

Business Processes Modelling

MPB (6 cfu, 295AA)

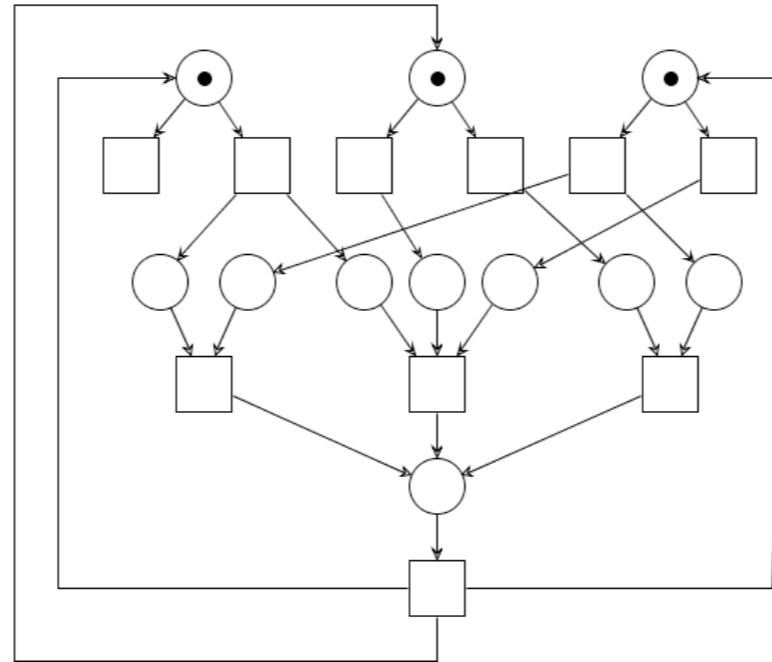
Roberto Bruni

<http://www.di.unipi.it/~bruni>

18 - Free-choice nets



Object



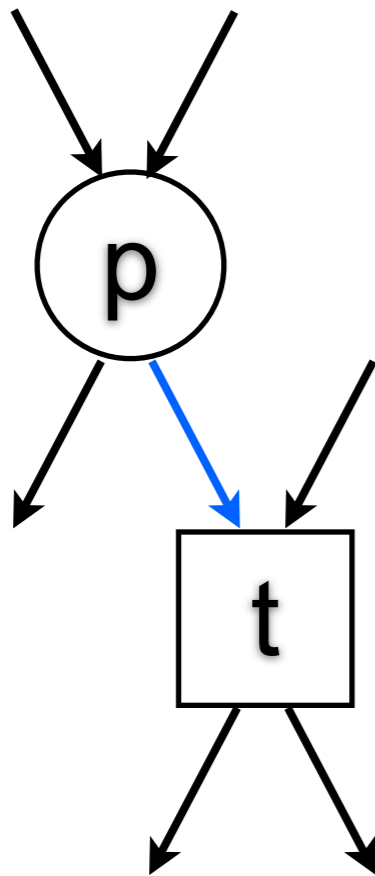
We study some “good” properties of free-choice nets

Free Choice Nets (book, optional reading)

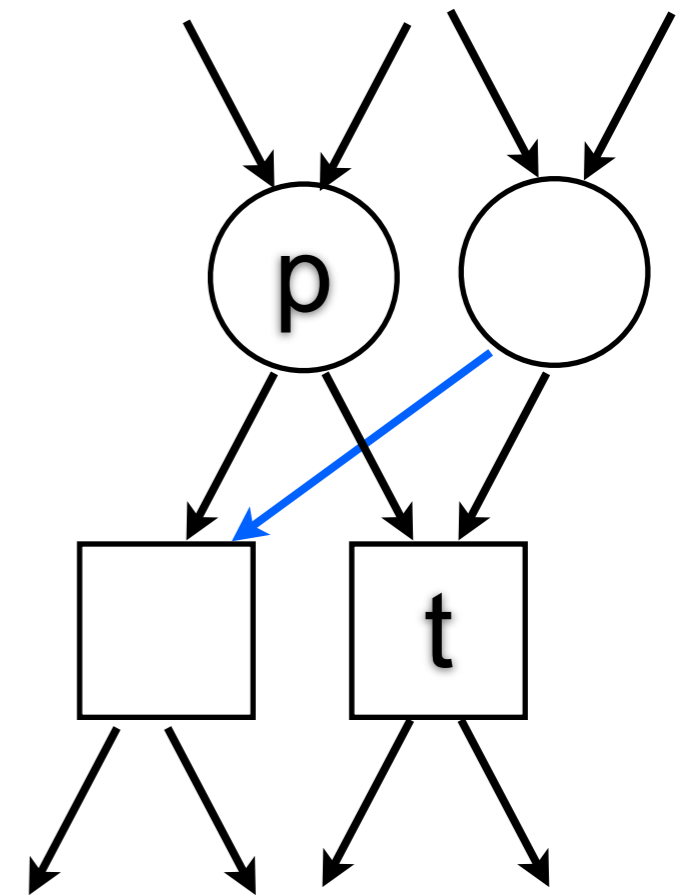
<https://www7.in.tum.de/~esparza/bookfc.html>

Free-choice net

Definition: We recall that a net N is **free-choice** if whenever there is an arc (p,t) , then there is an arc from any input place of t to any output transition of p



implies



Free-choice net: alternative definitions

Proposition: All the following definitions of free-choice net are equivalent.

1) A net (P, T, F) is free-choice if:

$$\forall p \in P, \forall t \in T, (p, t) \in F \text{ implies } \bullet t \times p \bullet \subseteq F.$$

2) A net (P, T, F) is free-choice if:

$$\forall p, q \in P, \forall t, u \in T, \{(p, t), (q, t), (p, u)\} \subseteq F \text{ implies } (q, u) \in F.$$

3) A net (P, T, F) is free-choice if:

$$\forall p, q \in P, \text{ either } p \bullet = q \bullet \text{ or } p \bullet \cap q \bullet = \emptyset.$$

4) A net (P, T, F) is free-choice if:

$$\forall t, u \in T, \text{ either } \bullet t = \bullet u \text{ or } \bullet t \cap \bullet u = \emptyset.$$

Free-choice net: my favourite definition

4) A net (P, T, F) is free-choice if:

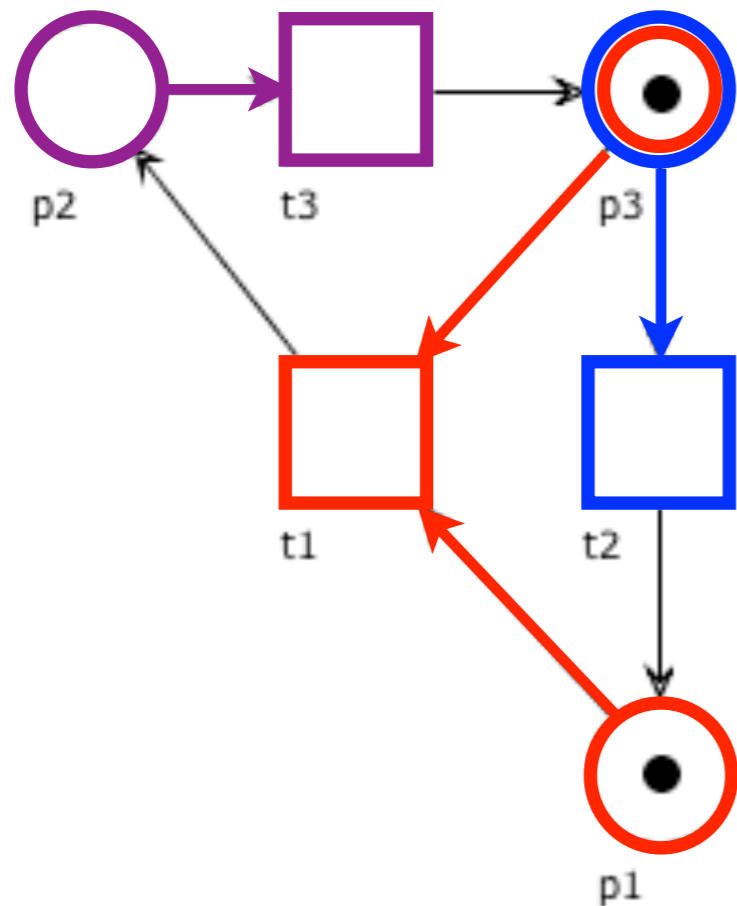
$\forall t, u \in T$, either $\bullet t = \bullet u$ or $\bullet t \cap \bullet u = \emptyset$.

A system (N, M_0) is **free-choice** if N is free-choice

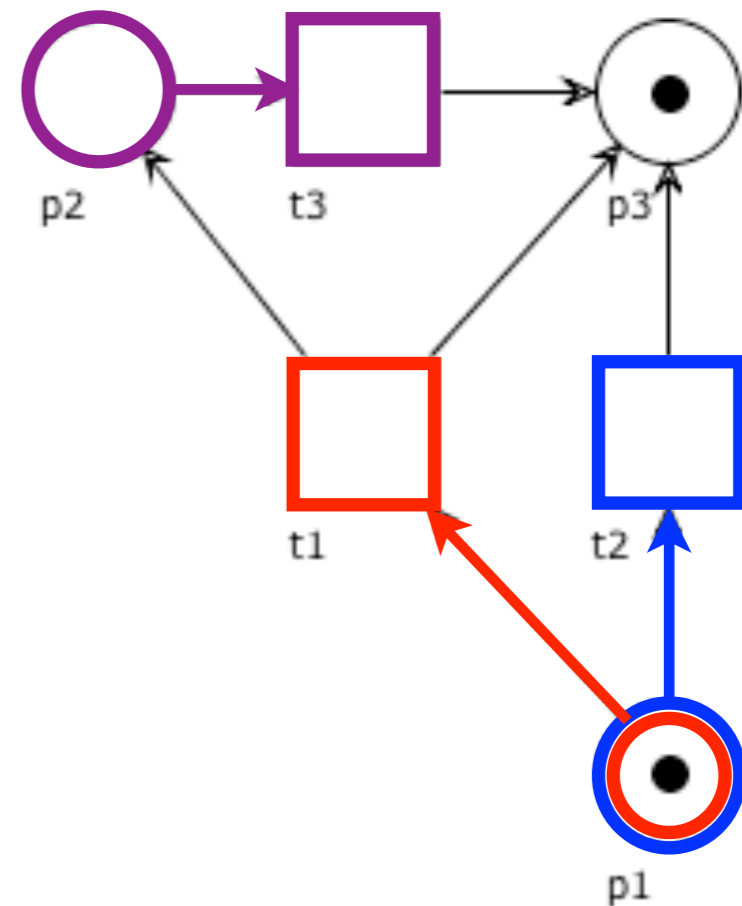
Example

$$\begin{aligned} \bullet t_1 &= \{p_1, p_3\} \\ \bullet t_2 &= \{p_3\} \\ \bullet t_1 &\neq \bullet t_2 \\ \bullet t_1 \cap \bullet t_2 &= \{p_3\} \neq \emptyset \end{aligned}$$

$$\begin{aligned} \bullet t_1 &= \bullet t_2 \\ \bullet t_1 \cap \bullet t_3 &= \emptyset \\ \bullet t_2 \cap \bullet t_3 &= \emptyset \end{aligned}$$



non free-choice



free-choice

Fundamental property of free-choice nets

Proposition: Let (P, T, F, M_0) be free-choice.

If $M \xrightarrow{t}$ and $t \in p\bullet$, then $M \xrightarrow{t'}$ for every $t' \in p\bullet$.

The proof is trivial, by definition of free-choice net

$$(t, t' \in p\bullet \text{ implies } \bullet t = \bullet t')$$

Free-choice N^*

Proposition: A workflow net N is free-choice
iff
 N^* is free-choice

N and N^* differ only for the reset transition,
whose pre-set (o) is disjoint
from the pre-set of any other transition

Liveness = Place liveness (in Free Choice systems)

In any system:

liveness implies place-liveness

p dead implies any transition t in its pre/post-set is dead

It can be shown that

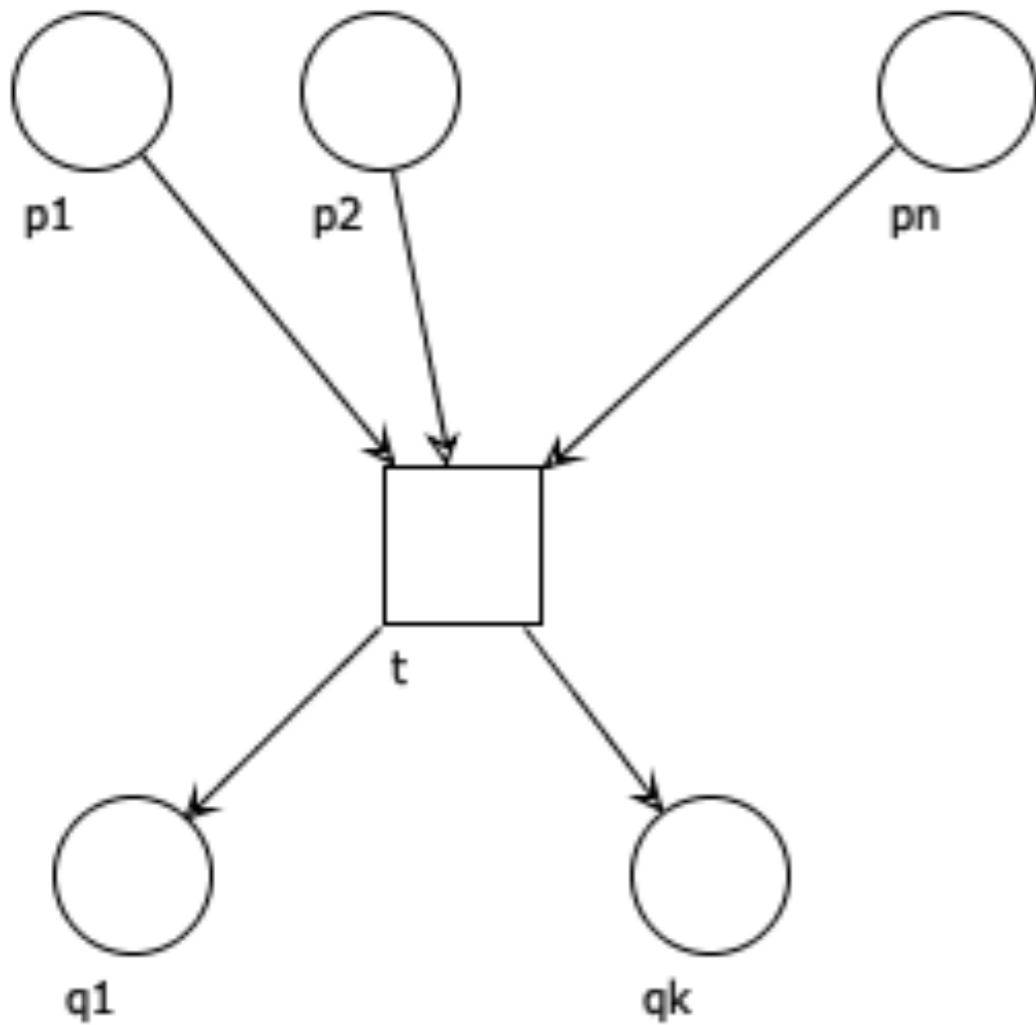
If a free-choice system is place-live, then it is live

