

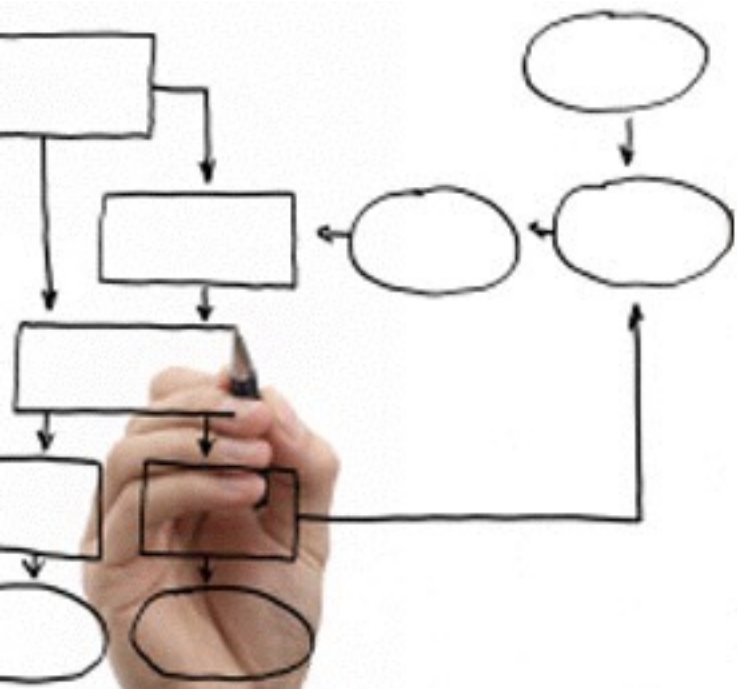
Methods for the specification and verification of business processes

MPB (6 cfu, 295AA)

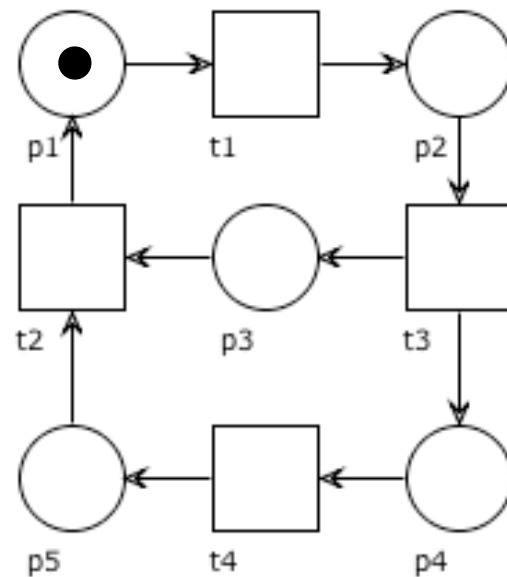
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17 - T-systems



Object



We study some “good” properties of T-systems

Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

T-systems

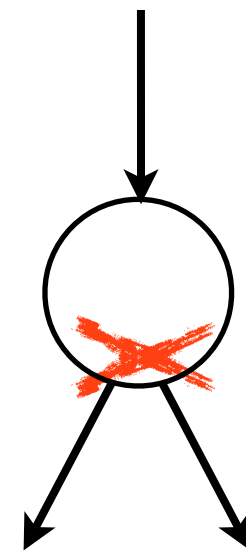
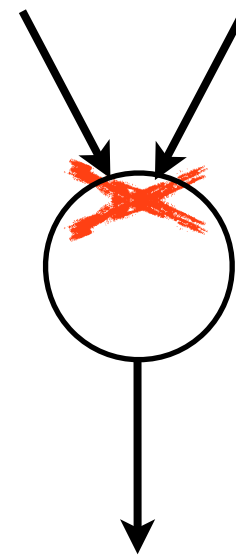
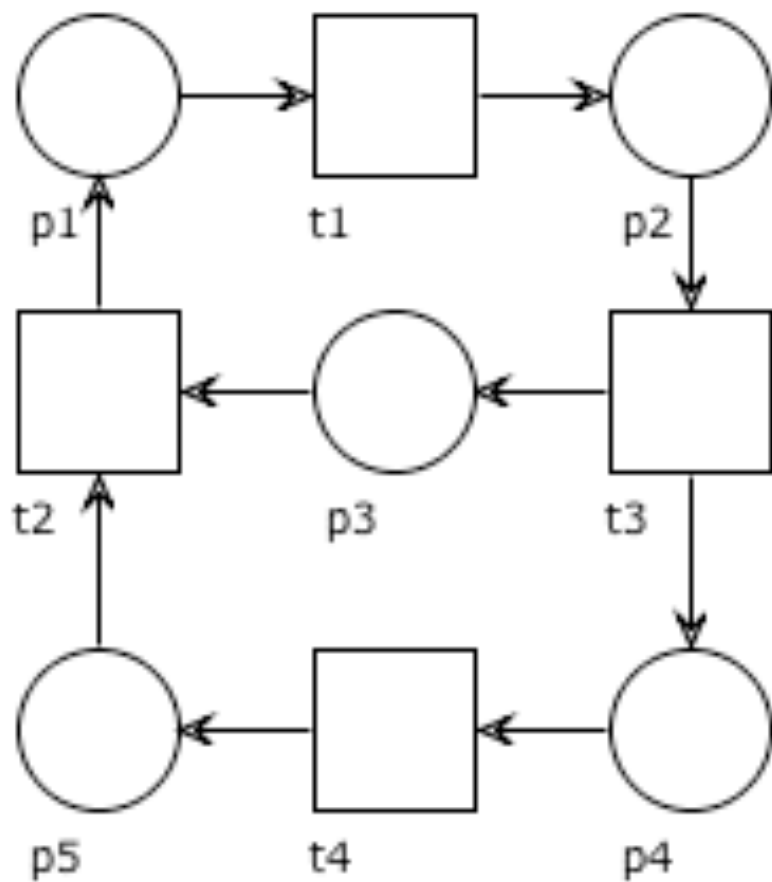
T-system

Definition: We recall that a net N is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \quad |\bullet p| = 1 = |p \bullet|$$

A system (N, M_0) is a **T-system** if N is a T-net

T-net: example



T-systems: an observation

Notably, computation in T-systems is concurrent,
but essentially deterministic:

the firing of a transition t in M cannot disable
another transition t' enabled at M

T-net N^*

Is the following conjecture true?

A workflow net N is a T-net
iff N^* is a T-net

No, a workflow net cannot be a T-net because
the place i has no incoming arc
and the place o has no outgoing arc

(N^* can be a T-net)

T-systems: another observation

Determination of control:

the transitions responsible for enabling t are
one for each input place of t

Notation: token count of a circuit

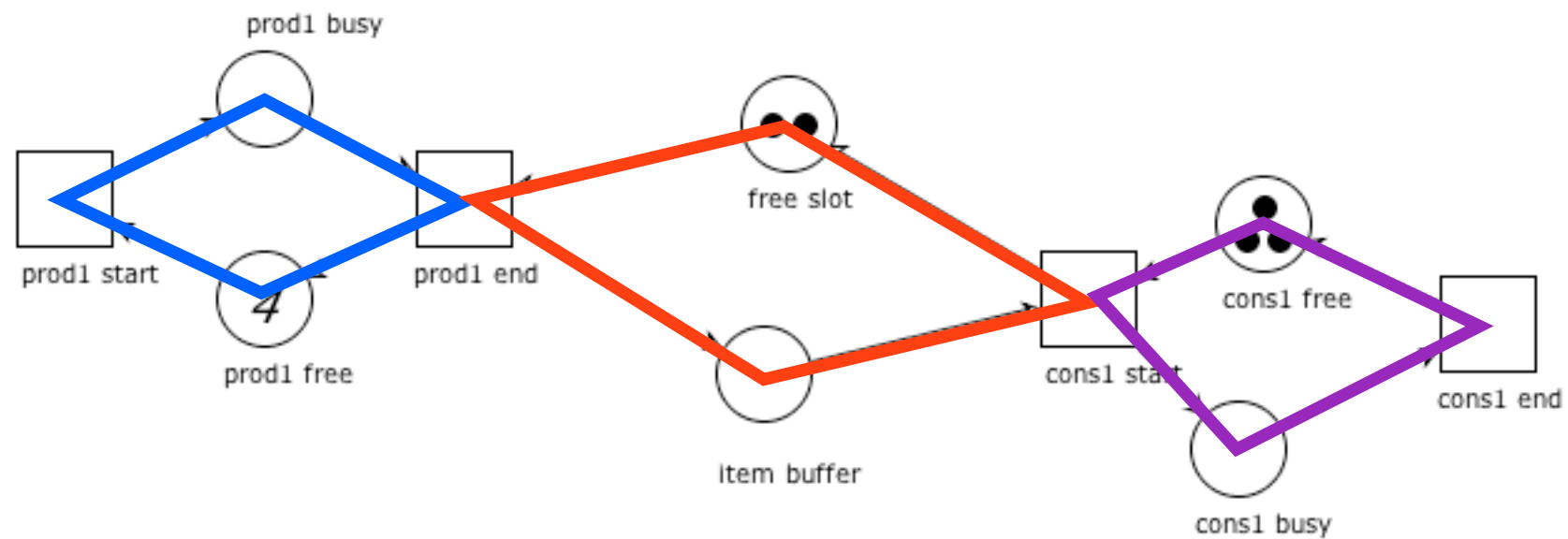
Let $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2)\dots(x_n, y_n)$ be a circuit.

