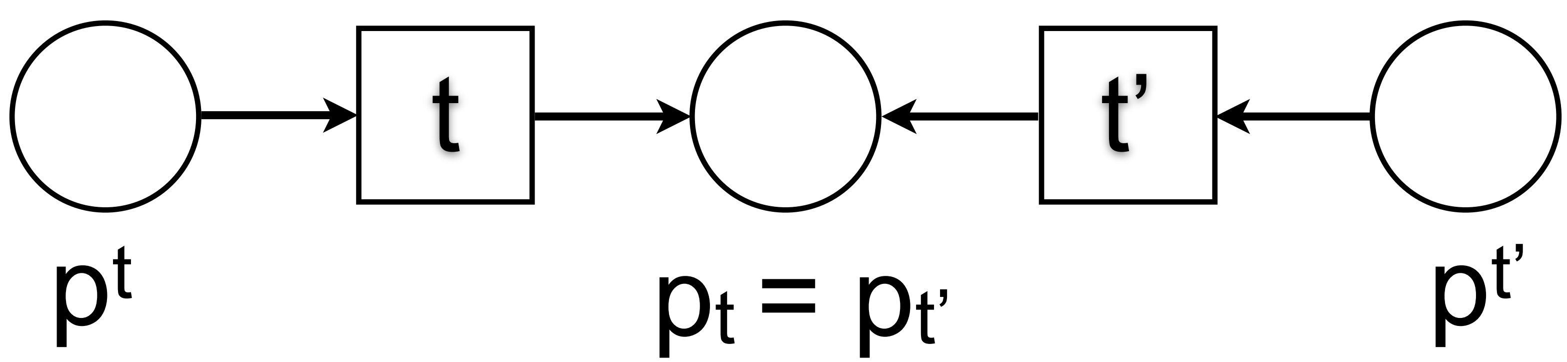

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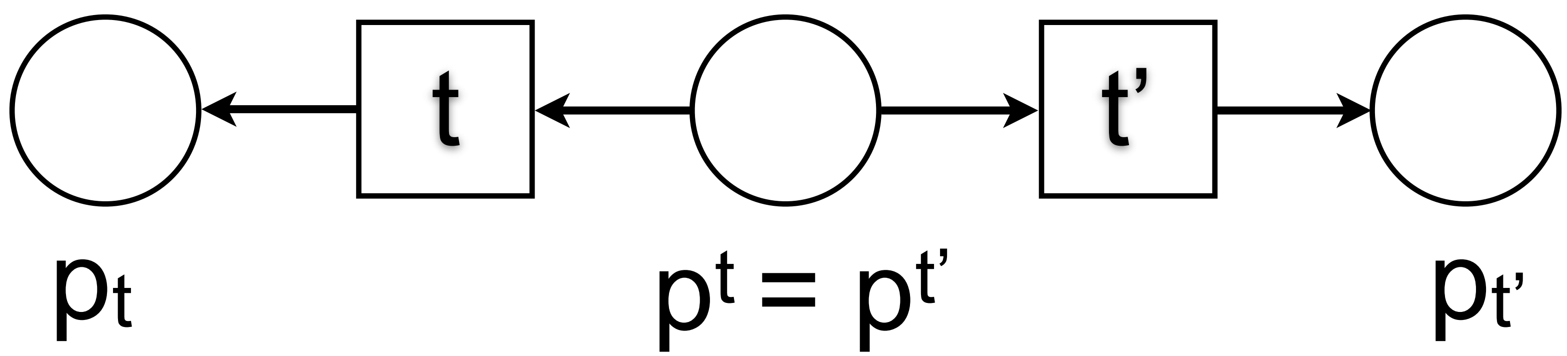


$$\mathbf{I}(p_t) = \mathbf{I}(p^t) = \mathbf{I}(p^{t'}) = \mathbf{I}(p_{t'})$$

S-invariants of S-nets

Proposition: Let N=(P,T,F) be a (connected) S-net.
I is an S-invariant of N **iff** **I**=[k ... k] for some value k

weak connectivity $\forall p_0, p_n \in P, \quad p_0\, t_1\, p_1\, t_2\, p_2\, t_3\, p_3 \ldots t_n\, p_n$