Corollary:

A free-choice system is live **iff** it is place-live

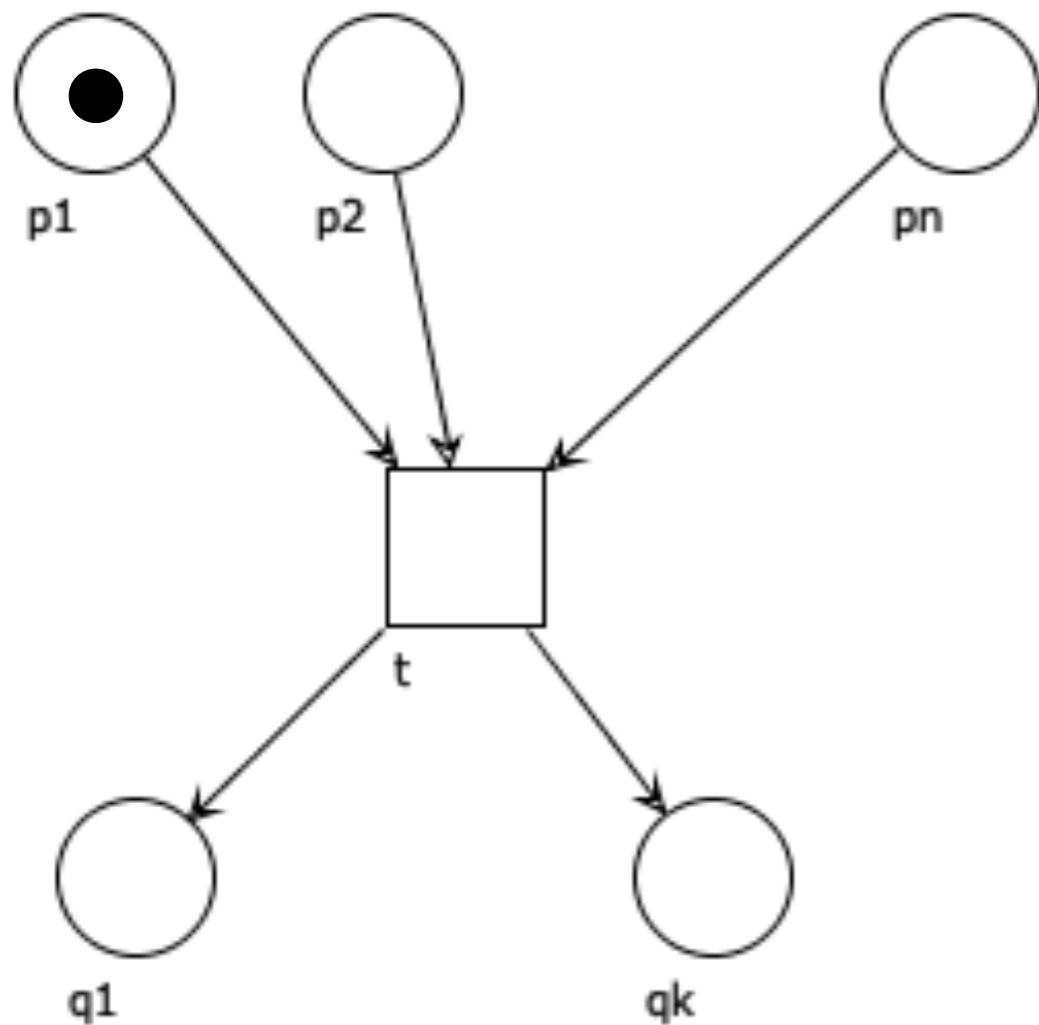
FC Place-live implies FC Live (intuition)

From a reachable marking M we would like to enable t



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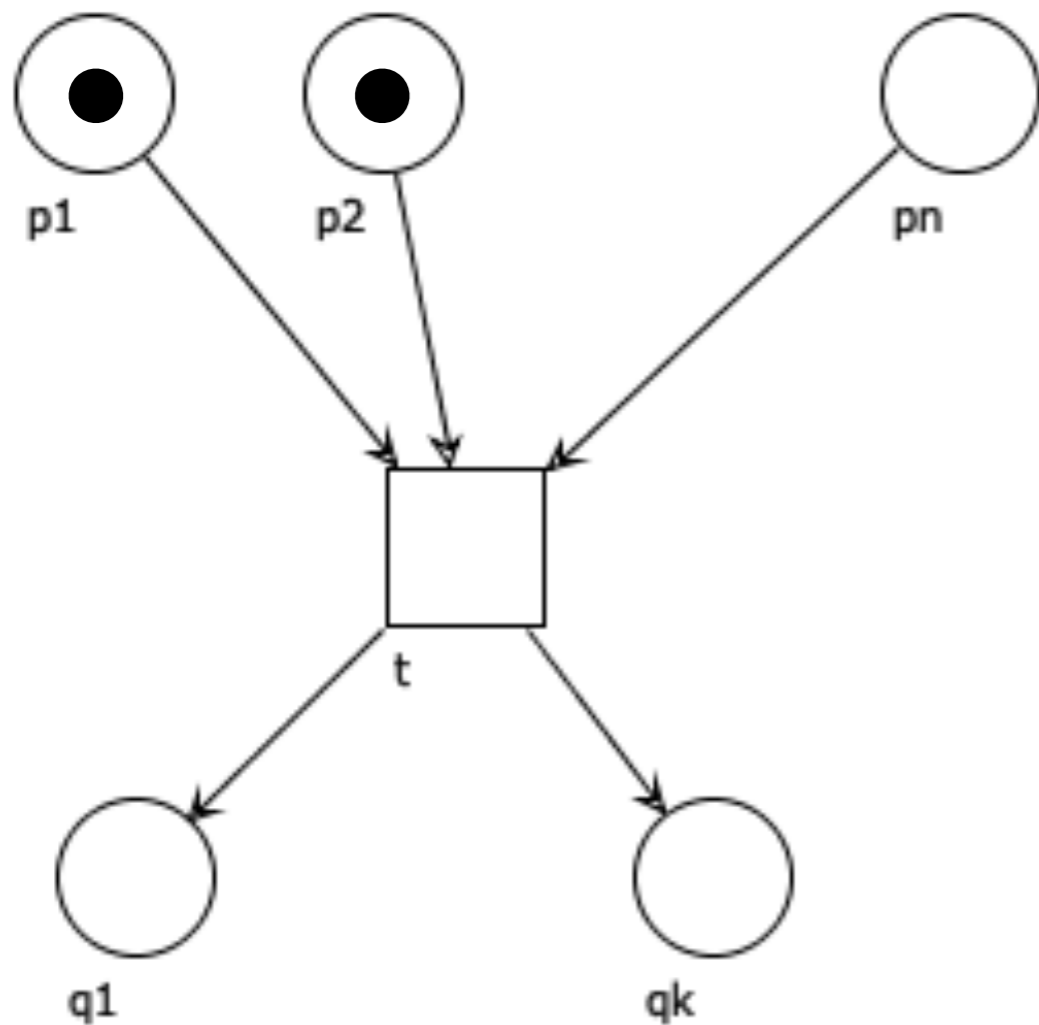
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from M we can reach M_1 that marks p_1 (because place-live)

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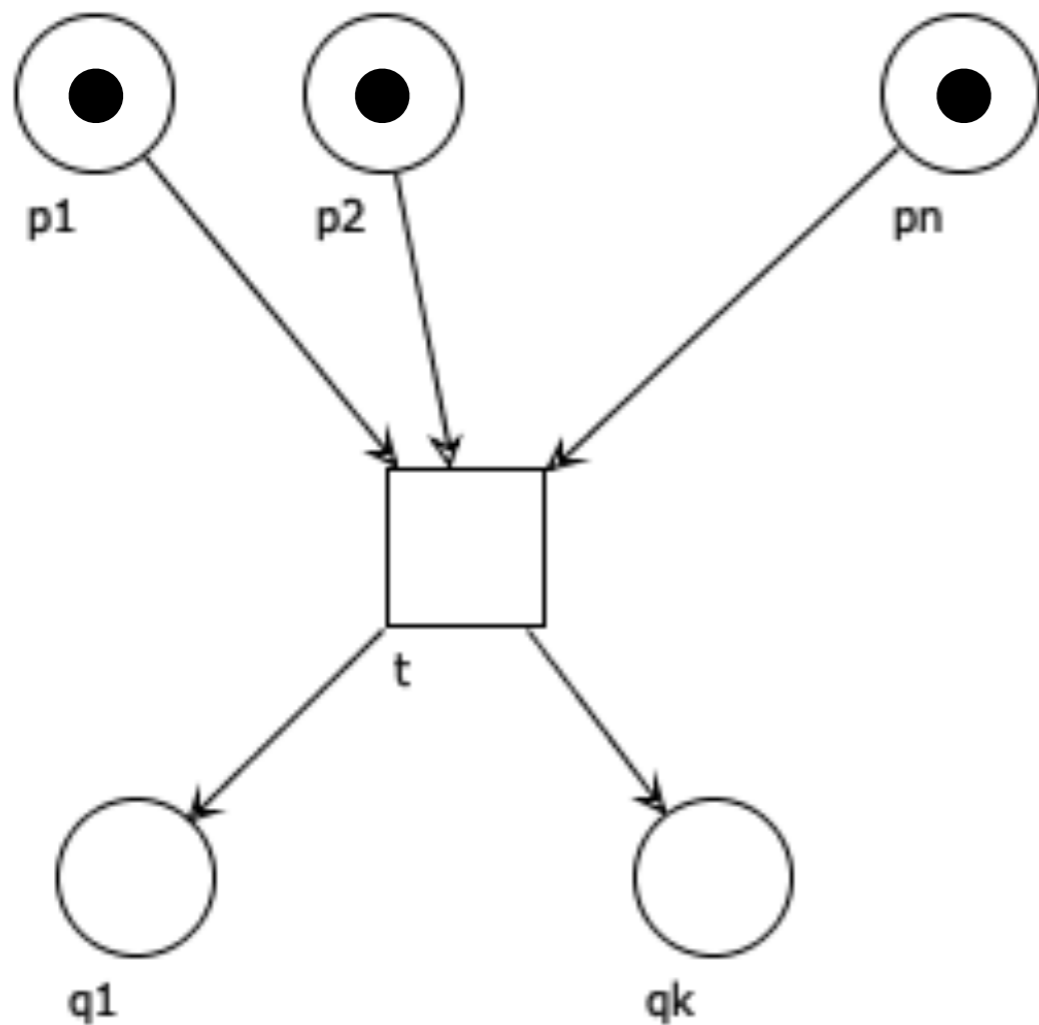


from M we can reach M_1 that marks p_1 (because place-live)
from M_1 we can reach M_2 that marks p_2 (because place-live)

Note: the token remains in p_1 (fundamental property of FC: if t' can remove a token from p_1 , then t' has the same preset as t)

FC Place-live implies FC Live (intuition)