Let $P|_{\gamma} \subseteq P$ be the set of places in γ .

$$M(\gamma) = M(P|_{\gamma}) = \sum_{p \in P|_{\gamma}} M(p)$$

We say that γ is **marked at** M if $M(\gamma) > 0$

Example



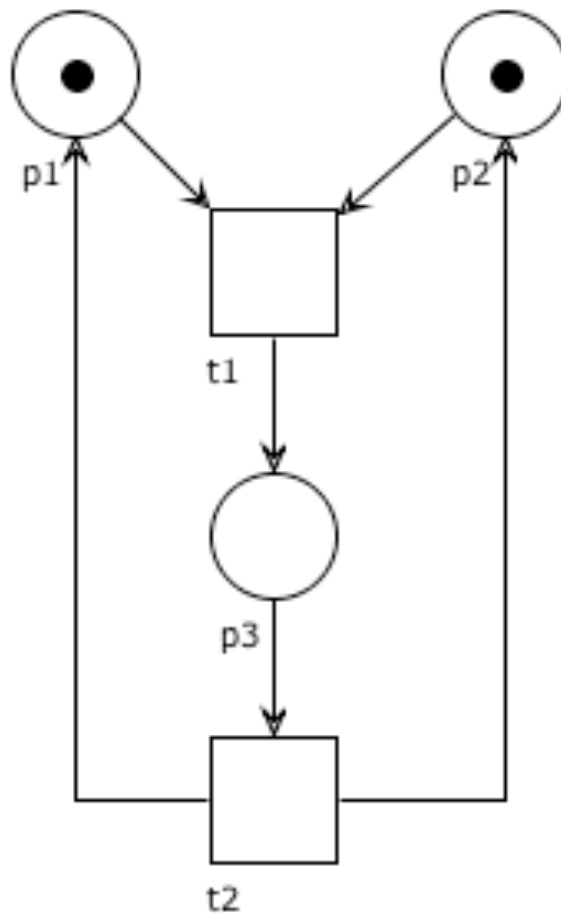
$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

Question time

Trace two circuits over the T-system below



Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

Fundamental property of T-systems

Proposition: Let γ be a circuit of a T-system (P, T, F, M_0) .
If M is a reachable marking, then $M(\gamma) = M_0(\gamma)$

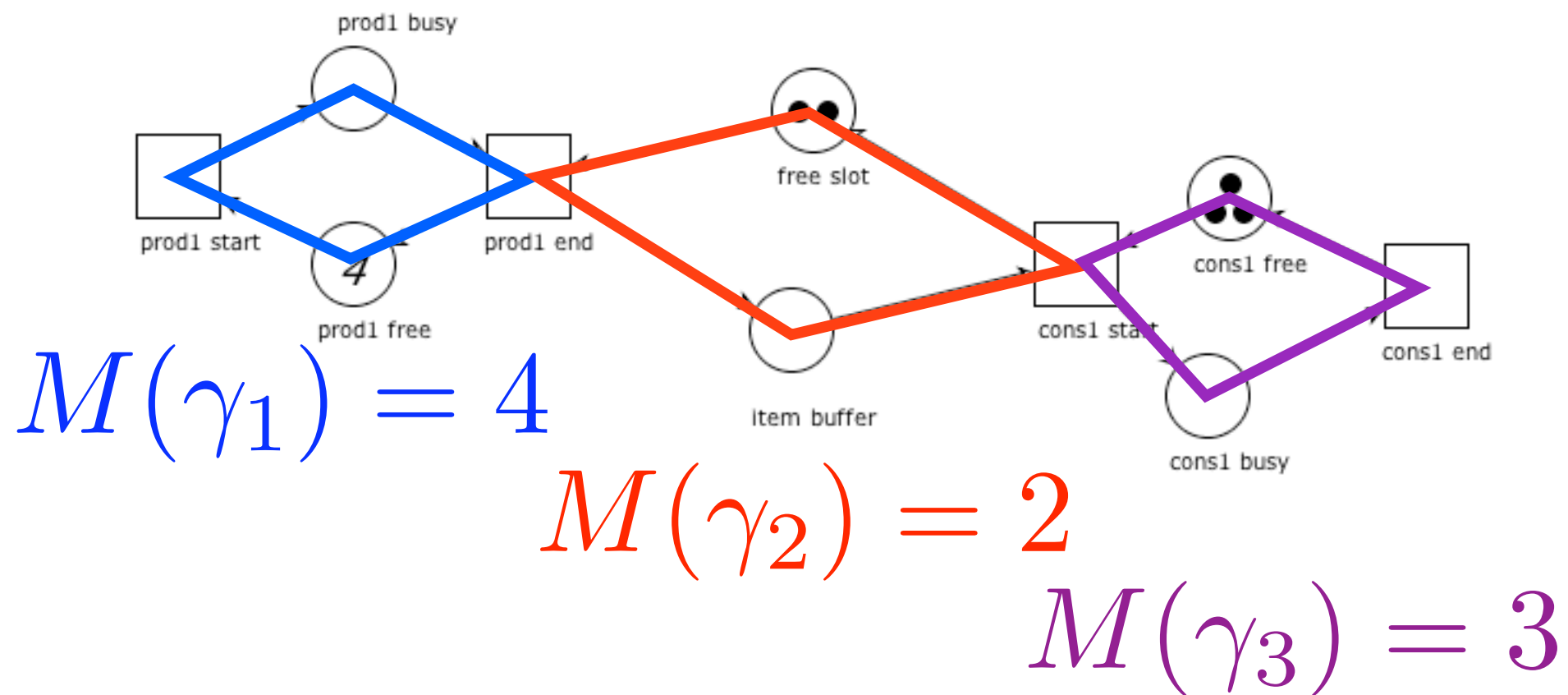
Take any $t \in T$: either $t \notin \gamma$ or $t \in \gamma$.

If $t \notin \gamma$, then no place in $\bullet t \cup t \bullet$ is in γ
(otherwise, by definition of T-nets, t would be in γ).

Then, an occurrence of t does not change the token count of γ .

If $t \in \gamma$, then exactly one place in $\bullet t$ and one place in $t \bullet$ are in γ .
Then, an occurrence of t does not change the token count of γ .