S-net ($\forall t_i$, either $(p_i, t_i)(t_i, p_{i+1})$ or $(t_i, p_i)(p_{i+1}, t_i)$)

$$\forall p_0, p_n \in P, \mathbf{I}(p_0) = \mathbf{I}(p_n)$$

A note on S-invariants and S-nets

S-invariance

$$\forall M \in [\, M_0 \,\rangle, \quad \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$$

S-invariant of S-nets

$$\mathbf{I} = [\, 1\ 1\ ...\ 1 \,]$$

consequence

$$\forall M, \quad \mathbf{I} \cdot M = \sum_{p \in P} 1 \cdot M(p) = \sum_{p \in P} M(p) = M(P)$$

We recover the Fundamental property of S-nets

$$\forall M \in [\, M_0 \,\rangle, \quad M(P) = \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 = M_0(P)$$

Liveness theorem for S-systems

Theorem: An S-system (N,M_0) is live **iff**
N is strongly connected and M_0 marks at least one place

$\Rightarrow$) (quite obvious)
(N, M_0) is live by hypothesis and bounded (because S-system).
By the strong connectedness theorem, N is strongly connected.

Since (N, M_0) is live, then $M_0 \xrightarrow{t}$ for some t.

Assume $\bullet t = \{p\}$. Thus, $M_0(p) \geq 1$.

Liveness theorem for S-systems

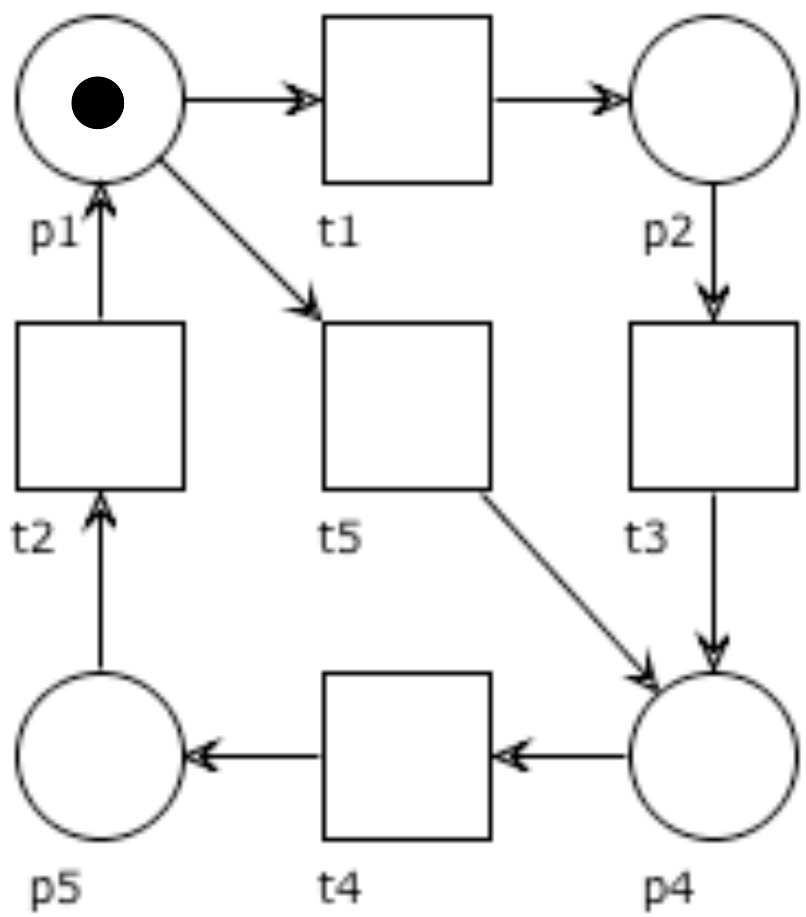
Theorem: An S-system (N,M_0) is live **iff**
N is strongly connected and M_0 marks at least one place

$\Leftarrow$) (more interesting)
Take any $M \in [\, M_0 \,\rangle$ and $t \in T$.
We want to find $M' \in [\, M \,\rangle$ such that $M' \stackrel{t}{\longrightarrow}$.