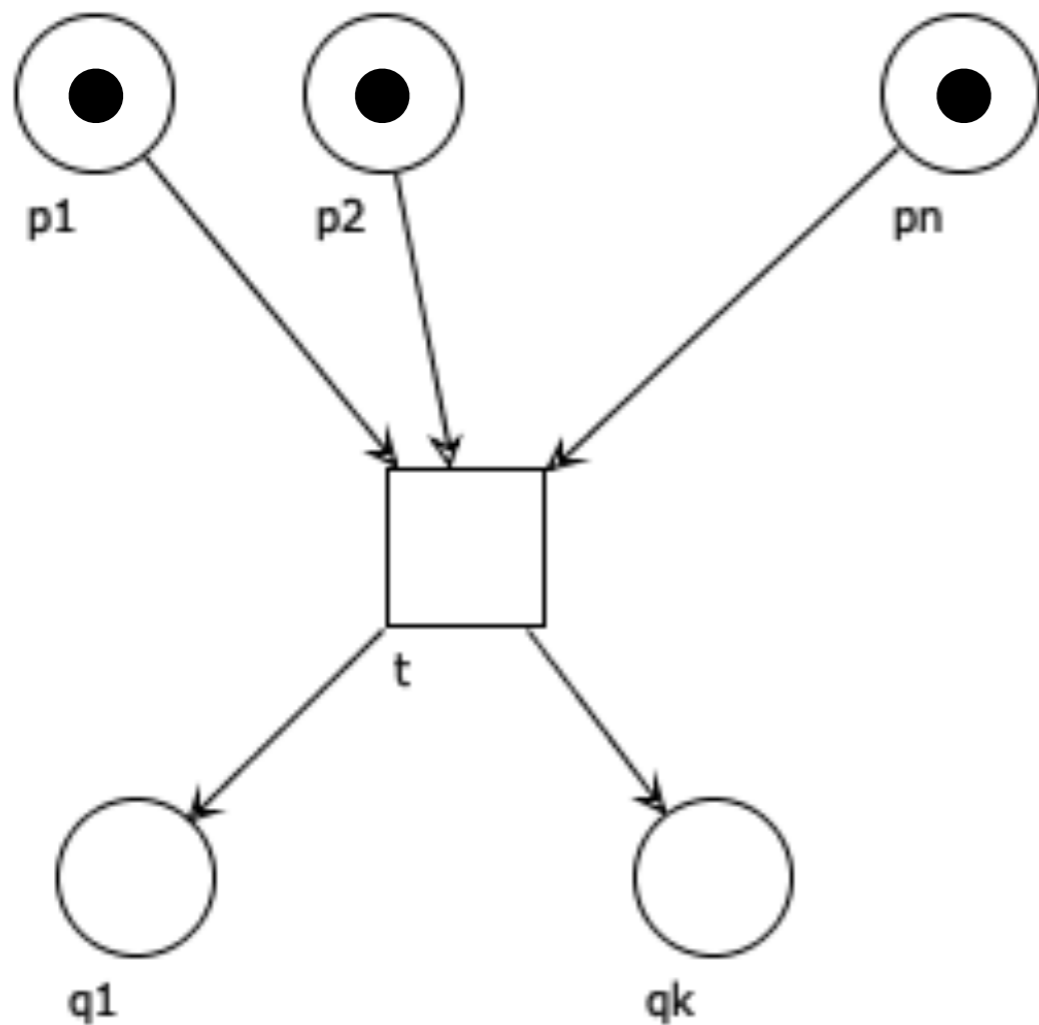
From a reachable marking M we would like to enable t



from M we can reach M_1 that marks p_1 (because place-live)
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...
from M_{n-1} we can reach M_n that marks p_n (because place-live)

FC Place-live implies FC Live (intuition)

From a reachable marking M we would like to enable t



from M we can reach M_1 that marks p_1 (because place-live)
from M_1 we can reach M_2 that marks p_2 (because place-live)
...
from M_{n-1} we can reach M_n that marks p_n (because place-live)
from M we reach M_n that enables t !

Commoner's theorem (proof omitted)

Theorem:

A free-choice system is live
iff

every proper **siphon** includes an initially marked **trap**

Rank Theorem

(main result, proof omitted)

Theorem:

A free-choice system (P, T, F, M_0) is live and bounded
iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper **siphon**
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |\mathbf{C}_N| - 1$

(where \mathbf{C}_N is the set of **clusters**)

Coming next

What is a cluster?

What is a siphon?

What is a trap?

Is it hard to show that a free-choice net is live?

Is it hard to show that a free-choice net is live and bounded?

What is a cluster?

Cluster

Let x be the node of a net $N = (P, T, F)$
(not necessarily free-choice)

Definition:

The **cluster** of x , written $[x]$, is the least set s.t.

1. $x \in [x]$

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1. $x \in [x]$

2. if $p \in [x] \cap P$ then $p \bullet \subseteq [x]$ (if a place p is in the cluster,
then all transitions in the
post-set of p are in the cluster)

Cluster

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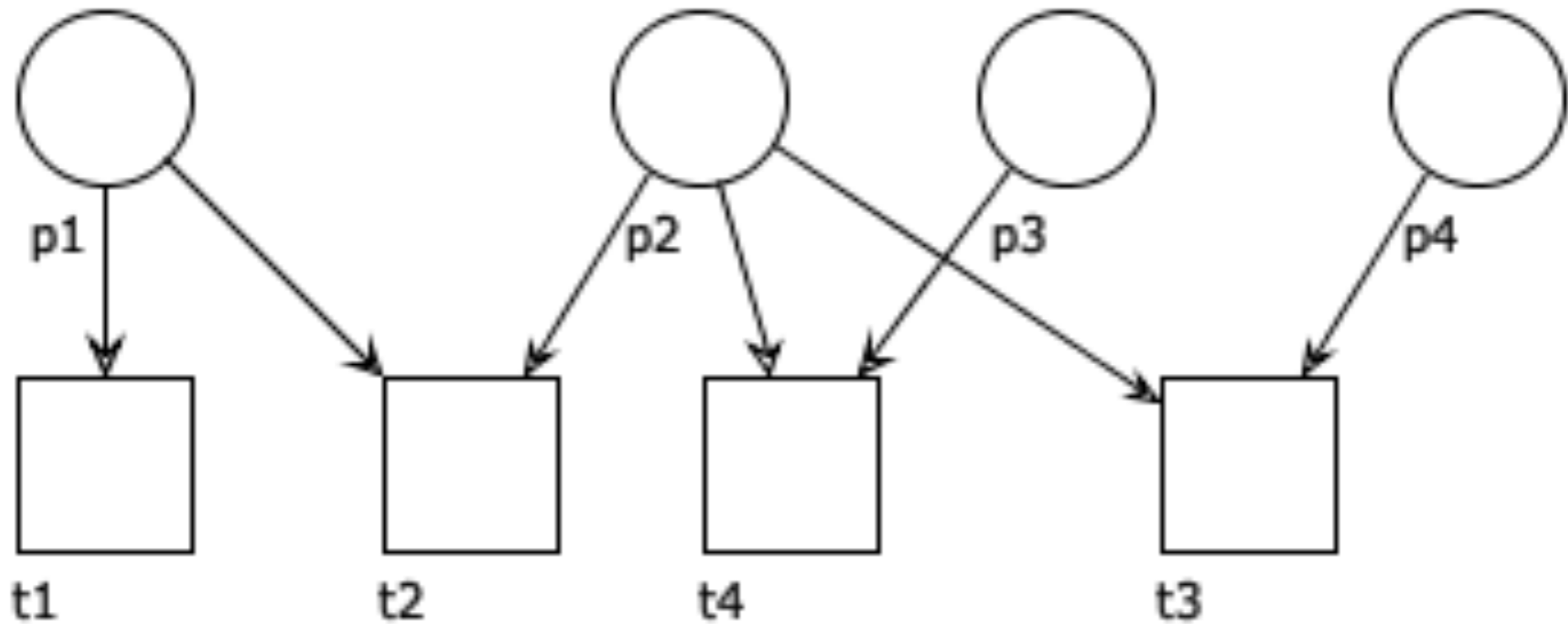
Definition:

The **cluster** of x , written $[x]$, is the least set s.t.

1. $x \in [x]$
2. if $p \in [x] \cap P$ then $p \bullet \subseteq [x]$ (if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)
3. if $t \in [x] \cap T$ then $\bullet t \subseteq [x]$ (if a transition t is in the cluster, then all places in the pre-set of t are in the cluster)

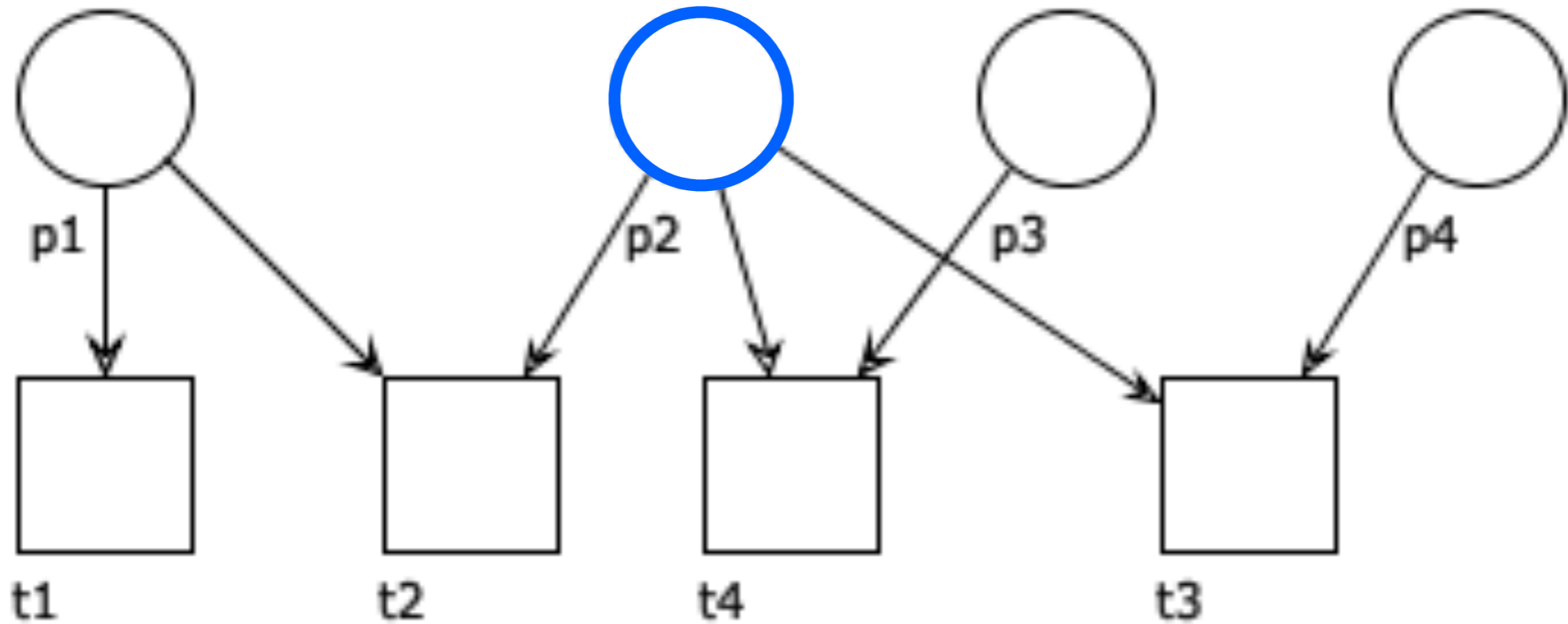
Cluster: intuition

[p2] = ?