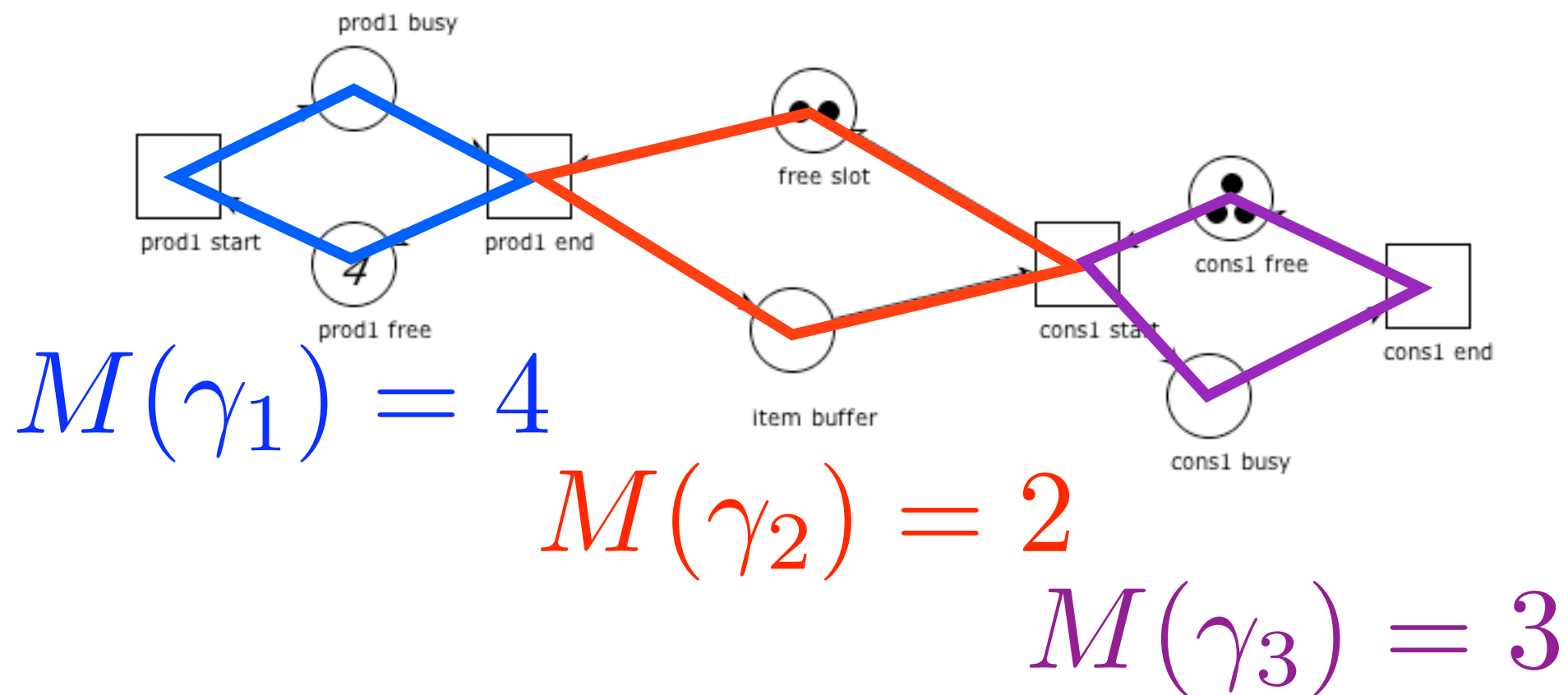
Example



$$M_0 = [0 \ 4 \ 2 \ 0 \ 3 \ 0]$$

$$M = [2 \ 2 \ 1 \ 2 \ 2 \ 1]$$

Example

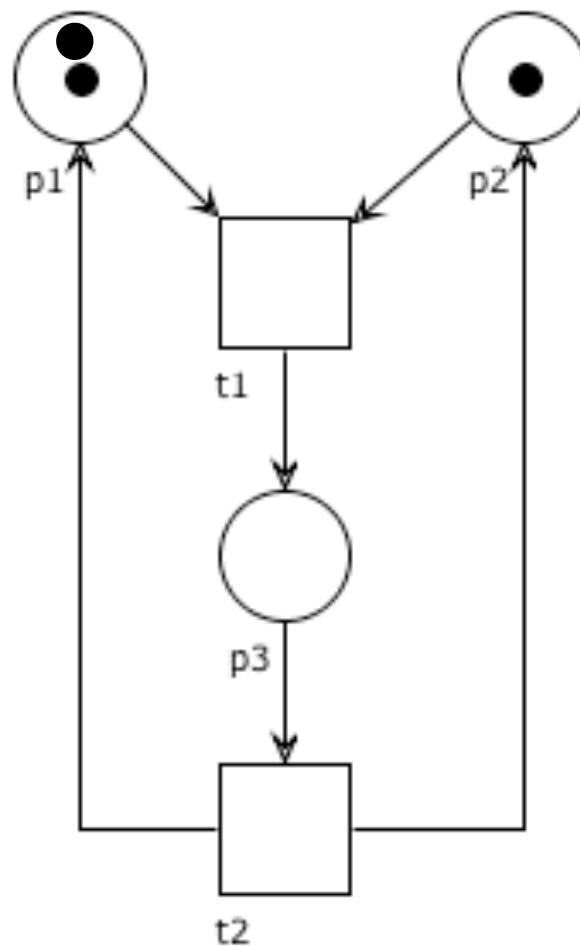


$$M_0 = [0 \ 4 \ 2 \ 0 \ 3 \ 0]$$

$$M' = [2 \ 1 \ 1 \ 1 \ 2 \ 2]$$

Question time

Is the marking $p_1 + 2p_2$ reachable? (why?)



T-invariants of T-nets

Proposition: Let $N=(P,T,F)$ be a (connected) T-net. J is a T-invariant of N iff $J=[k \dots k]$ for some value k

(the proof is dual to the analogous proposition for S-invariants of S-nets)

Boundedness in strongly connected T-systems

Lemma: If a T-system (N, M_0) is strongly connected, then it is bounded

Let Γ be the set of the circuits of N and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$.

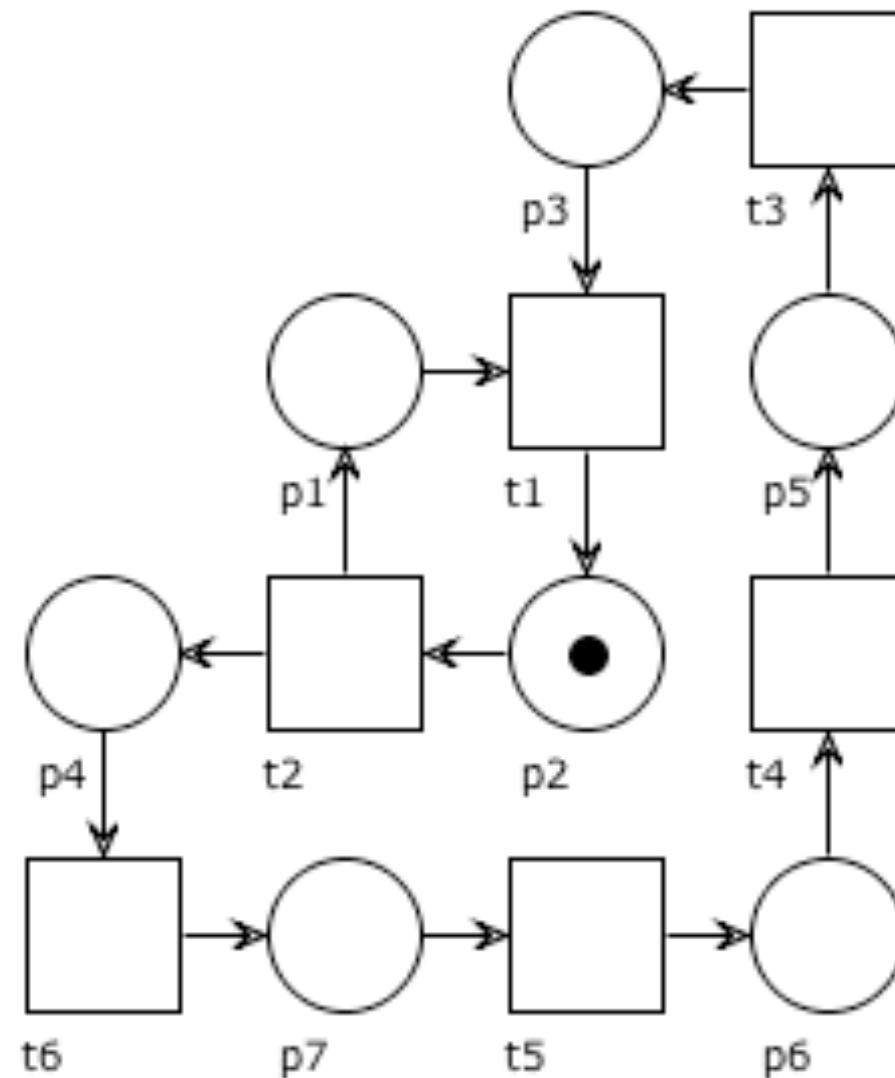
Since N is strongly connected, every place p belongs to some circuit γ_p .

By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M , we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$. Hence the net is k -bounded.

Question time

Is the T-systems below bounded? (why?)



Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
iff every circuit of N is marked at M_0

\Rightarrow) (quite obvious)

By contradiction, let γ be a circuit with $M_0(\gamma) = 0$.

By the fundamental property of T-systems: $\forall M \in [M_0 \rangle, M(\gamma) = 0$.

Take any $t \in T|_\gamma$ and $p \in P|_\gamma \cap \bullet t$.

For any $M \in [M_0 \rangle$, we have $M(p) = 0$.

Hence t is never enabled and the T-system is not live.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
iff every circuit of N is marked at M_0

\Leftarrow) (more involved)

Take any $t \in T$ and $M \in [M_0 \rangle$.

We need to show that some marking M' reachable from M enables t .

The key idea is to collect the places that control the firing of t :

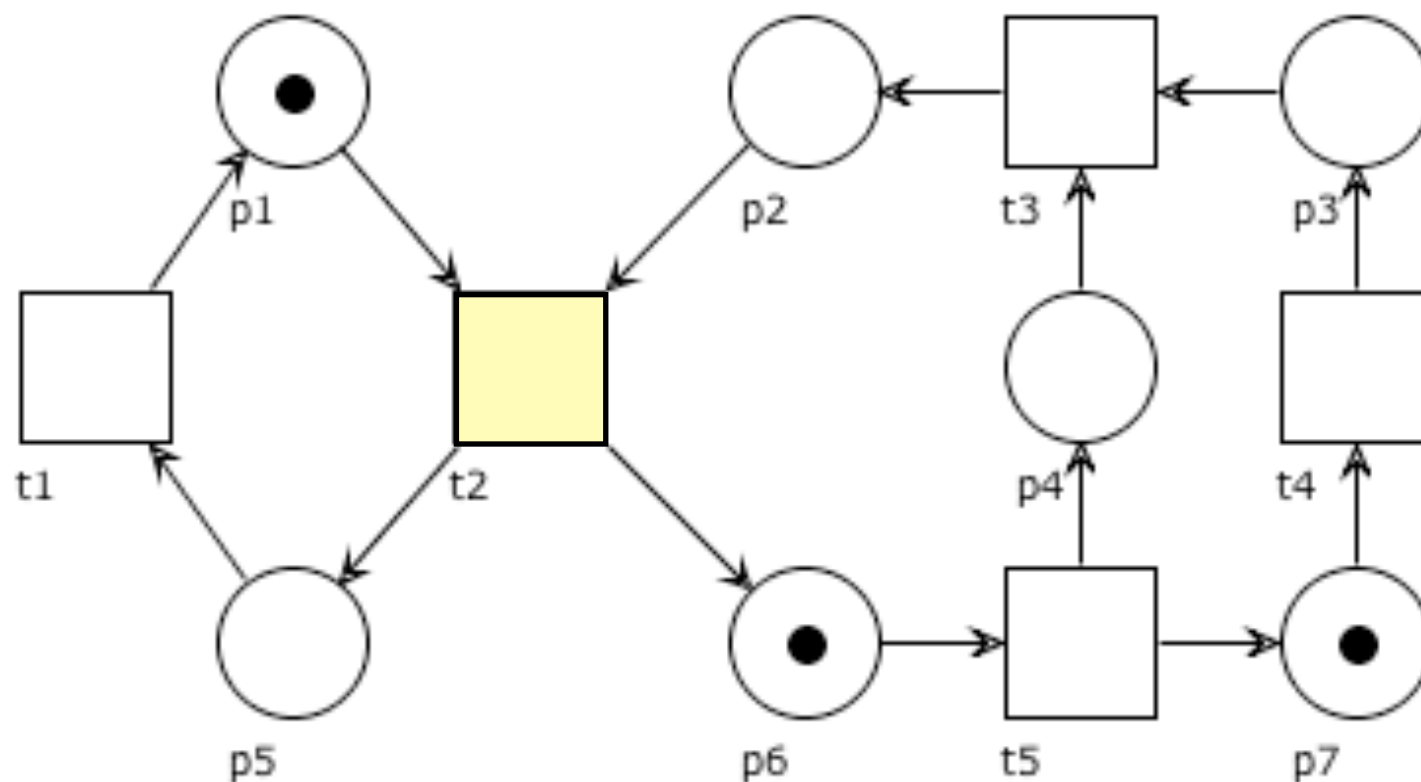
$p \in P_{M,t}$ if there is a path from p to t through places unmarked at M .