Take $p_1 \in P$ such that $M(p_1) \geq 1$ (it exists, because $M(P) = M_0(P) \geq 1$).
By strong connectedness: there is a path from p_1 to $t_n = t$
$(p_1, t_1)(t_1, p_2)(p_2, t_2) ... (p_n, t_n)$

By definition of S-system: $\bullet t_i = \{p_i\}$ and $t_i \bullet = \{p_{i+1}\}$.
Thus, $M \stackrel{\sigma}{\longrightarrow} M' \stackrel{t}{\longrightarrow}$ for $\sigma = t_1 t_2 ... t_{n-1}$.

Question time



Is this system live?

Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
If M(P) = M'(P), then M' is reachable from M

We proceed by induction on $M(P)$

base ($M(P) = M'(P) = 0$)**:** trivial ($M' = M$)

induction ($M(P) = M'(P) > 0$)**:**

Let $p, p' \in P$ be such that $M(p) > 0$ and $M'(p') > 0$.

Let $K = M - p$ and $K' = M' - p'$.

Clearly $K'(P) = K(P) < M(P) = M'(P)$.

By inductive hypothesis: $\exists \sigma$, $K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from $p_0 = p$ to $p_n = p'$

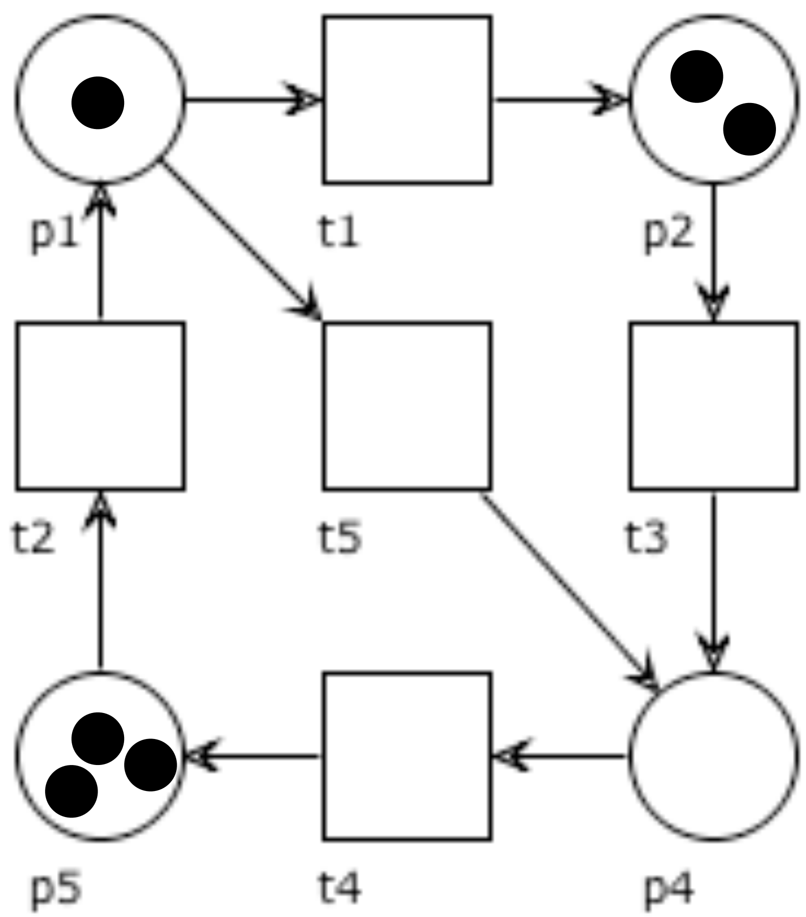
$(p_0, t_1)(t_1, p_1)(p_1, t_2)...(t_n, p_n)$

By definition of S-system: $\bullet t_i = \{p_{i-1}\}$ and $t_i \bullet = \{p_i\}$.