Cluster: intuition

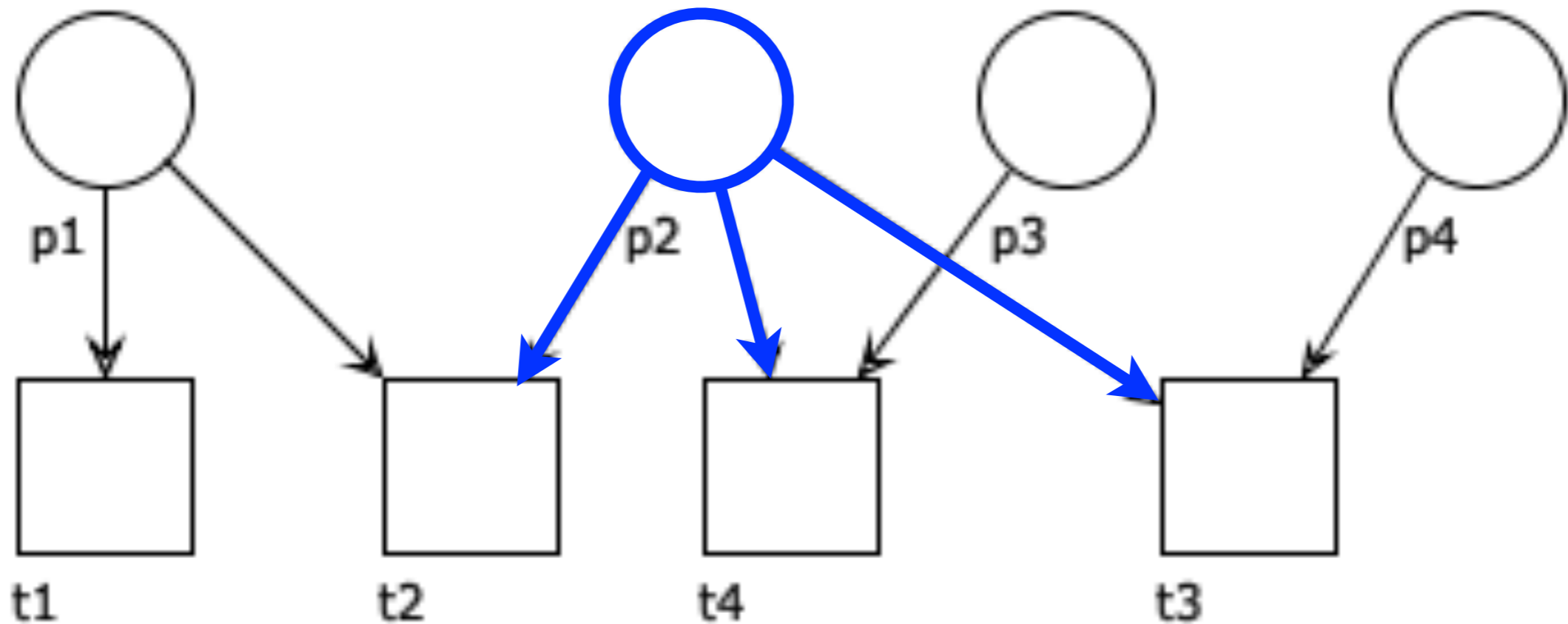
$[p2] = \{p2, \dots\}$



Cluster: intuition

$$[p_2] = \{p_2, \dots\}$$

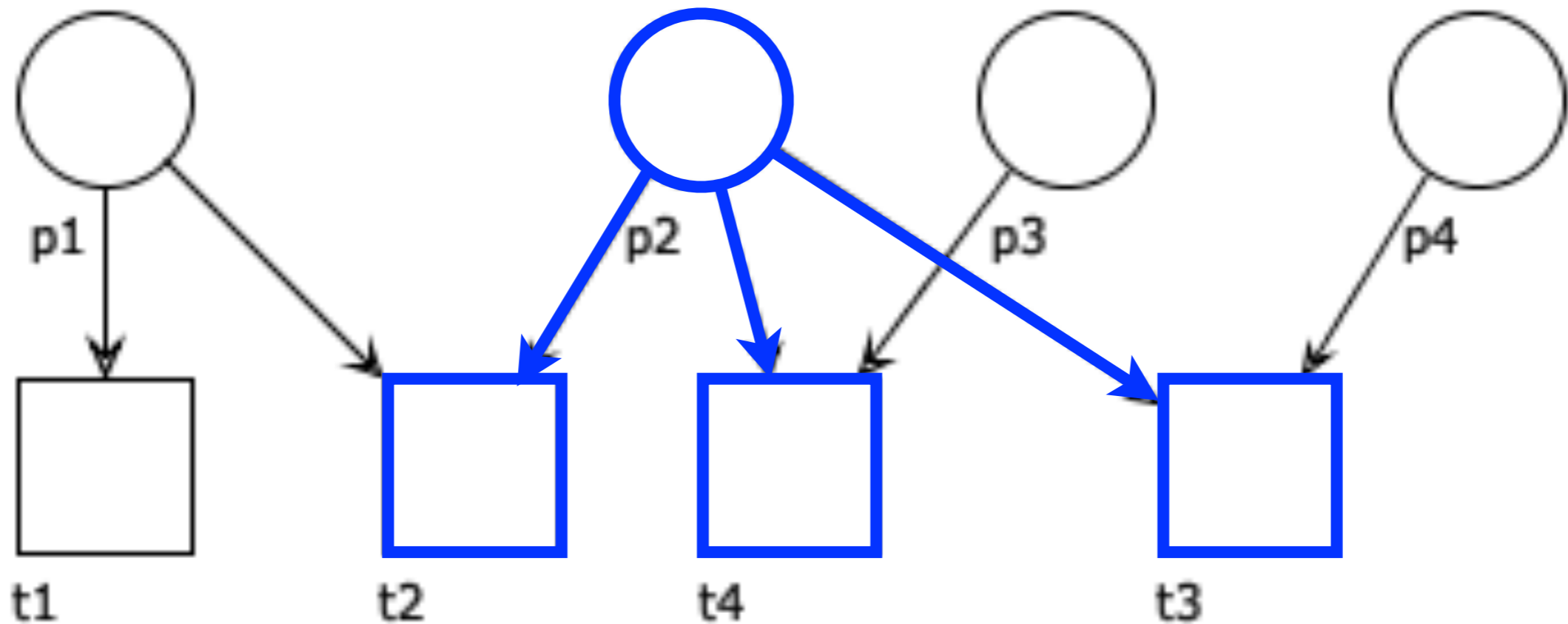
(if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)



Cluster: intuition

$$[p2] = \{ p2 , t2, t4, t3, \dots \}$$

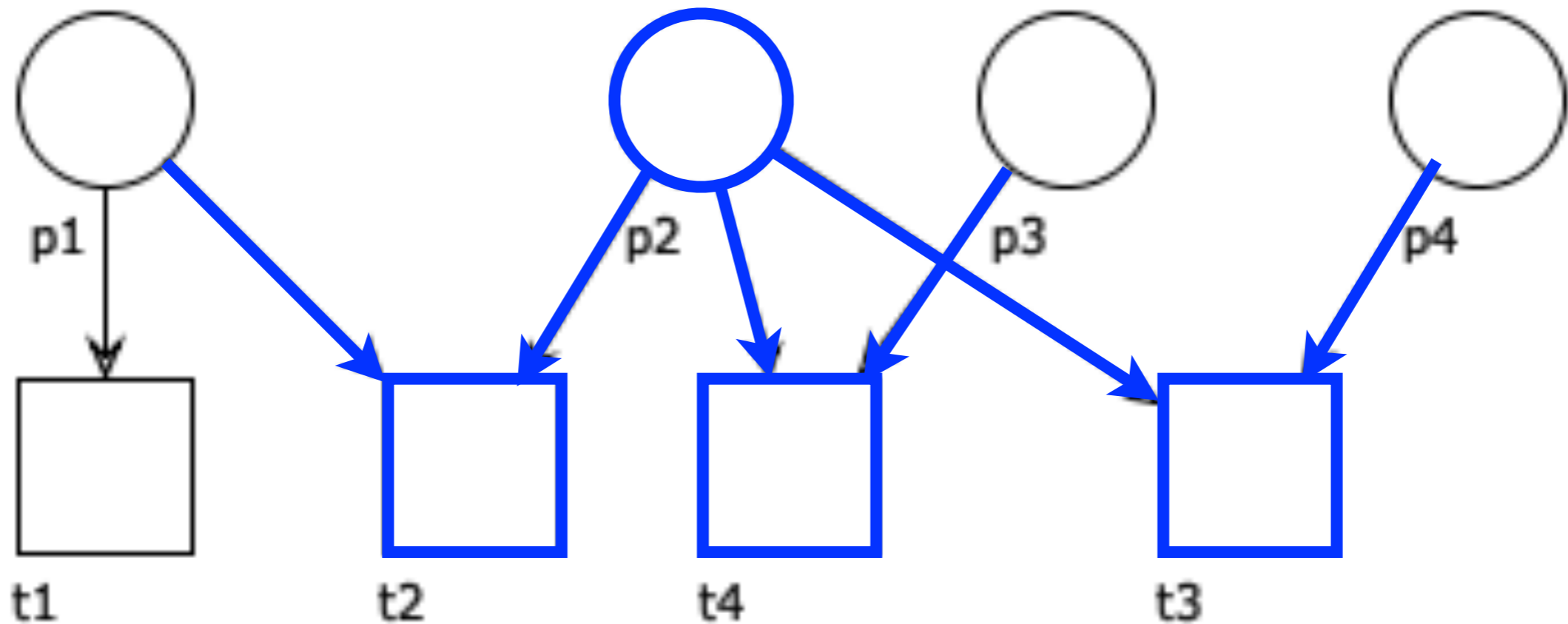
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Cluster: intuition

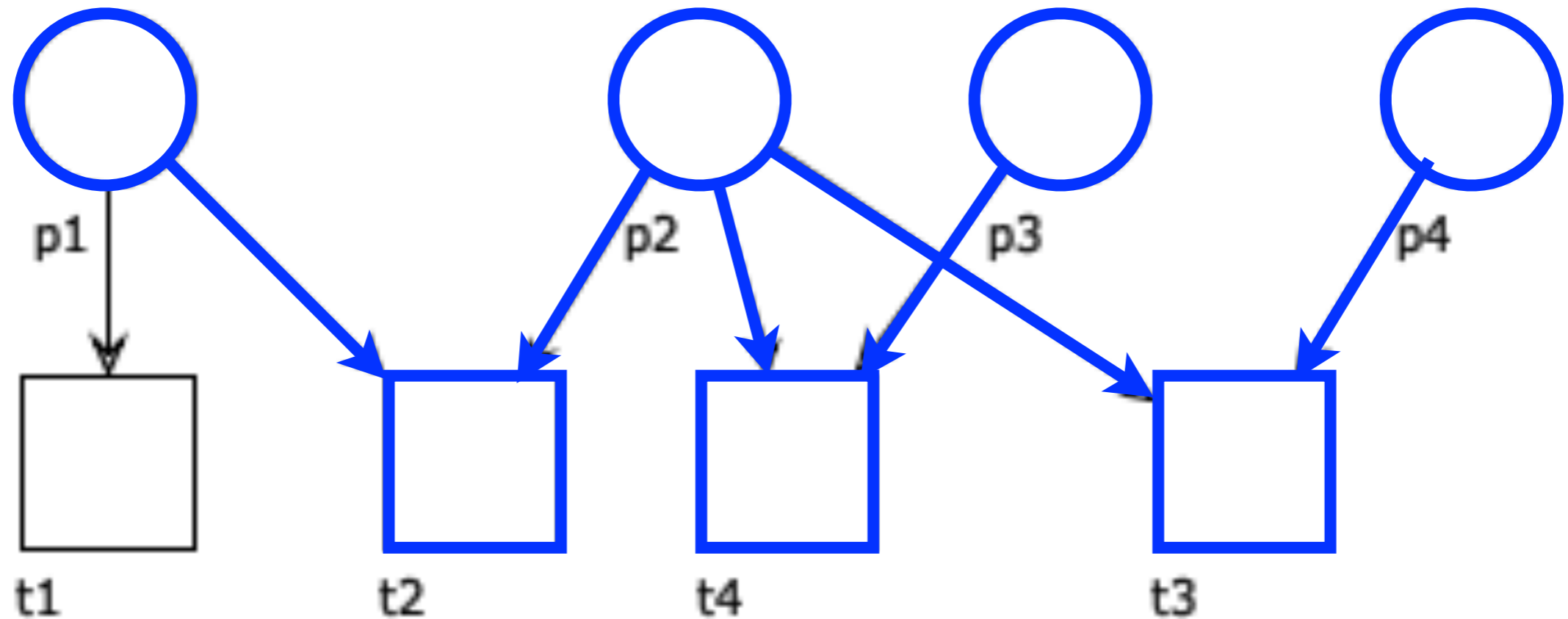
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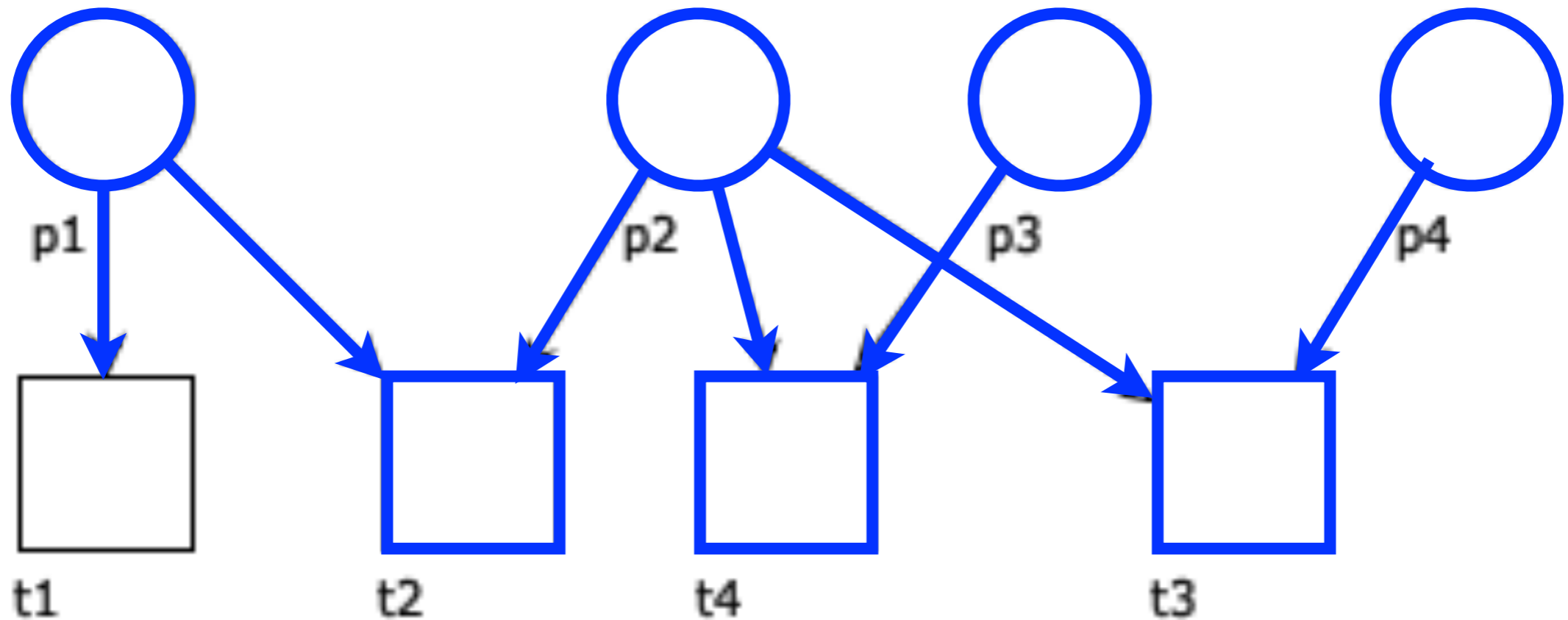
Cluster: intuition