We then proceed by induction on the size of $P_{M,t}$.

We just sketch the key idea of the proof over a T-system.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0



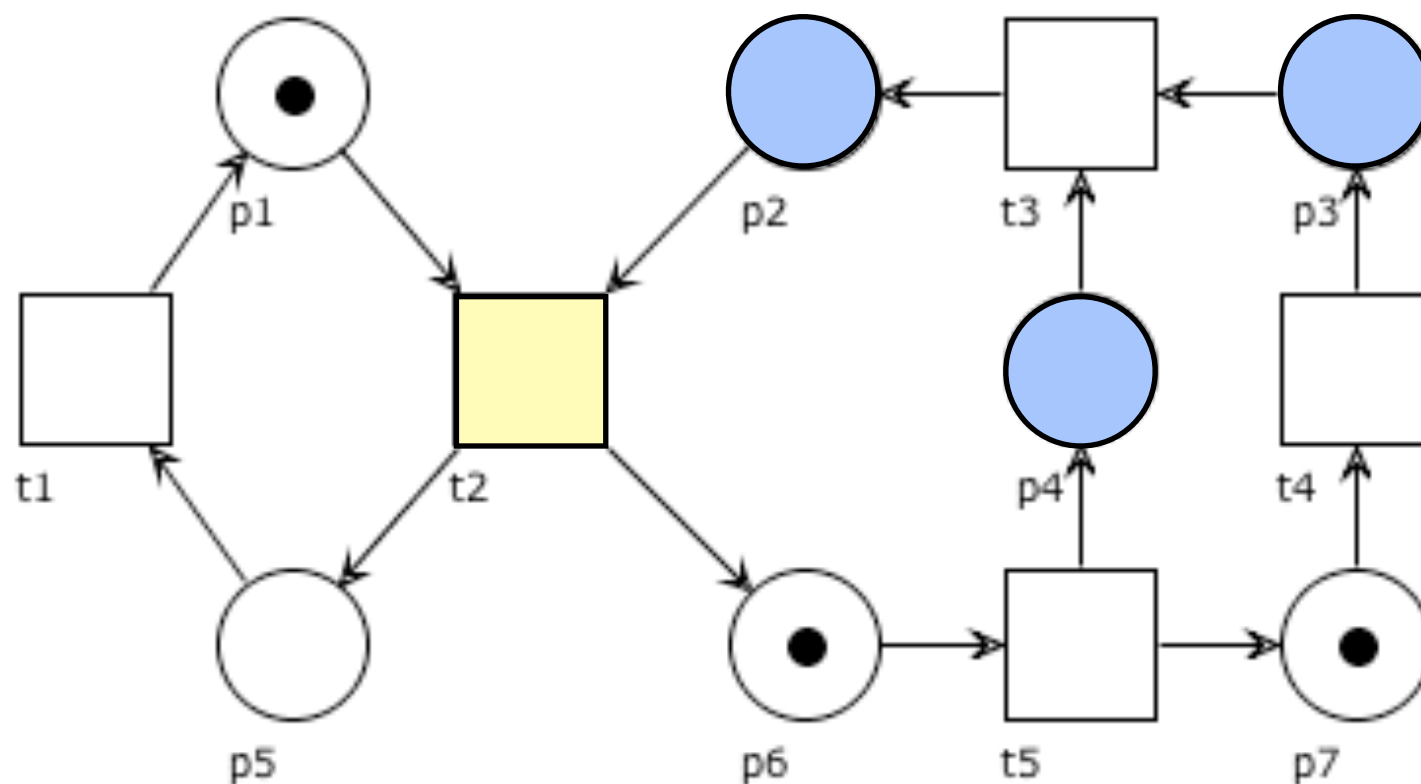
$$M = p_1 + p_6 + p_7$$

M' enabling t_2 ?

$p \in P_{M,t}$ if there is a path from p to t through places unmarked at M .

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live \Leftrightarrow every circuit of N is marked at M_0



$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (continued proof sketch)

Base case: $|P_{M,t}| = 0$.

Every place in $\bullet t$ is already marked at M .

Hence t is enabled at M .

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (continued proof sketch)

Inductive case: $|P_{M,t}| > 0$.

Therefore t is not enabled at M .

We look for a path π of maximal length necessary for firing t .

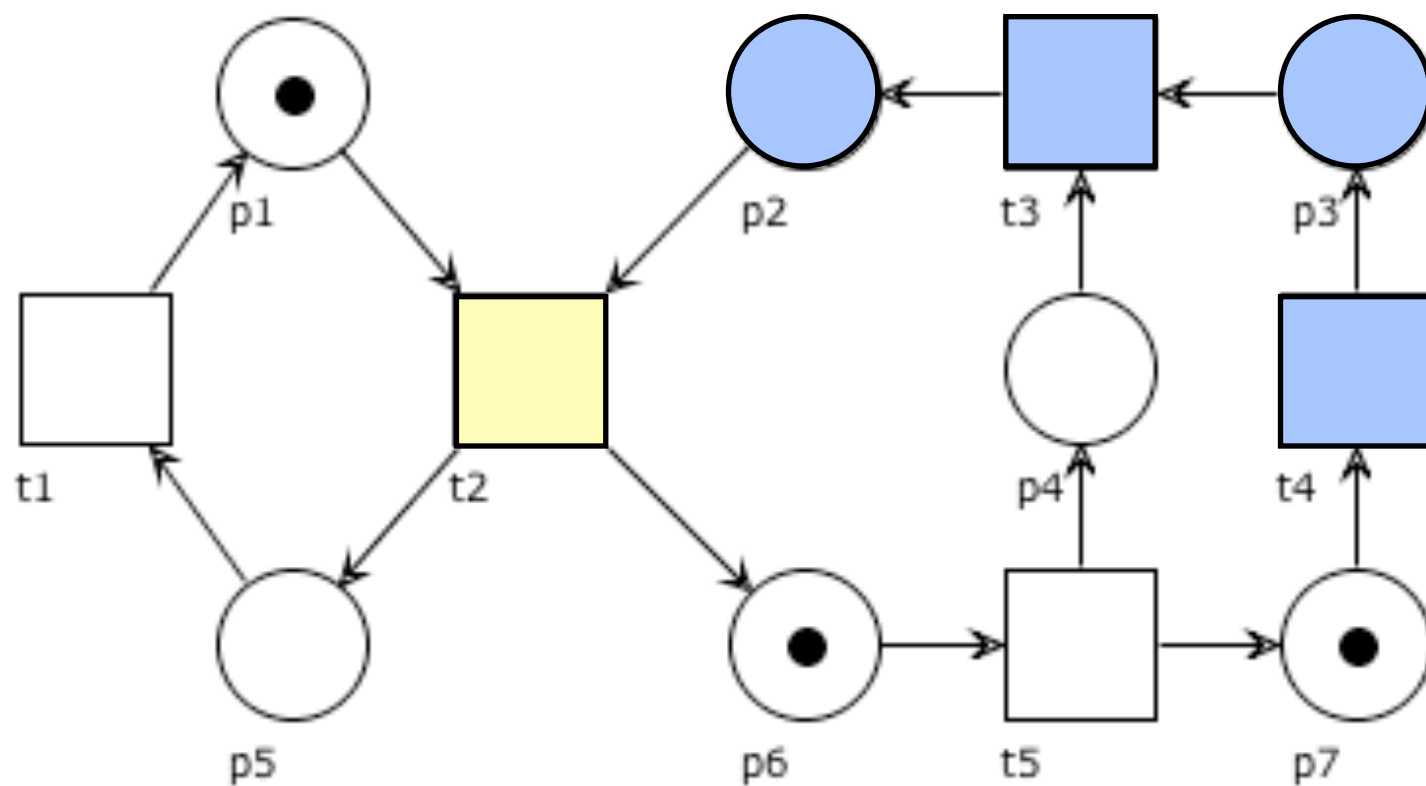
π must contain only places unmarked at M .