Thus, $p = p_0 \xrightarrow{\sigma'} p_n = p'$ for $\sigma' = t_1 t_2 ... t_n$.

By the monotonicity lemma: $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

Question time



Is the marking
$M = p_2 + 4p_4 + p_5$
reachable?

Reachability Theorem for S-systems

Theorem: Let (P,T,F,M_0) be a live S-system.
A marking M is reachable **iff** $M(P)=M_0(P)$

=>) Follows from the fundamental property of S-systems

<=) By the previous liveness theorem, the S-net is strongly connected.
We conclude by applying the reachability lemma for S-systems.

S-systems: recap

S-system => bounded

S-system: strong conn. + $M_0(P)>0$ <=> live

S-system + M reachable => $M(P) = M_0(P)$

S-system + str. conn.: $M(P)=M_0(P)$ <=> M reachable

S-system + live: $M(P)=M_0(P)$ <=> M reachable

S-system: S-invariant **I** <=> **I** = [k k ... k]

Workflow S-nets

Theorem: If a workflow net N is an S-system then it is safe and sound

N is S-system <=> N* is S-system

N and N* S-systems => N and N* bounded

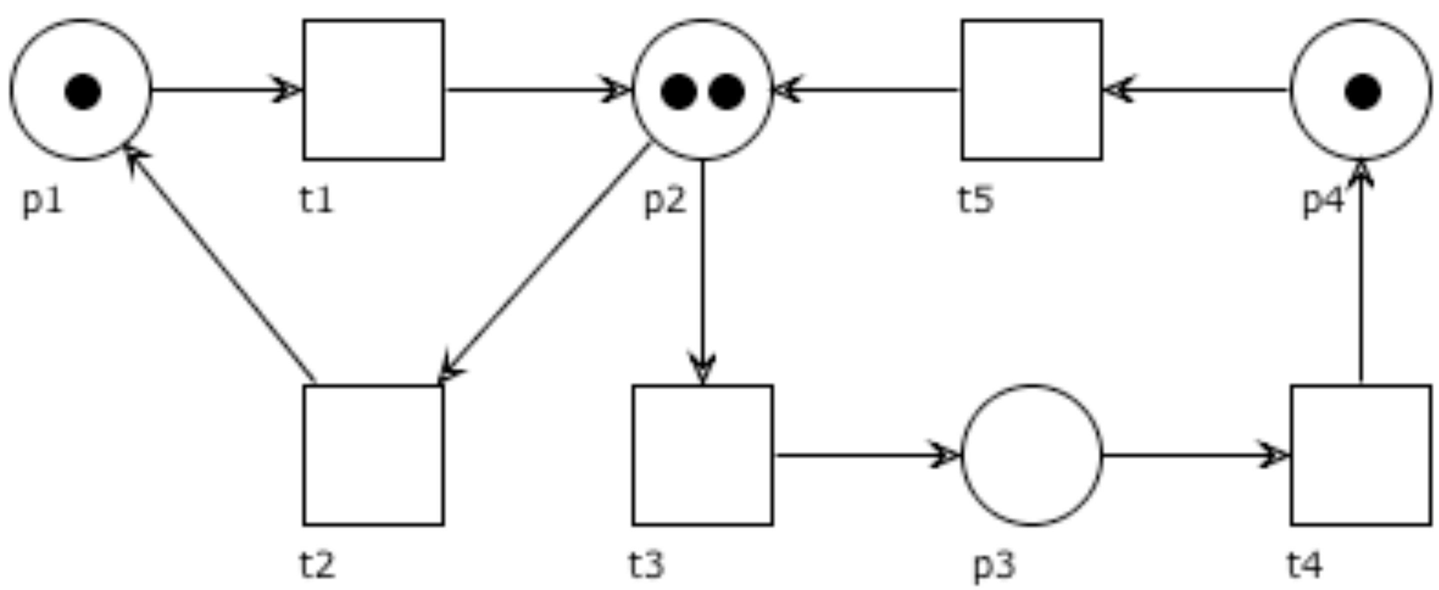
$M_0(P)=1$ (initially one token in place i)
N and N* S-systems + $M_0(P)=1$ => N and N* safe