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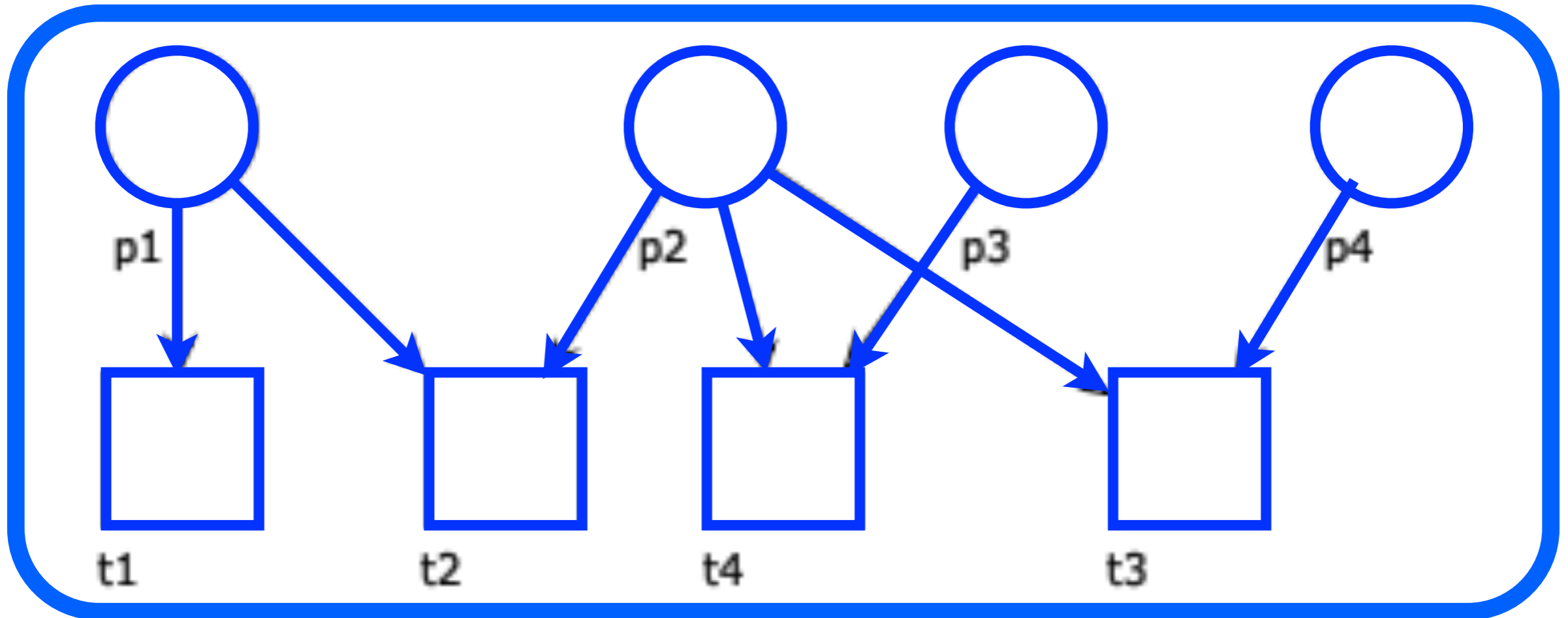
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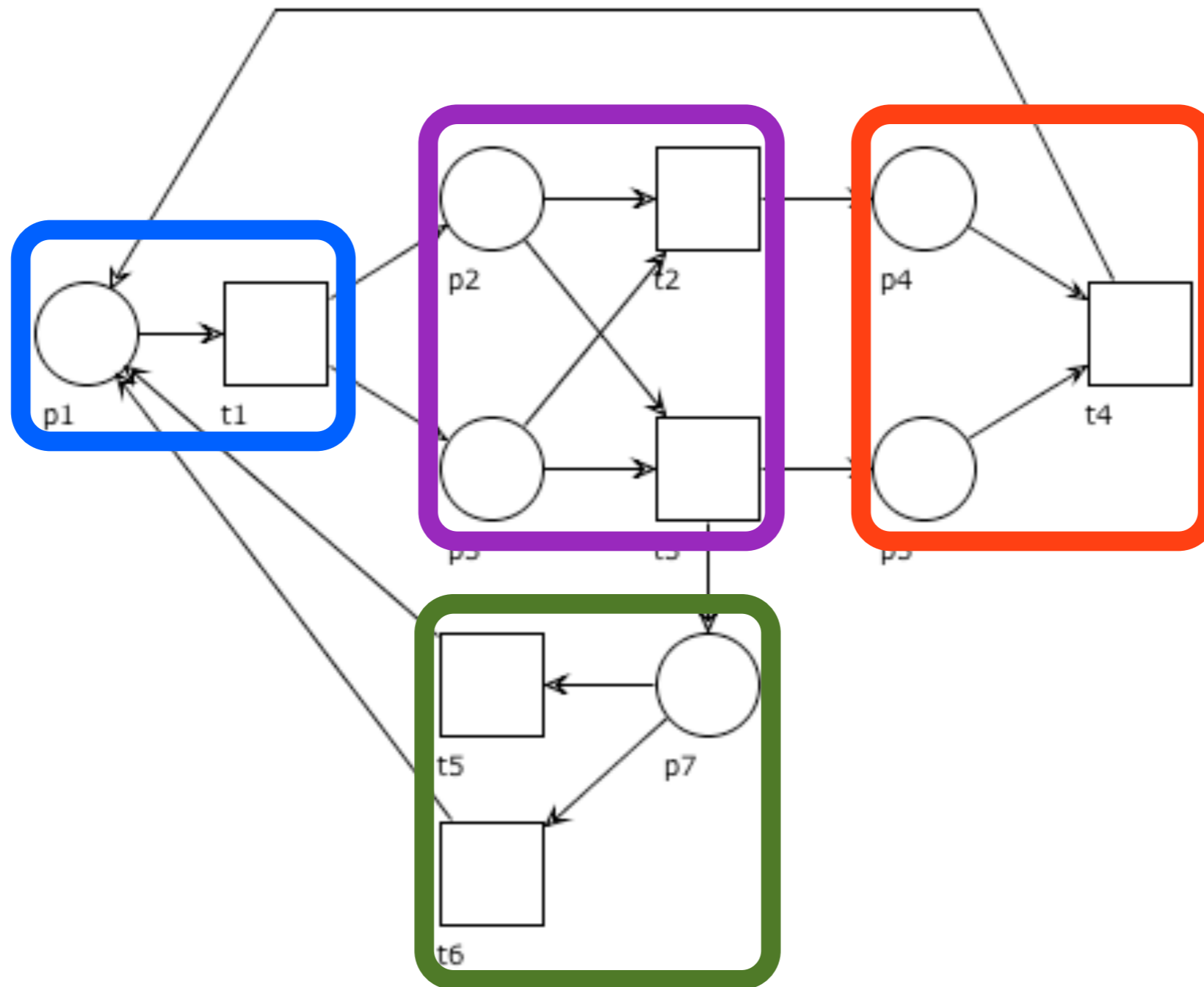


Cluster: intuition

$[p_2] = \{ p_2, t_2, t_4, t_3, p_1, p_3, p_4, t_1 \}$ (if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)

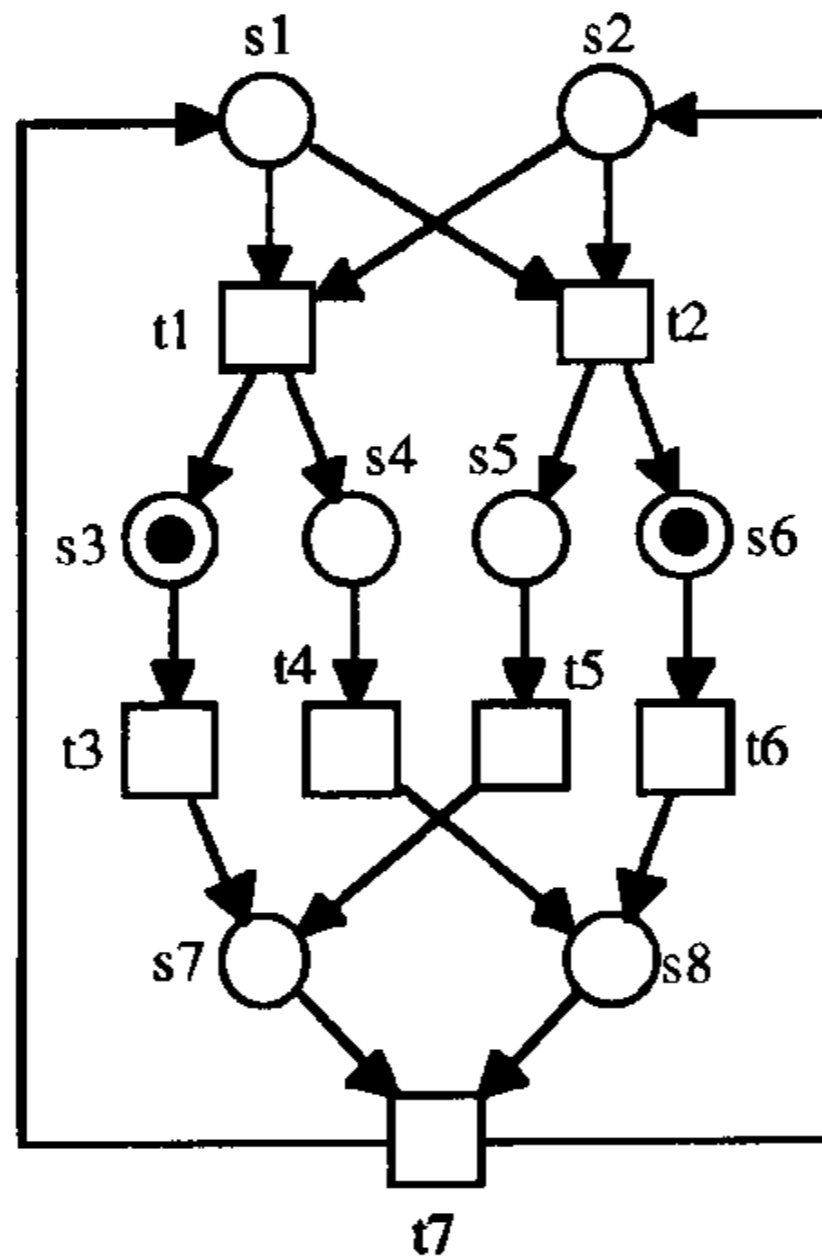


Clusters: example



Exercise

Draw all clusters in the free-choice net below



Clusters and Rank

Theorem

Theorem:

A free-choice system (P, T, F, M_0) is live and bounded
iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |C_N| - 1$

(where C_N is the set of clusters)

A convenient concept:
Stable set of markings

Stable set of markings

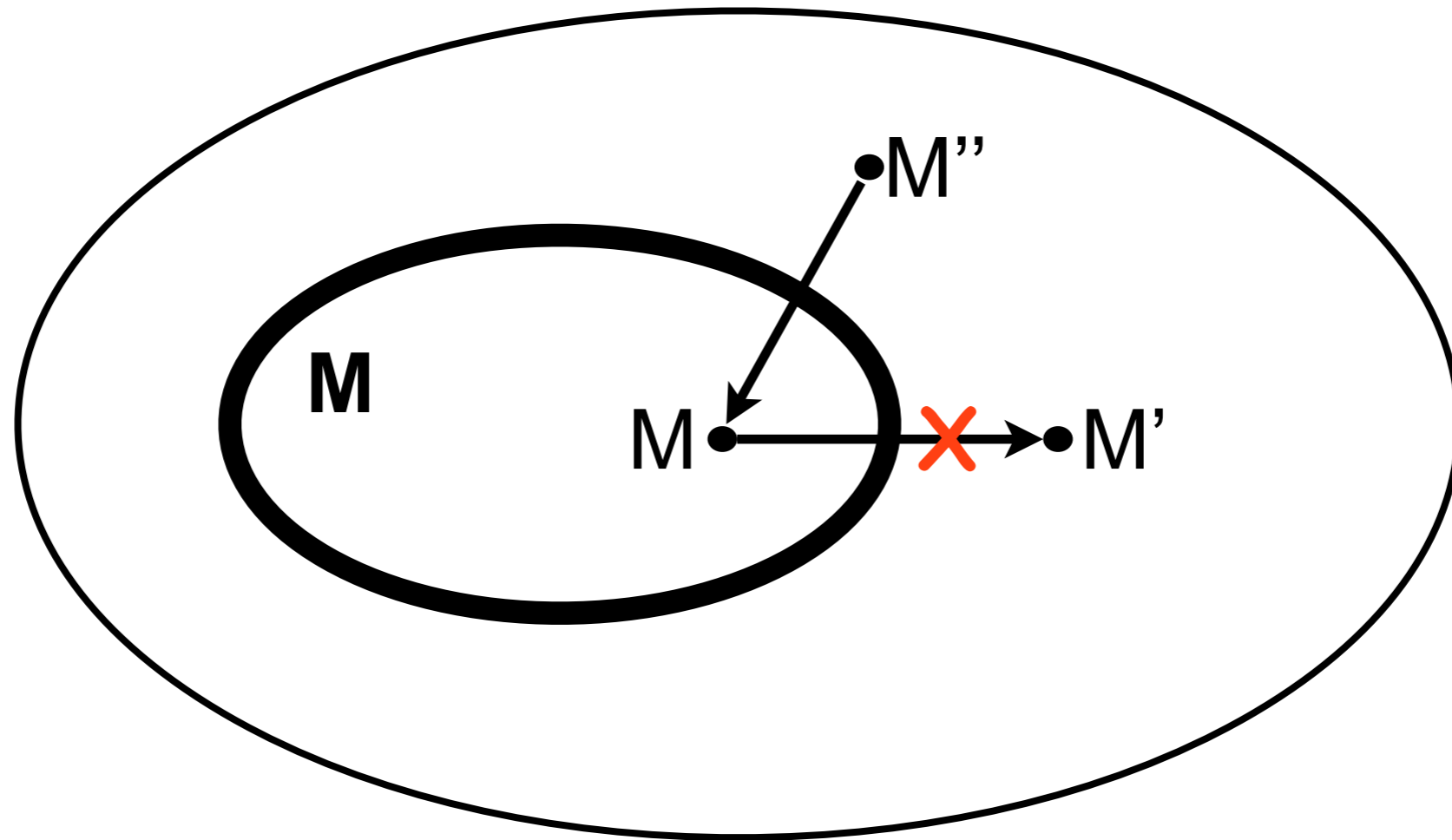
Definition: A set of markings \mathbf{M} is called **stable** if

$$M \in \mathbf{M} \quad \text{implies} \quad [M \rangle \subseteq \mathbf{M}$$

(starting from any marking in the stable set \mathbf{M} ,
no marking outside \mathbf{M} is reachable)

$[M_0 \rangle$ is the least stable set that includes the marking M_0

Stable set of markings



(starting from any marking **M** in the stable set **M**,
no marking **M'** outside **M** is reachable)

Stability check

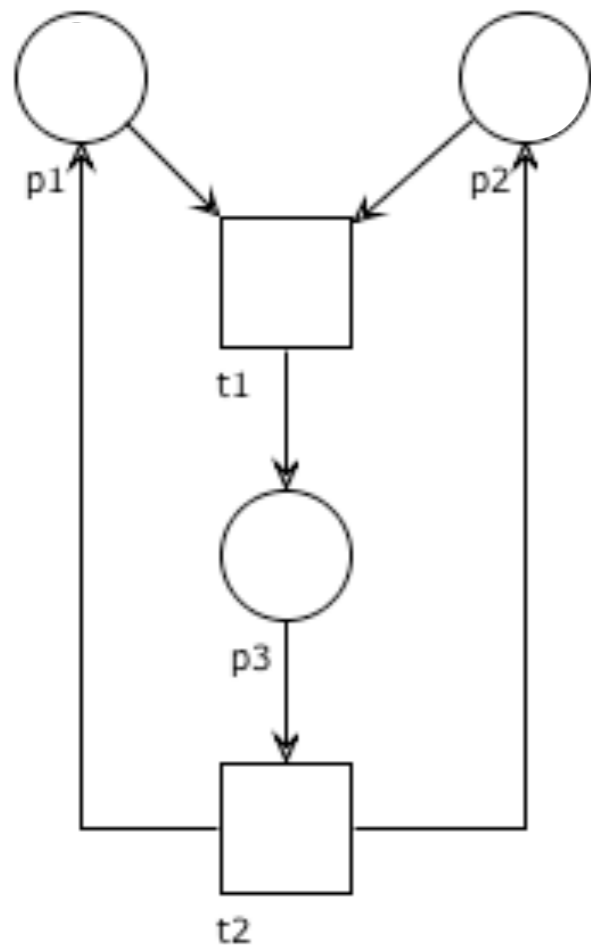
\mathbf{M} is stable iff

$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$

Example

Which of the following is a stable set of markings?