By the fundamental property of T-systems: all circuits are marked at M .

π is not necessarily unique, but exists (no cycle in it).

Liveness theorem for T-systems

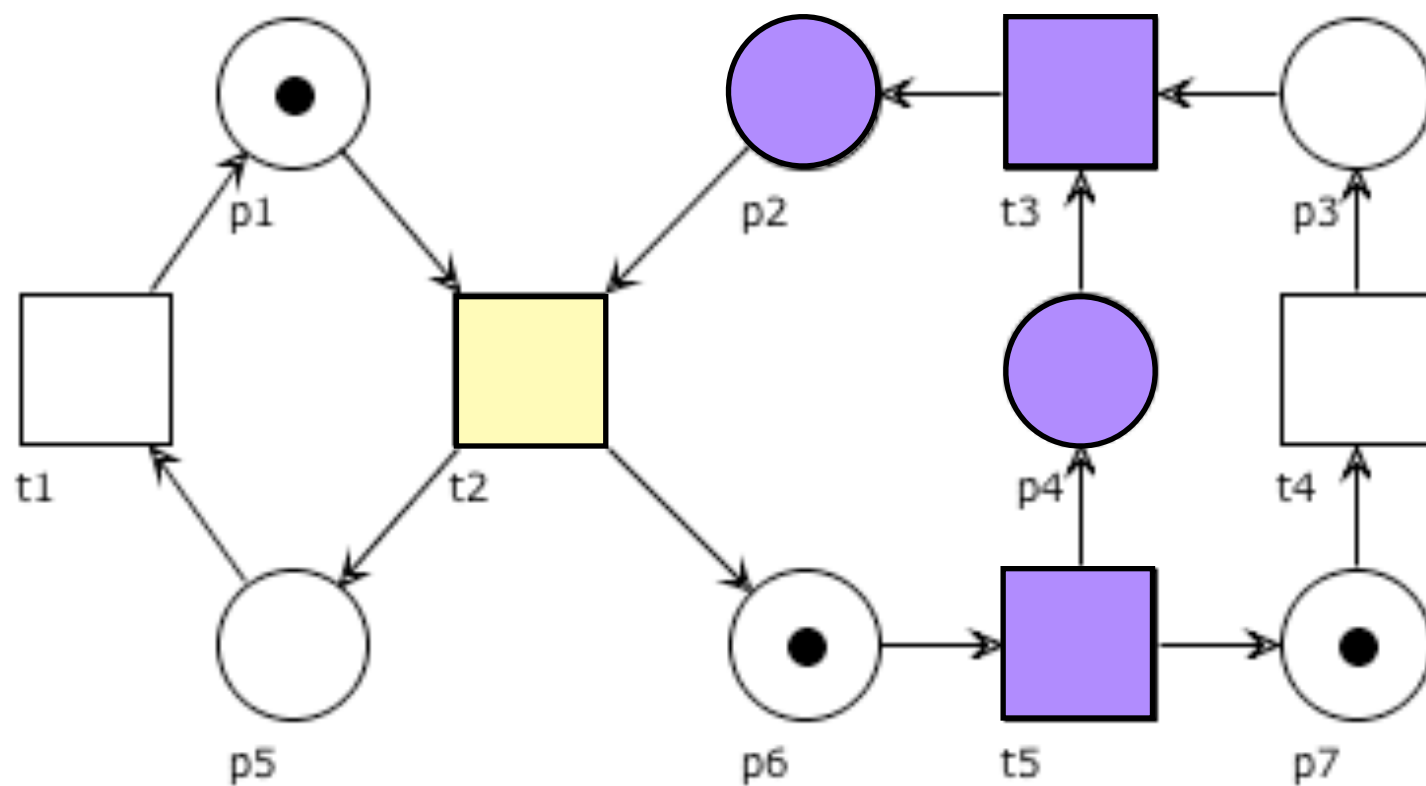
Theorem: A T-system (N, M_0) is live \Leftrightarrow every circuit of N is marked at M_0



$$\pi = t_4 p_3 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live \Leftrightarrow every circuit of N is marked at M_0



$$\pi = t_5 p_4 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

\Leftarrow) (Inductive case: $|P_{M,t}| > 0$, continued proof sketch)

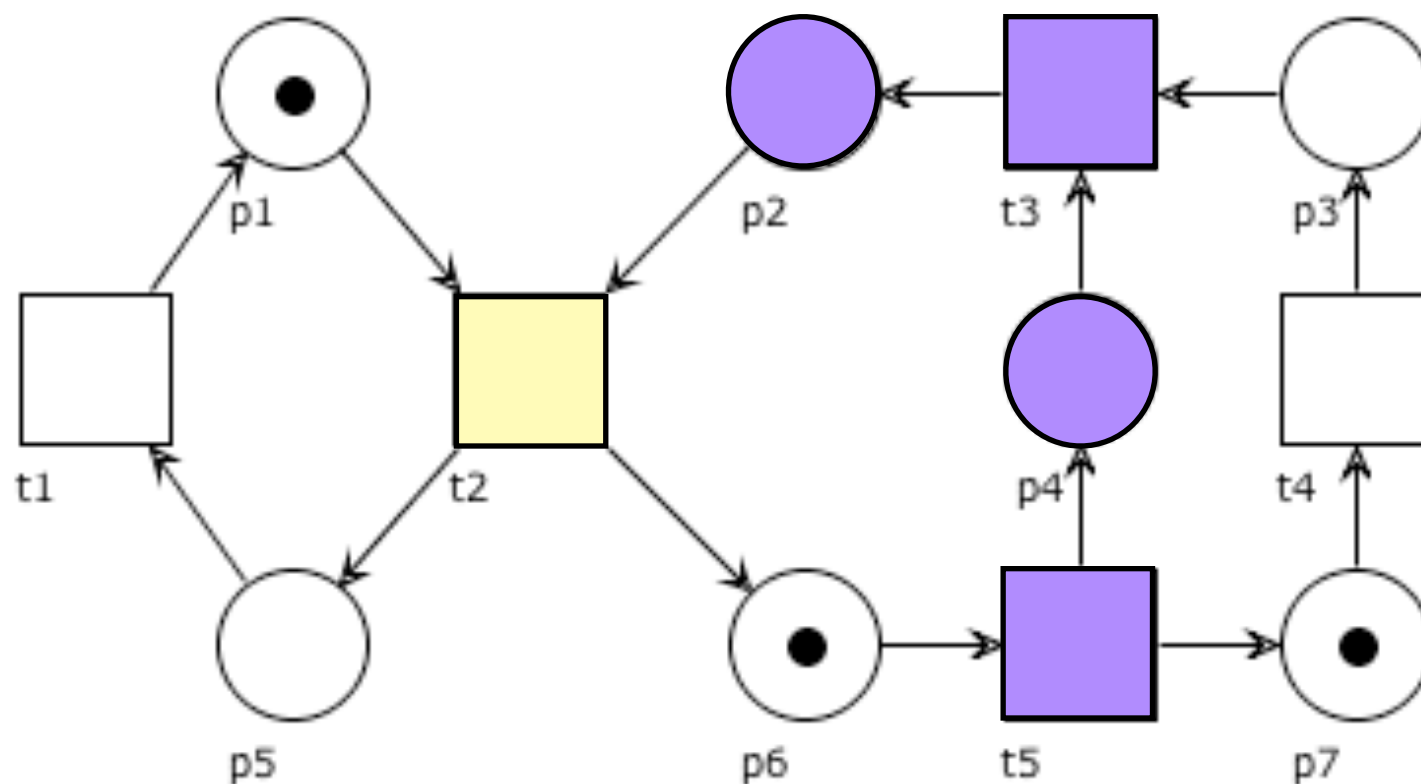
π begins with a transition t' enabled at M .
(otherwise a longer path could be found).

By firing t' we reach a marking M'' such that $P_{M'',t} \subset P_{M,t}$.