N workflow net => N* strong connected
N* strong connected + $M_0(P) = 1$ <=> N* live

N* bounded and live <=> N sound

Question time

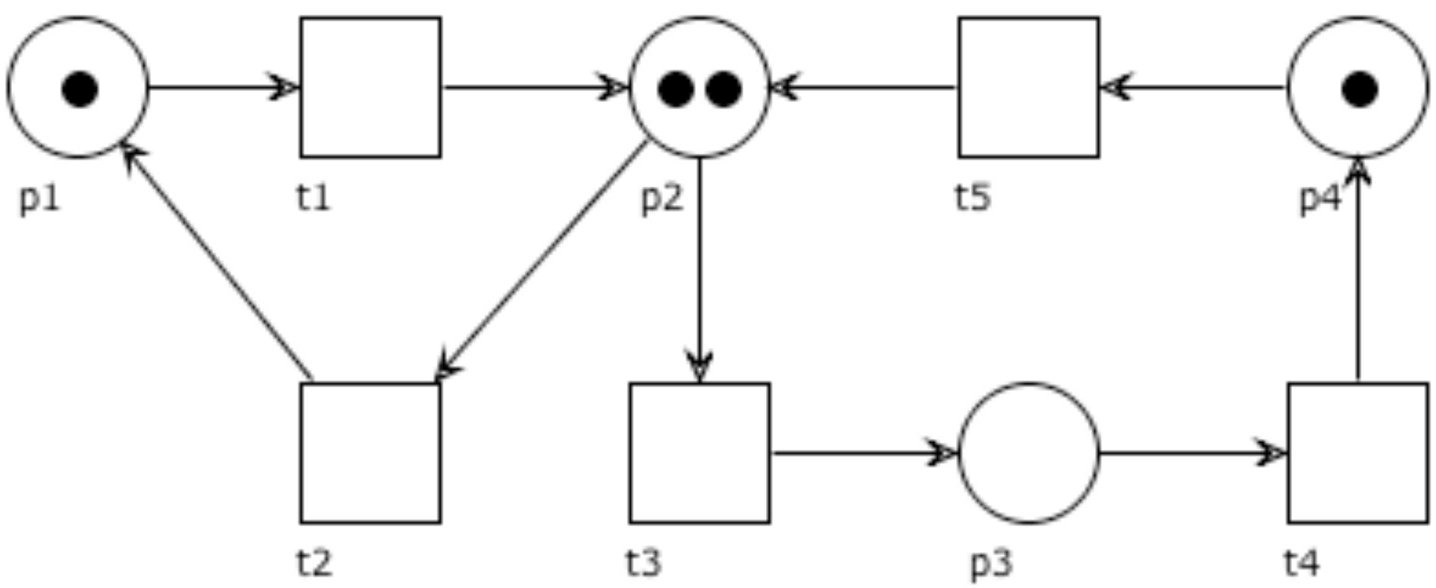
Which of the following markings are reachable? (why?)