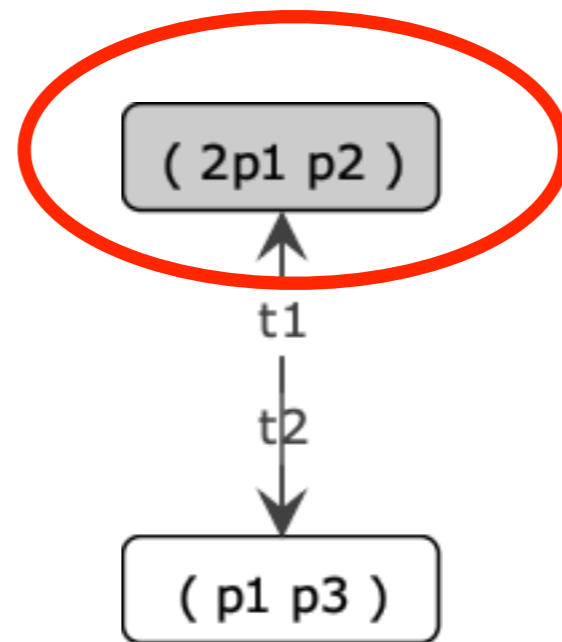
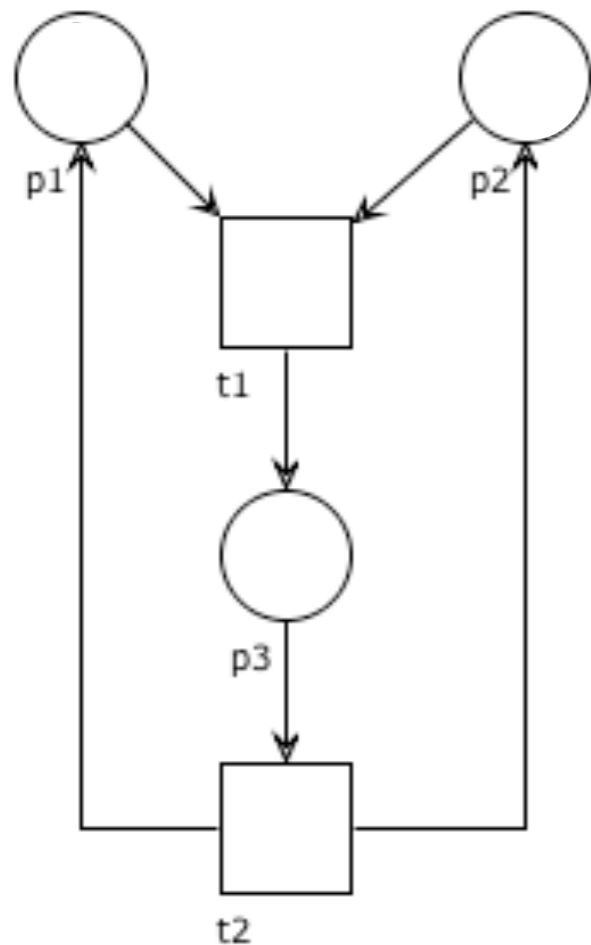
$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$



- $\{ 2p_1+p_2 \}$
- $\{ 2p_1+p_2, p_1+2p_3 \}$
- $\{ p_1, p_2 \}$
- $\{ p_1+p_2, p_3 \}$

Example

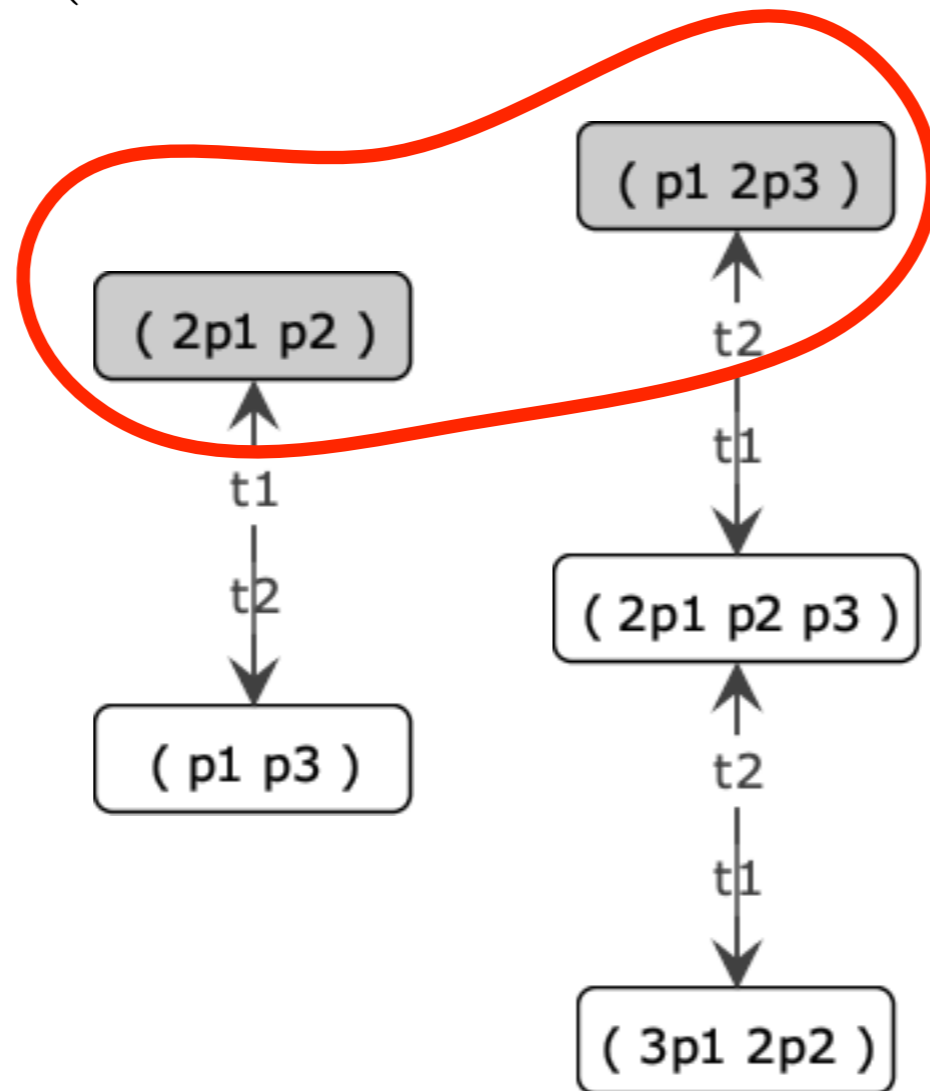
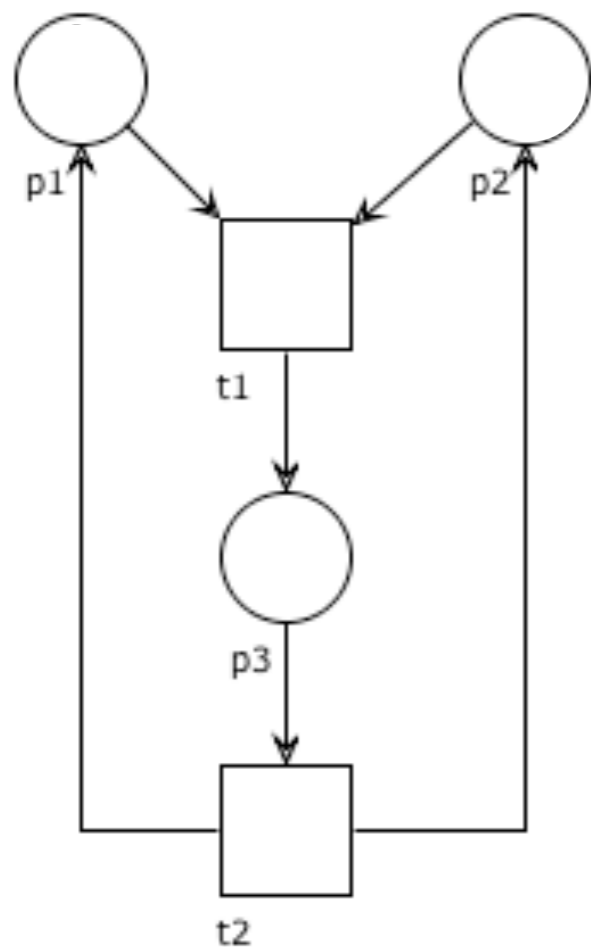
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Example

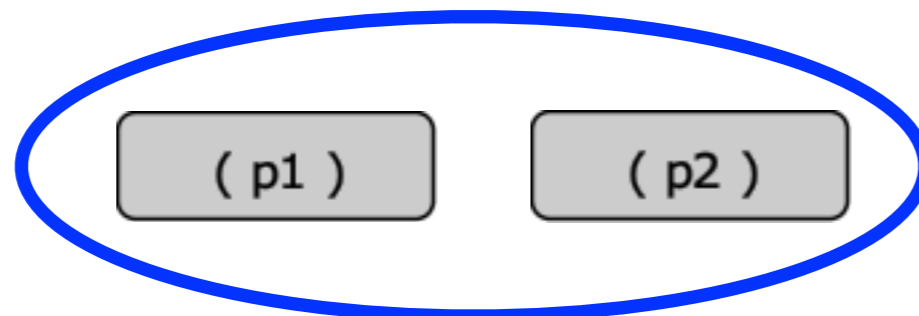
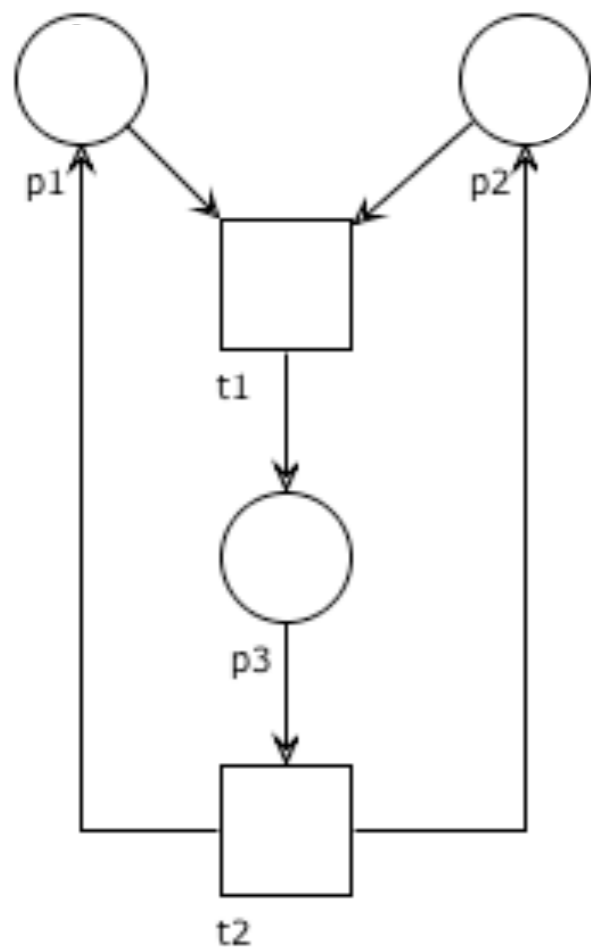
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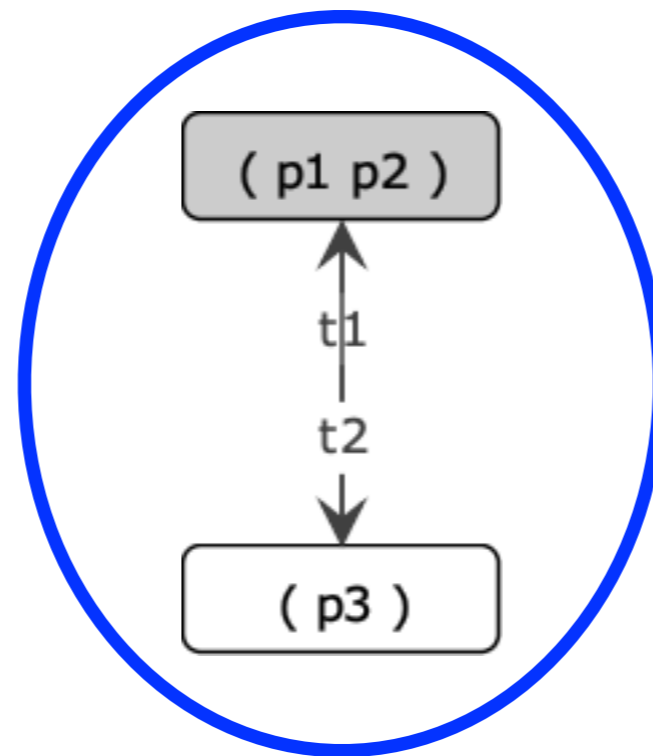
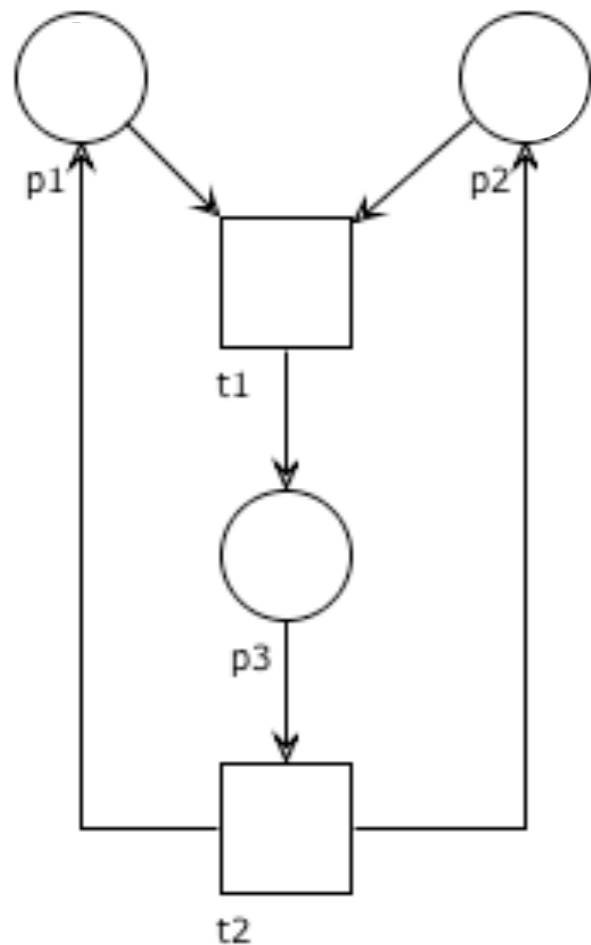
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 $\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$