Hence $|P_{M'',t}| < |P_{M,t}|$ and we conclude by inductive hypothesis.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
 \Leftrightarrow every circuit of N is marked at M_0

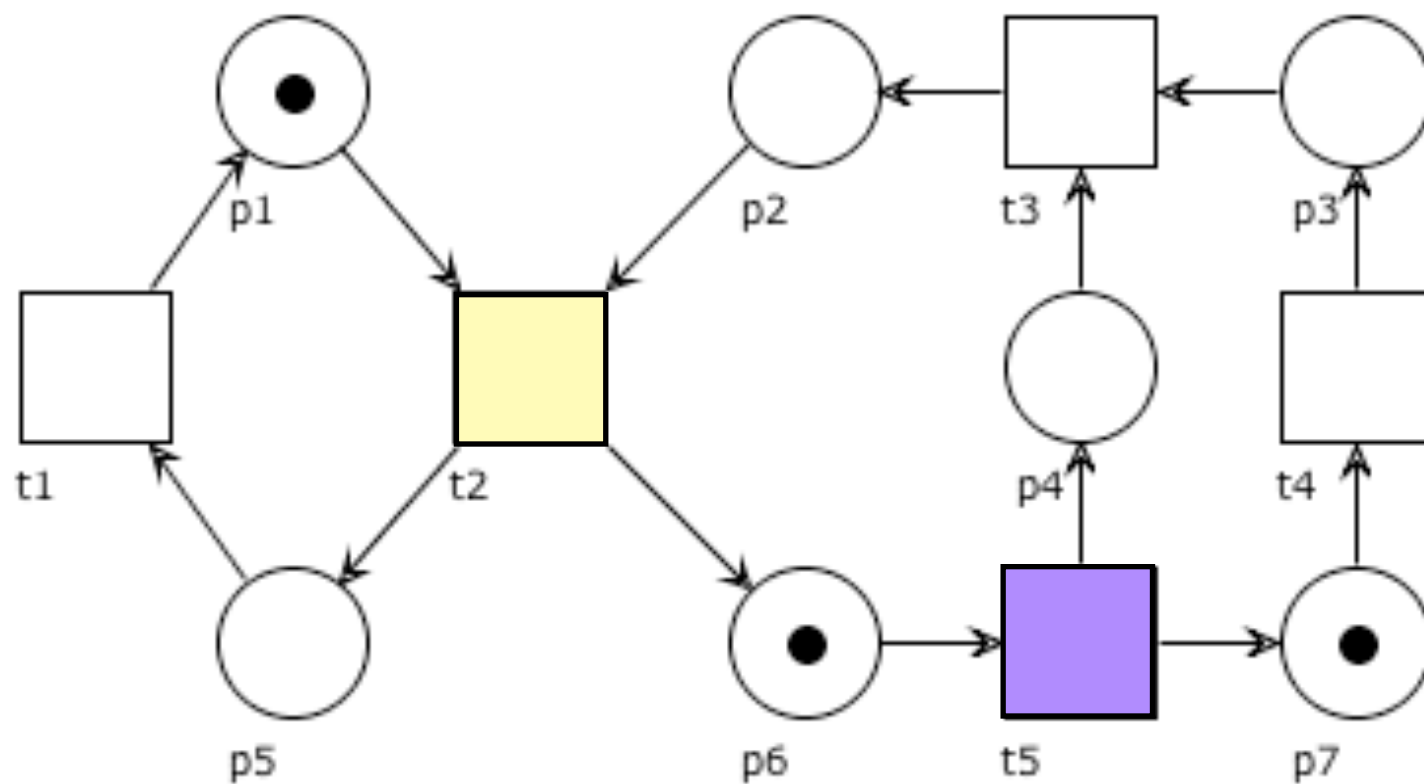


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$\pi = t_5 p_4 t_3 p_2 t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
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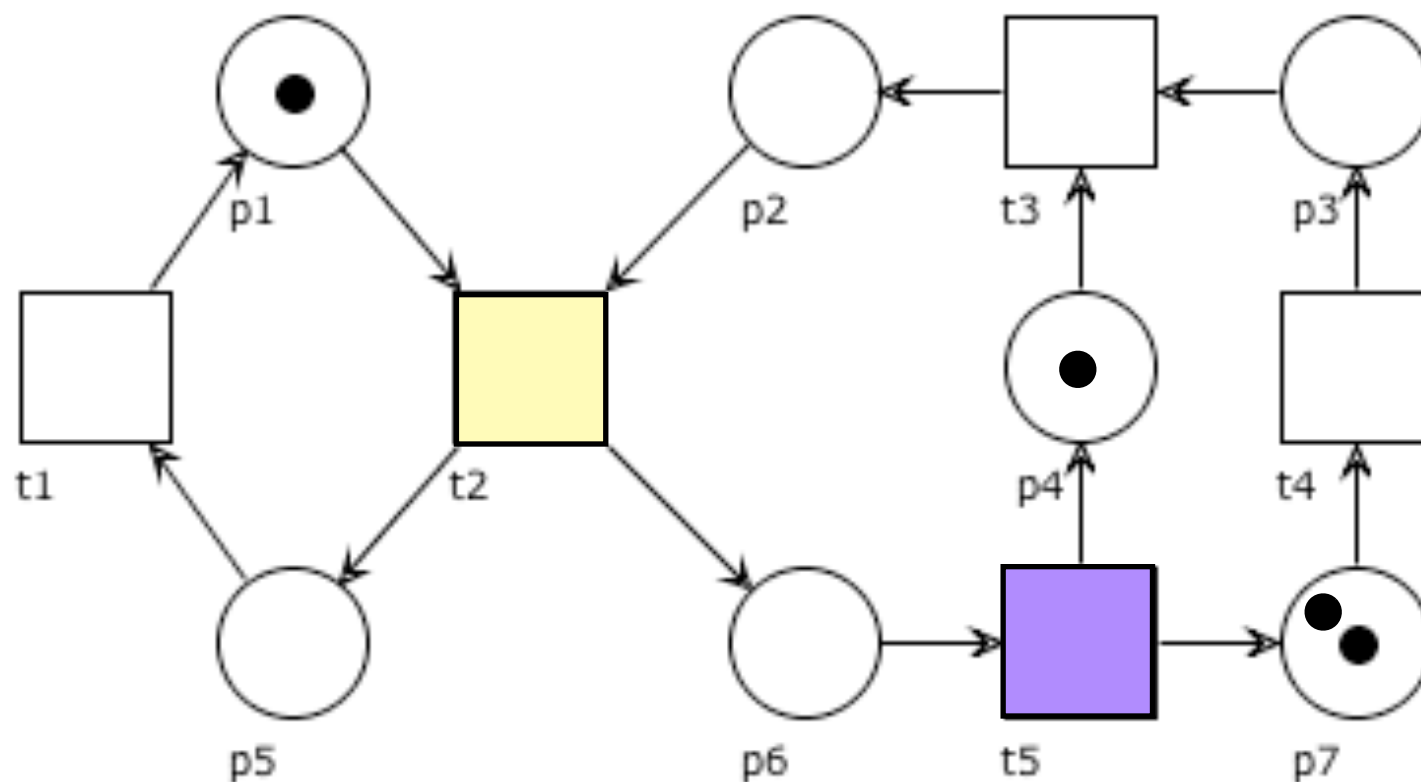


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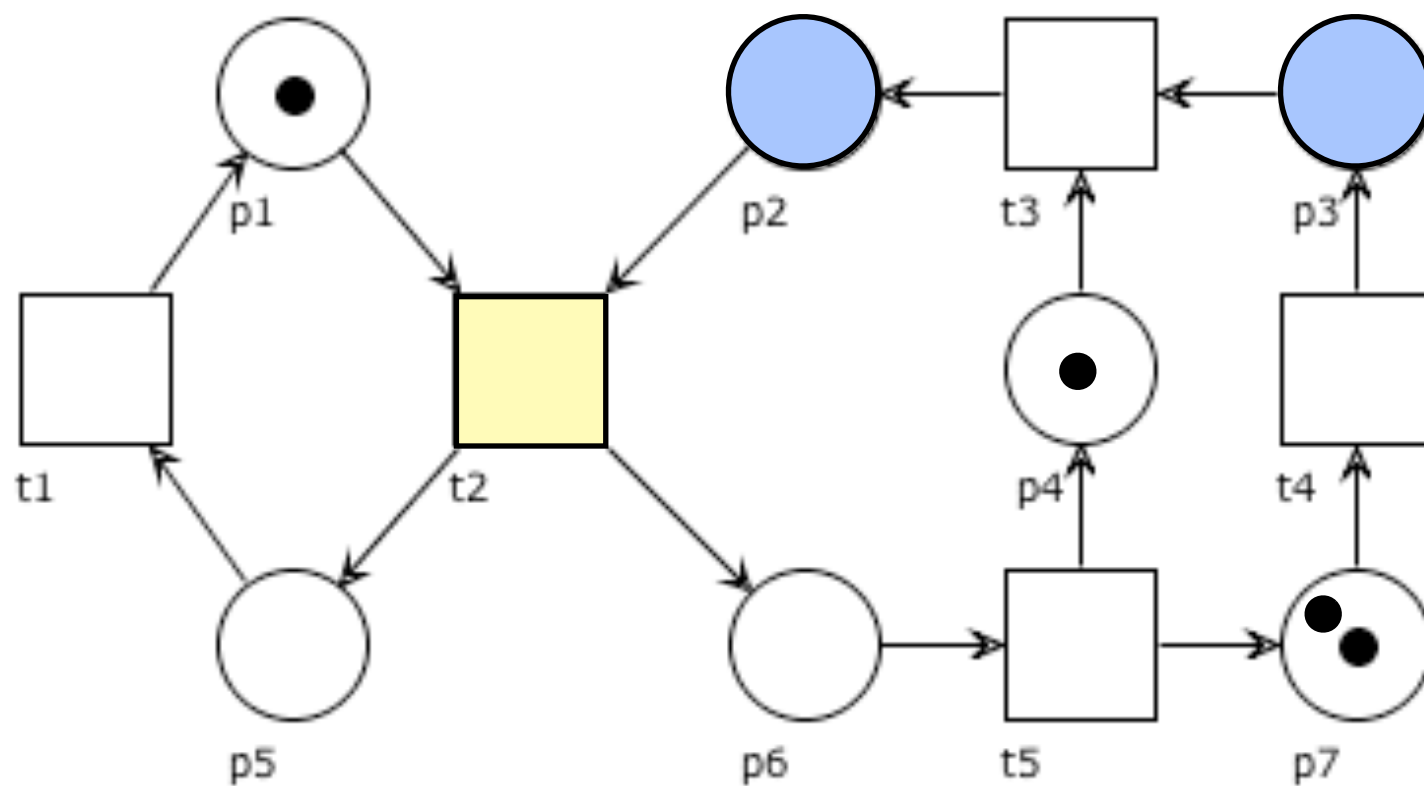