[1 1 1 1]
[2 0 2 0]
[1 2 1 2]
[4 0 0 0]
[0 4 0 4]
[0 3 2 1]
[0 0 4 0]
[0 3 0 0]
[0 3 0 1]

Question time

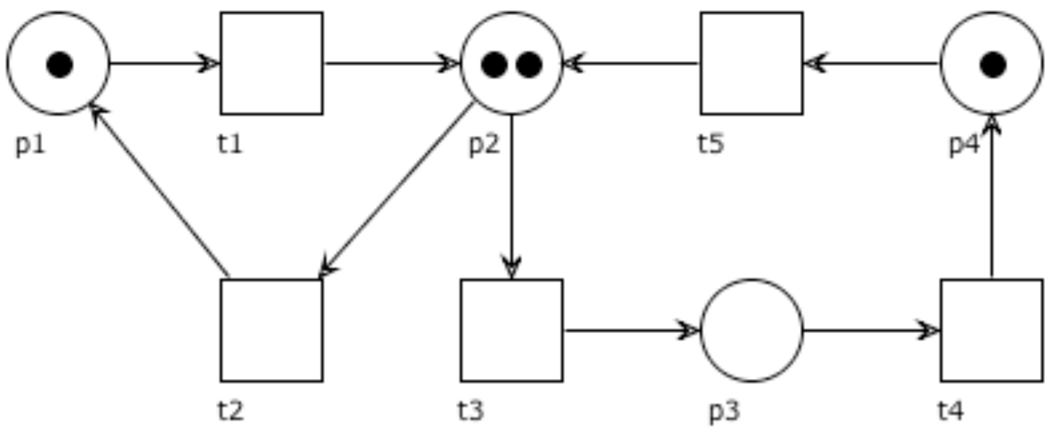
Which of the following are S-invariants? (why?)