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Example

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- $\{ 2p_1 + p_2 \}$
- $\{ 2p_1 + p_2, p_1 + 2p_3 \}$
- $\{ p_1, p_2 \}$
- $\{ p_1 + p_2, p_3 \}$

Example

$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$

Given a net system:

empty marking

Is the singleton set $\{ \mathbf{0} \}$ a stable set?

Is the set of all markings a stable set?

Is the set of live markings a stable set?

Is the set of deadlock markings a stable set?

Example

$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$

Given a net system:

empty marking

Is the singleton set $\{ \mathbf{0} \}$ a stable set?

YES: no firing is possible

Is the set of all markings a stable set?

Is the set of live markings a stable set?

Is the set of deadlock markings a stable set?

Example

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Given a net system:

Is the singleton set $\{ \mathbf{0} \}$ a stable set?

YES

Is the set of all markings a stable set?

YES: it is not possible to leave the set of all markings

Is the set of live markings a stable set?

Is the set of deadlock markings a stable set?

Example

$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$

Given a net system:

Is the singleton set $\{ \mathbf{0} \}$ a stable set?

YES

Is the set of all markings a stable set?

YES

Is the set of live markings a stable set?

YES: liveness is an invariant

Is the set of deadlock markings a stable set?

Example

$\forall M, t, M'. (M \in \mathbf{M} \wedge M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$

Given a net system:

Is the singleton set $\{ \mathbf{0} \}$ a stable set?

YES

Is the set of all markings a stable set?

YES

Is the set of live markings a stable set?

YES

Is the set of deadlock markings a stable set?

YES: no firing is possible

Exercises

Given a net (P,T,F) :

Show that the set $\{ M \mid M(P)=1 \}$ is not necessarily stable.

Show that the set $\{ M \mid M(P)<k \}$ is not necessarily stable.

Exercises

Let I be an S -invariant for (P, T, F, M_0)

Is the set $\{ M \mid I \cdot M = I \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid I \cdot M \neq I \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid I \cdot M = 1 \}$ a stable set?

Is the set $\{ M \mid I \cdot M = 0 \}$ a stable set?

What is a siphon?

Proper siphon

Definition:

A set of places R is a **siphon** if $\bullet R \subseteq R\bullet$

It is a **proper siphon** if $R \neq \emptyset$

Siphons, intuitively

A set of places R is a siphon if

all transitions that can produce tokens in the places of R

$$\bullet R \subseteq R \bullet$$

require some place in R to be marked

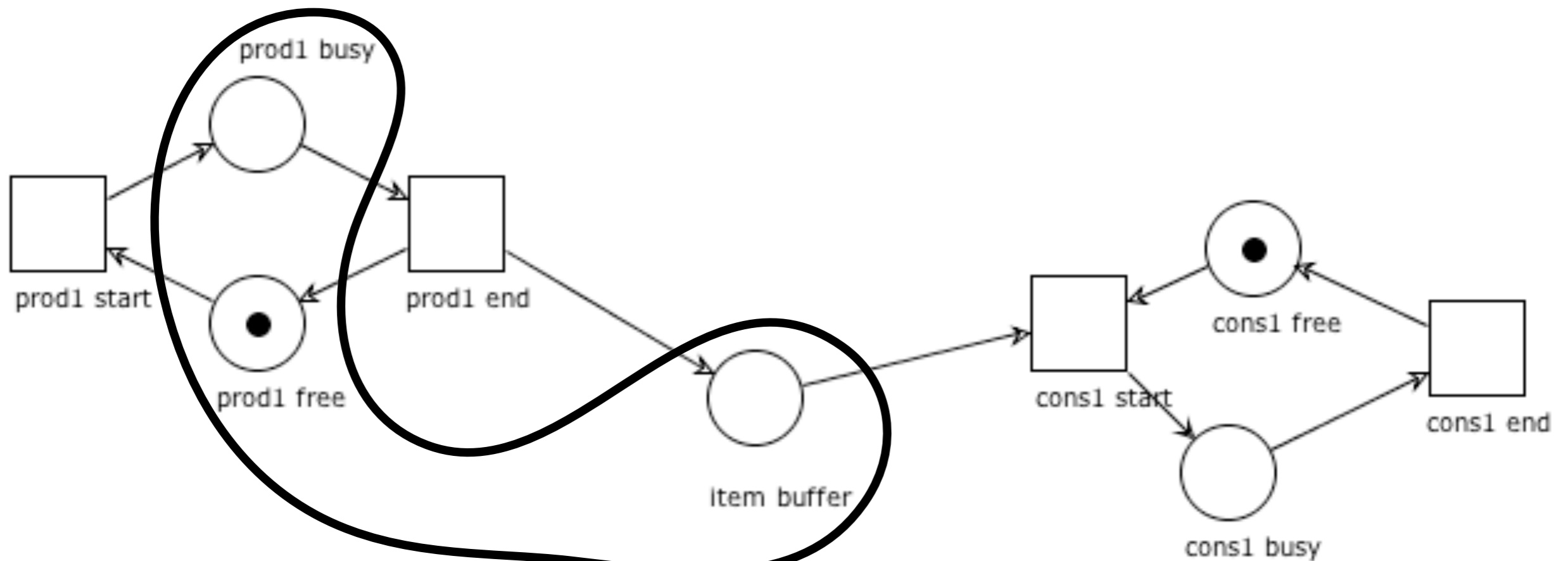
Therefore:

if no token is present in R ,
then no token will ever be produced in R

• $R \subseteq R$ •

Siphon check: example

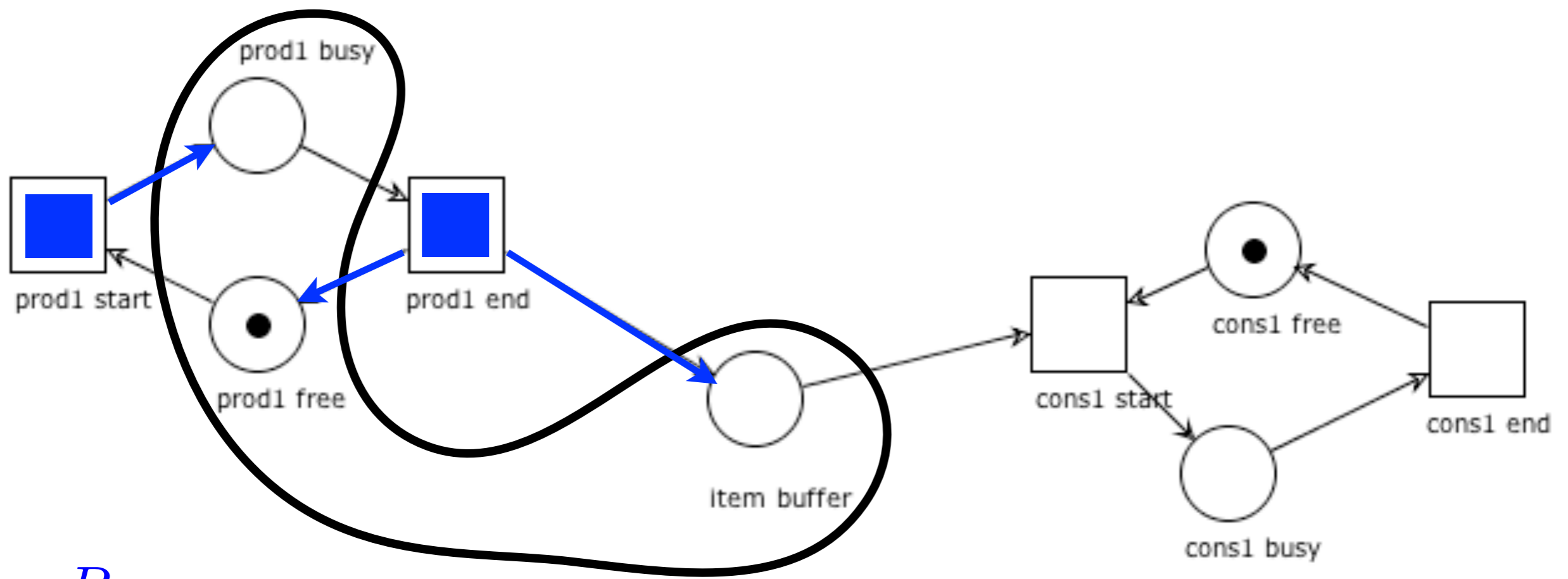
Is $R = \{ \text{prod1 busy}, \text{prod1 free}, \text{itembuffer} \}$ a siphon?



• $R \subseteq R$ •

Siphon check: example

Is $R = \{ \text{prod1 busy}, \text{prod1 free}, \text{itembuffer} \}$ a siphon?