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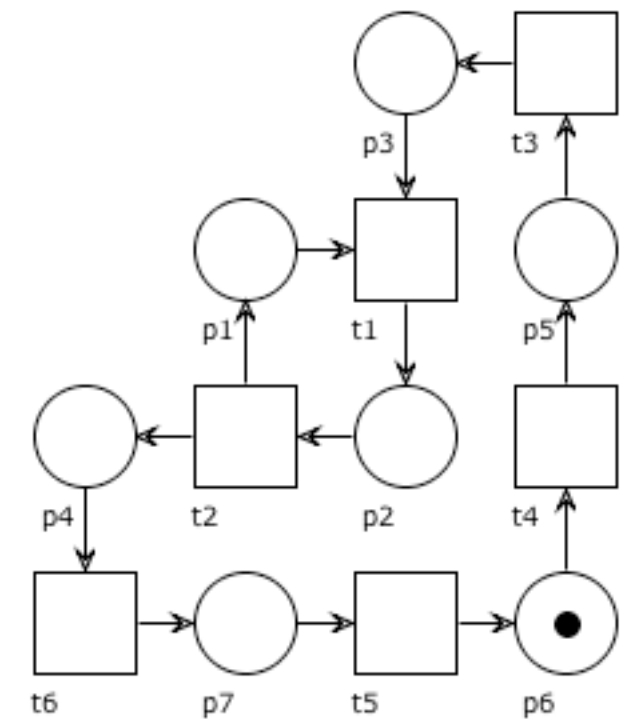
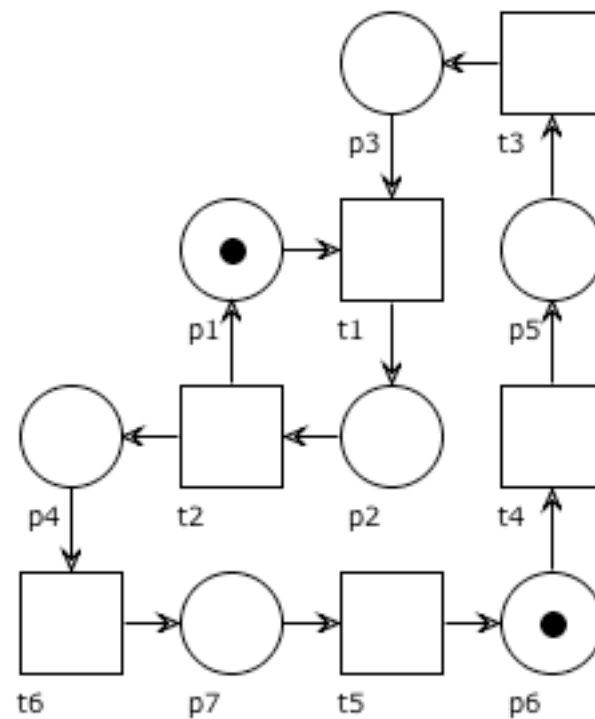
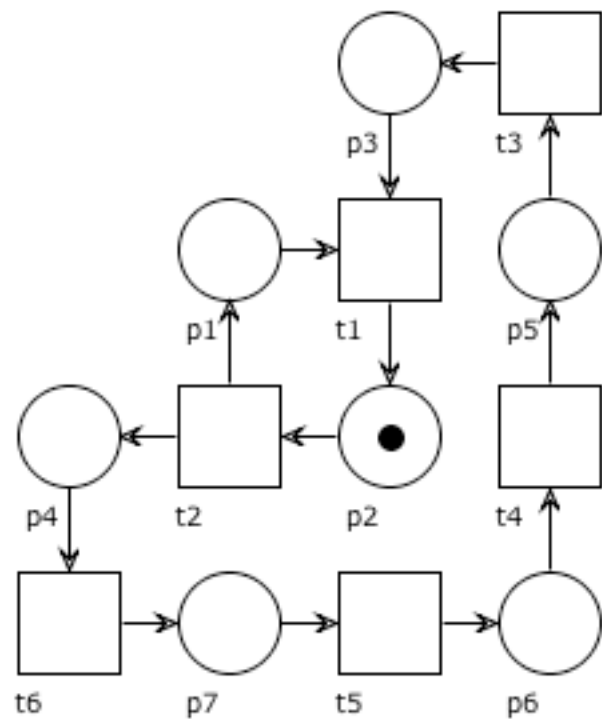


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$P_{M'', t_2} = \{ p_2, p_3 \}$$

Question time

Which of the T-systems below is live? (why?)



Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k -bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

\Leftarrow) Let $M \in [M_0 \rangle$ and take any $p \in P$.

By the fundamental property of T-systems:

$$M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$$

Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k -bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

\Rightarrow) Let $k_p \leq k$ be the bound of p .
Take $M \in [M_0]$ with $M(p) = k_p$.

Define $L = M - k_p p$ and note that the T-system (N, L) is not live.
(otherwise $L \xrightarrow{\sigma} L'$ with $L'(p) > 0$ for enabling $t \in p\bullet$. But then:
 $M = L + k_p p \xrightarrow{\sigma} L' + k_p p = M'$ with $M'(p) = L'(p) + k_p > k_p!$)

By the liveness theorem: some circuit γ is not marked at L .
Since (N, M) is live, the circuit γ is marked at $M \supset L$.
Since $M - L = k_p p$, the circuit γ contains p and
 $M_0(\gamma) = M(\gamma) = M(p) = k_p \leq k$.

Place bounds in live T-systems

Let (P, T, F, M_0) be a **live** T-system.

We can draw some easy consequences of the above results:

- 1) If $p \in P$ is bounded, then it belongs to some circuit.
(see part \Rightarrow of the proof of the boundedness theorem)
- 2) If $p \in P$ belongs to some circuit, then it is bounded.
(by the fundamental property of T-systems)
- 3) If (N, M_0) is bounded, then it is strongly connected.
(by strong connectedness theorem, holding for any system)
- 4) If N is strongly connected, then (N, M_0) is bounded.
(by 1, since any $p \in P$ belongs to a circuit by strong connectdness)

Place bounds in live T-systems

Let (P, T, F, M_0) be a **live** T-system.

We can draw some easy consequences of the above results:

1+2) $p \in P$ is bounded iff it belongs to some circuit.

3+4) (N, M_0) is bounded iff it is strongly connected.

T-systems: recap

T-system + γ circuit + M reachable $\Rightarrow M(\gamma) = M_0(\gamma)$

T-system + γ circuit + $M(\gamma) \neq M_0(\gamma) \Rightarrow M$ not reachable

T-system + $\gamma_1 \dots \gamma_n$ circuits: $\exists i. p \in \gamma_i \Leftrightarrow p$ bounded

T-system: $M_0(\gamma) > 0$ for all circuits $\gamma \Leftrightarrow$ live

T-system: strongly connected \Rightarrow bounded

T-system + live: strongly connected \Leftrightarrow bounded

T-system: T-invariant $\mathbf{J} \Leftrightarrow \mathbf{J} = [k \ k \ \dots \ k]$

Consequences on workflow nets

Theorem: If N is a workflow net s.t. N^* is a T-system then
 N is safe and sound **iff**
every circuit of N^* is marked

N workflow net $\Rightarrow N^*$ strongly connected

all circuits of N^* are marked $\Leftrightarrow N^*$ live

N^* strongly connected + N^* T-system $\Rightarrow N^*$ bounded

$i \in \gamma \Leftrightarrow M_0(\gamma)=1 \Leftrightarrow \gamma$ marked circuit