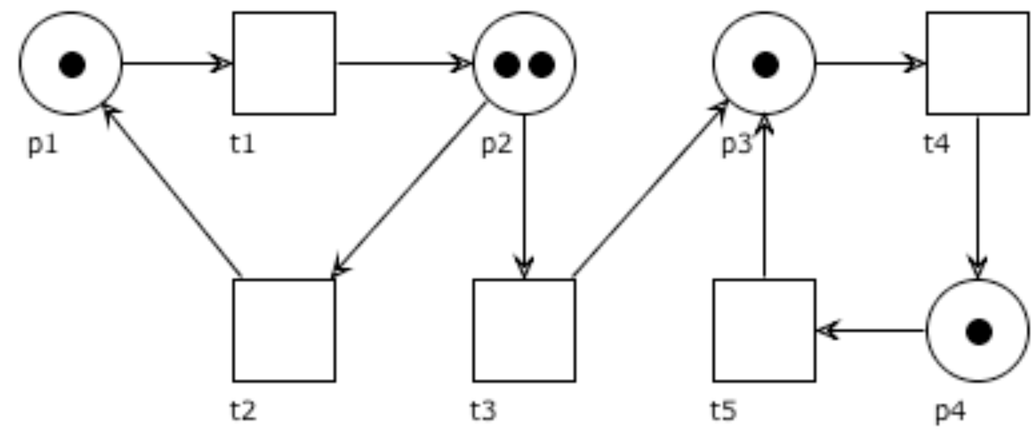


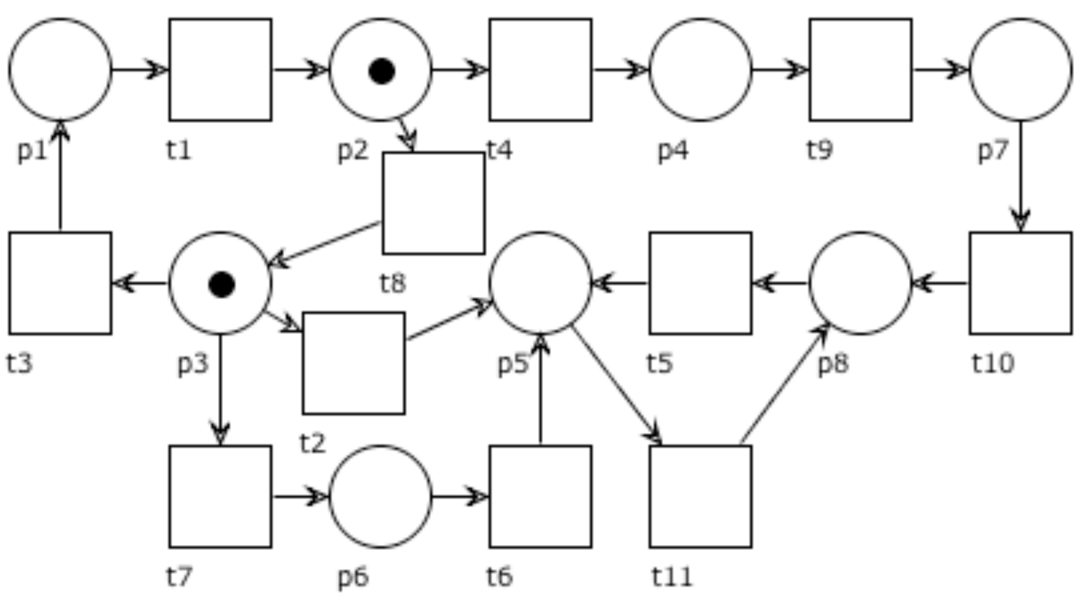
[1 1 0 0]
[0 0 2 2]
[1 1 1 1]
[2 2 1 1]
[2 2 2 2]
[1 2 2 1]

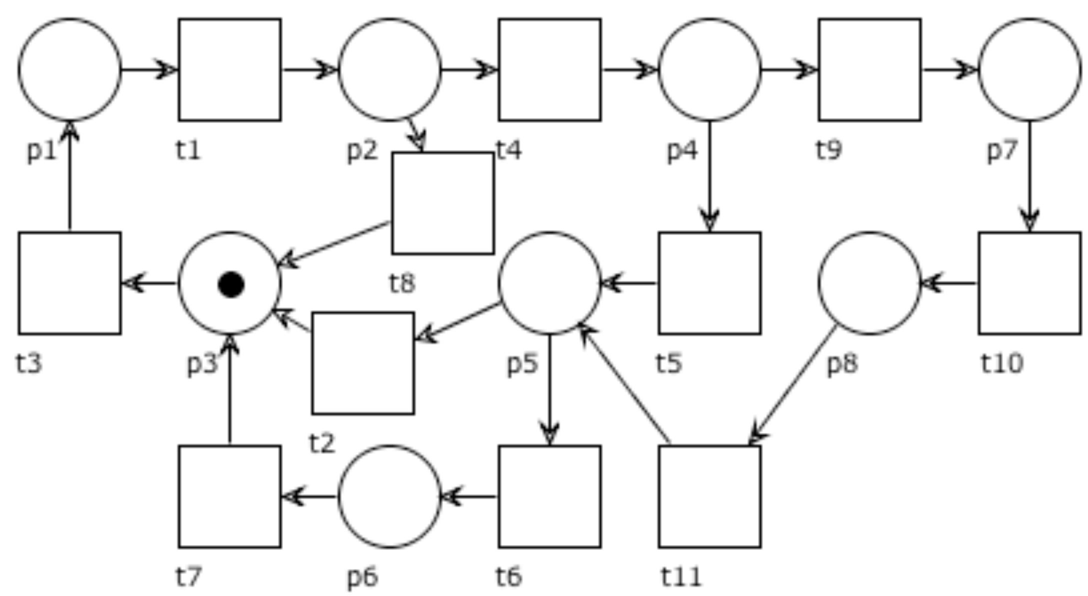
Exercises

Which of the following S-systems are live? (why?)









Boundedness Theorem for S-systems

Theorem:
A live S-system (P, T, F, M_0) is k-bounded iff $M_0(P) \leq k$

Exercise

Prove the boundedness theorem for live S-systems

Exercise

Is the net below a workflow net?
Is it sound?