γ marked circuit + M reachable $\Rightarrow M(\gamma)=1$

N^* bounded \Leftrightarrow any place p belongs to a circuit of N^*

p belongs to a circuit of N^* $\Rightarrow p$ is safe

all places belong to marked circuits $\Rightarrow N^*$ safe $\Rightarrow N$ safe

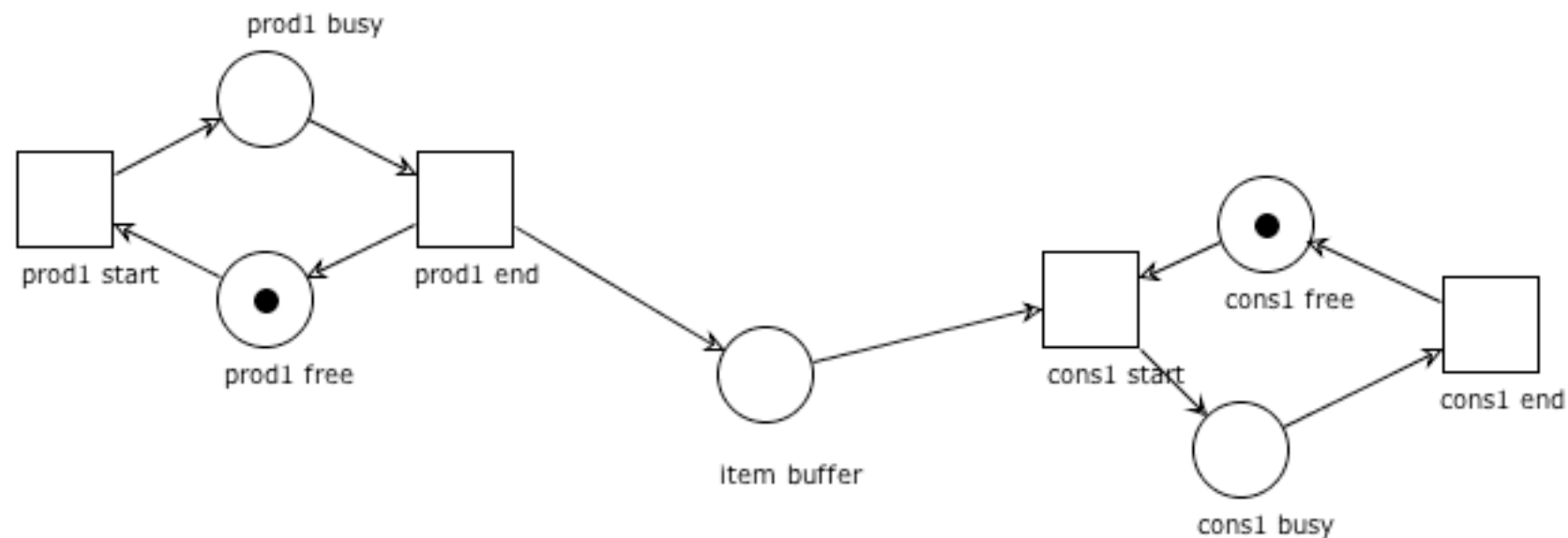
Exercises

Which are the circuits of the T-system below?

Is the T-system below live? (why?)

Which places are bounded? (why?)

Assign a bound to each bounded place.



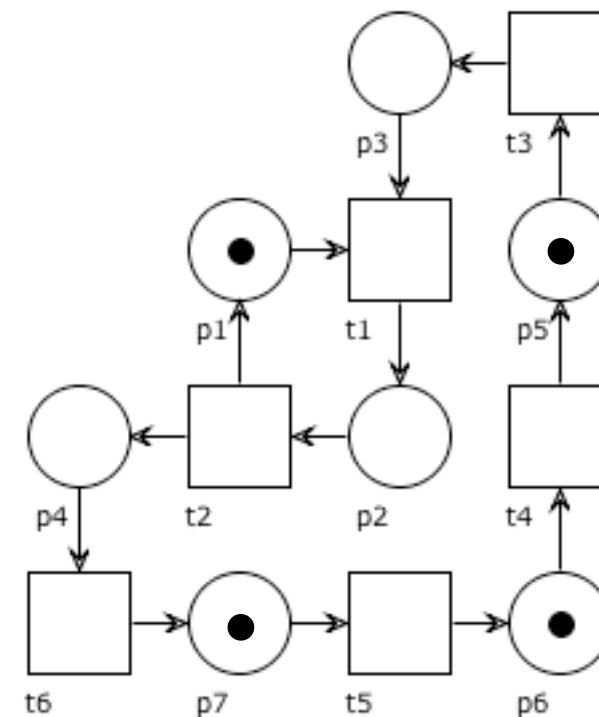
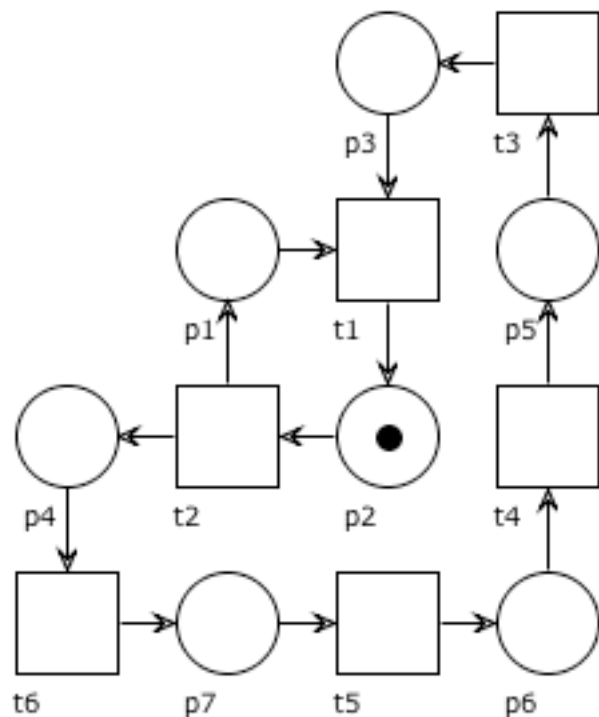
Exercises

Which are the circuits of the T-systems below?

Are the T-systems below live? (why?)

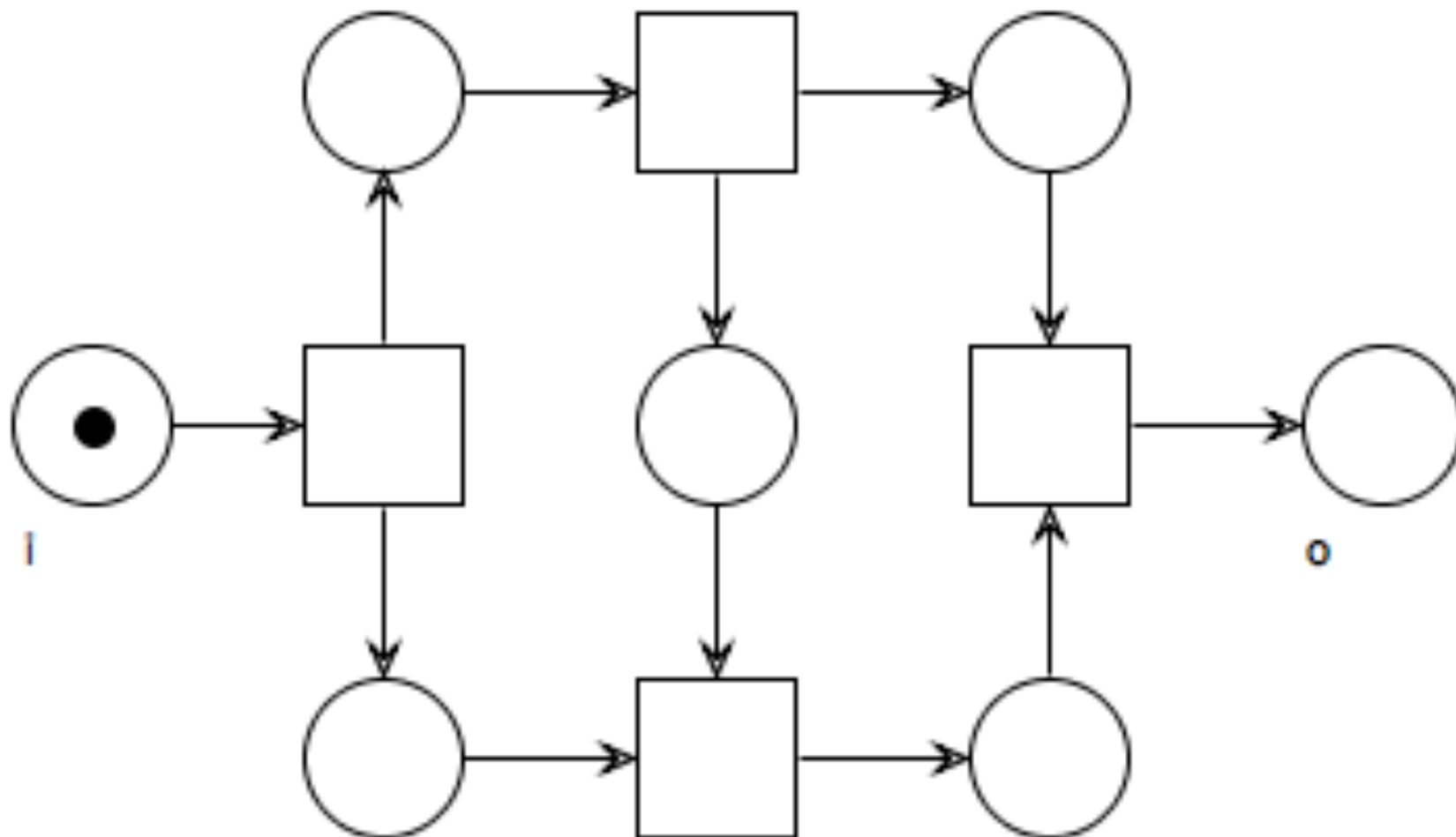
Which places are bounded? (why?)

Assign a bound to each bounded place.



Exercise

Is the net below a workflow net?
Is it sound?



Exercise

Is the net below a workflow net?

Is it sound?