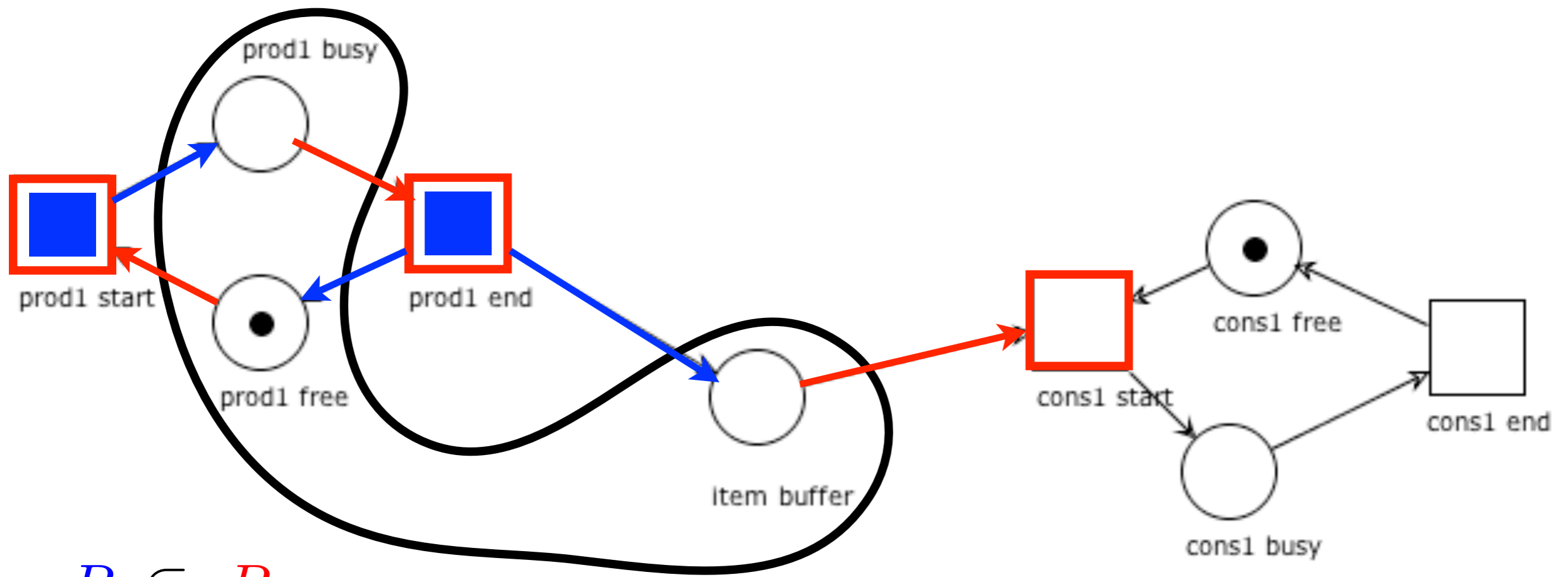


• R

$$\bullet R \subseteq R \bullet$$

Siphon check: example

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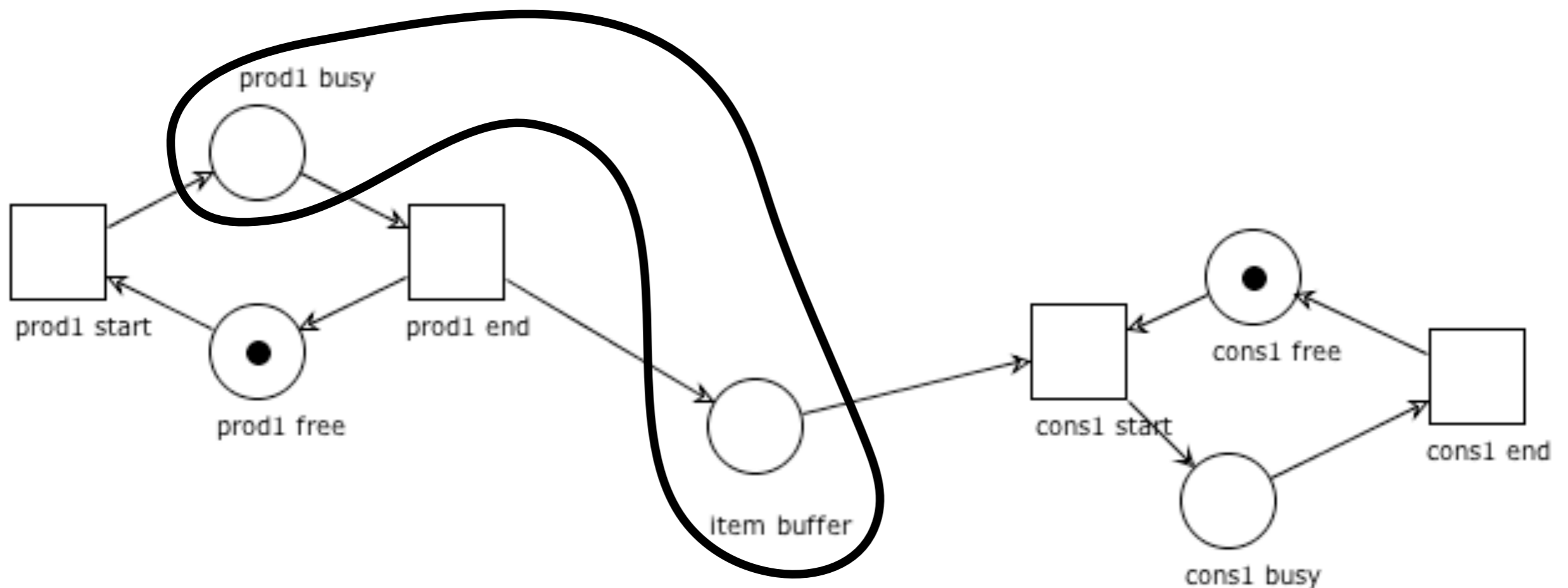


$$\bullet R \subseteq R \bullet$$

• $R \subseteq R$ •

Siphon check: example

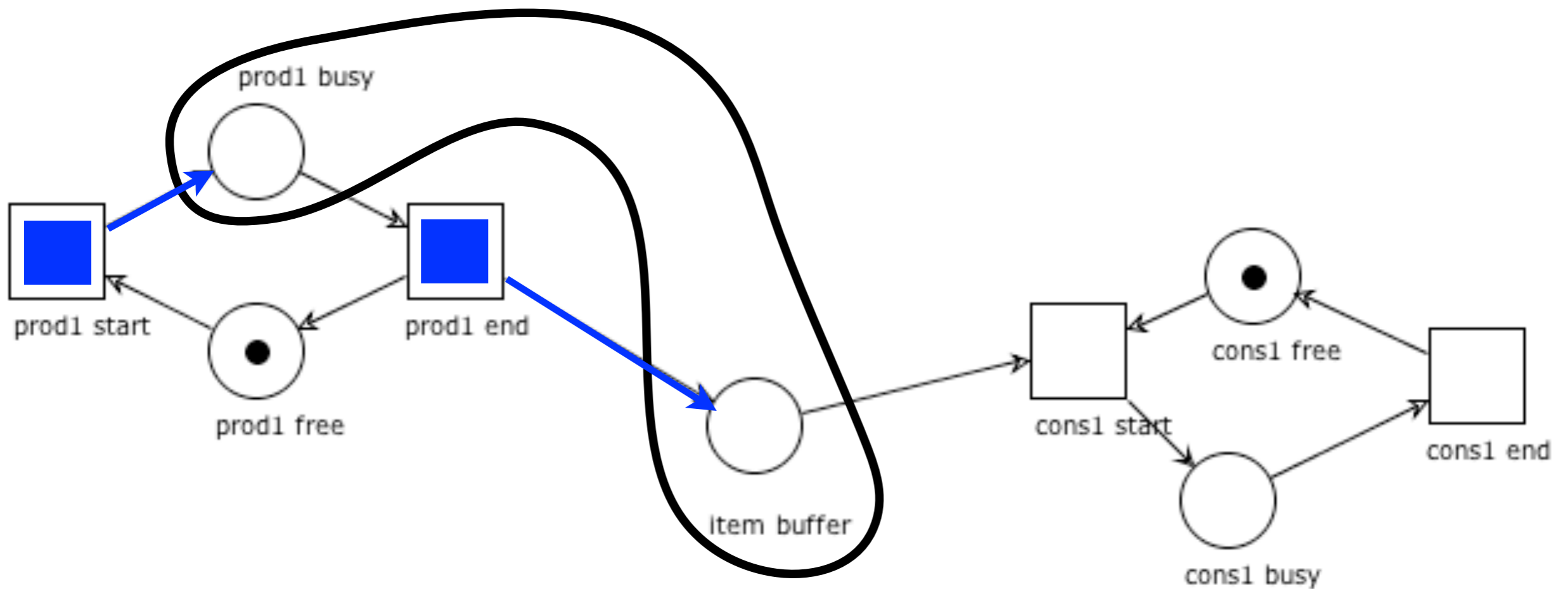
Is $R = \{ \text{prod1 busy}, \text{itembuffer} \}$ a siphon?



• $R \subseteq R$ •

Siphon check: example

Is $R = \{ \text{prod1 busy, itembuffer} \}$ a siphon?

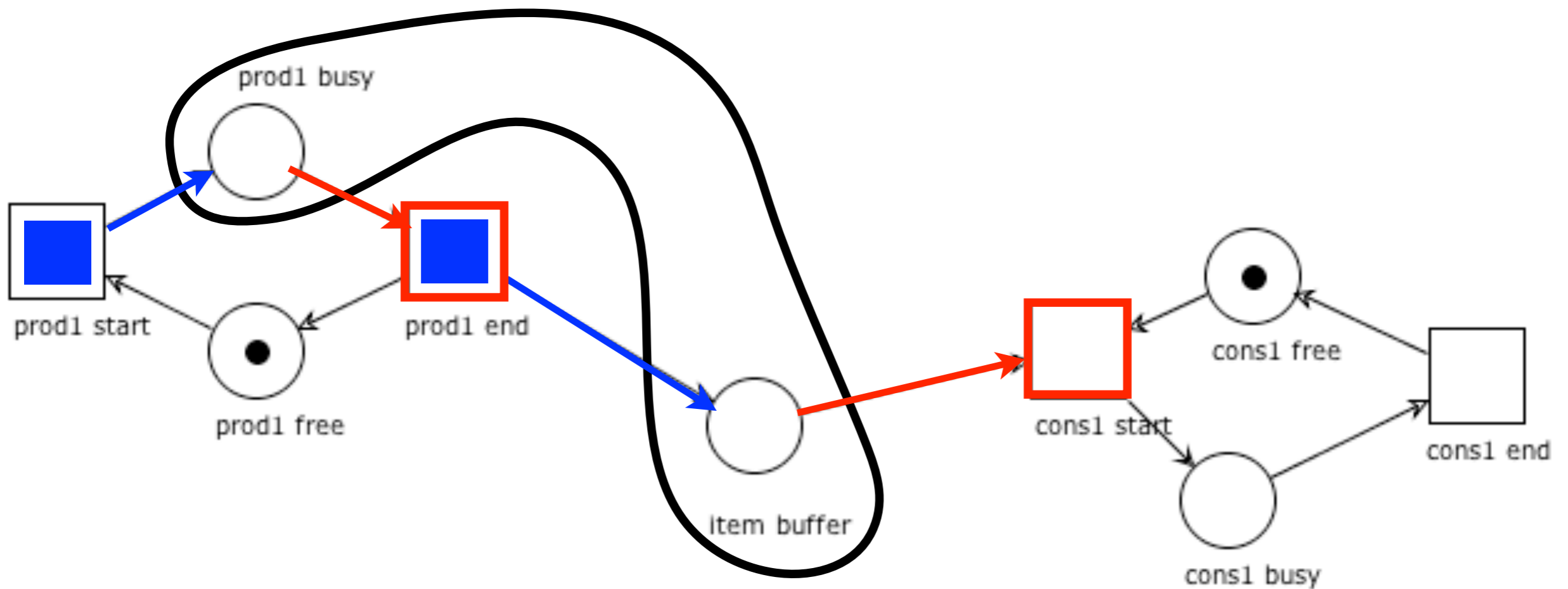


• R

• $R \subseteq R$ •

Siphon check: example

Is $R = \{ \text{prod1 busy, itembuffer} \}$ a siphon?



• $R \not\subseteq R$ •

Fundamental property of siphons

Proposition: Unmarked siphons remain unmarked

Take a siphon R .

We just need to prove that the set of markings

$$\mathbf{M} = \{ M \mid M(R)=0 \}$$

is stable, which is immediate by definition of siphon

Corollary:

If a siphon R is marked at some reachable marking M ,
then it was initially marked at M_0

Siphons and liveness

Prop.: If a system is live any proper siphon R is marked

Take $p \in R$ and let $t \in \bullet p \cup p \bullet$

Since the system is live, then there are $M, M' \in [M_0 \rangle$ such that

$$M \xrightarrow{t} M'$$

Therefore p is marked at either M or M'

Therefore R is marked at either M or M'

Therefore R was initially marked (at M_0)

Siphons and liveness

Corollary: If a system has an unmarked proper siphon
then it is not live

Siphons and Rank Theorem

Theorem:

A free-choice system (P, T, F, M_0) is live and bounded
iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper **siphon**
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |C_N| - 1$

(where C_N is the set of clusters)

Siphons and Commoner's theorem

Theorem:

A free-choice system is live
iff

every proper **siphon** includes an initially marked **trap**

What is a trap?

Proper trap

Definition:

A set of places R is a **trap** if $\bullet R \supseteq R\bullet$

It is a **proper trap** if $R \neq \emptyset$

Traps, intuitively

A set of places R is a trap if

all transitions that can consume tokens from R

$$\bullet R \supseteq R \bullet$$

produce some token in some place of R

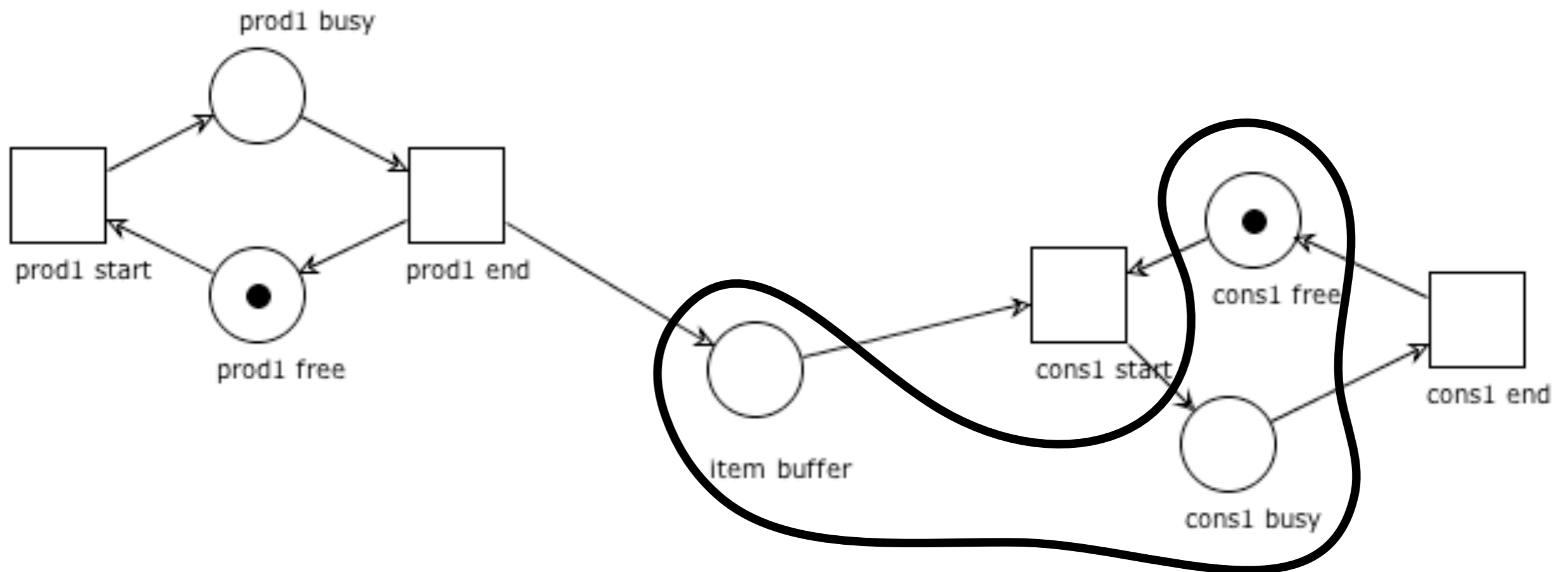
Therefore:

if some token is present in R ,
then it is never possible for R to become empty

• $R \supseteq R$ •

Trap check: example

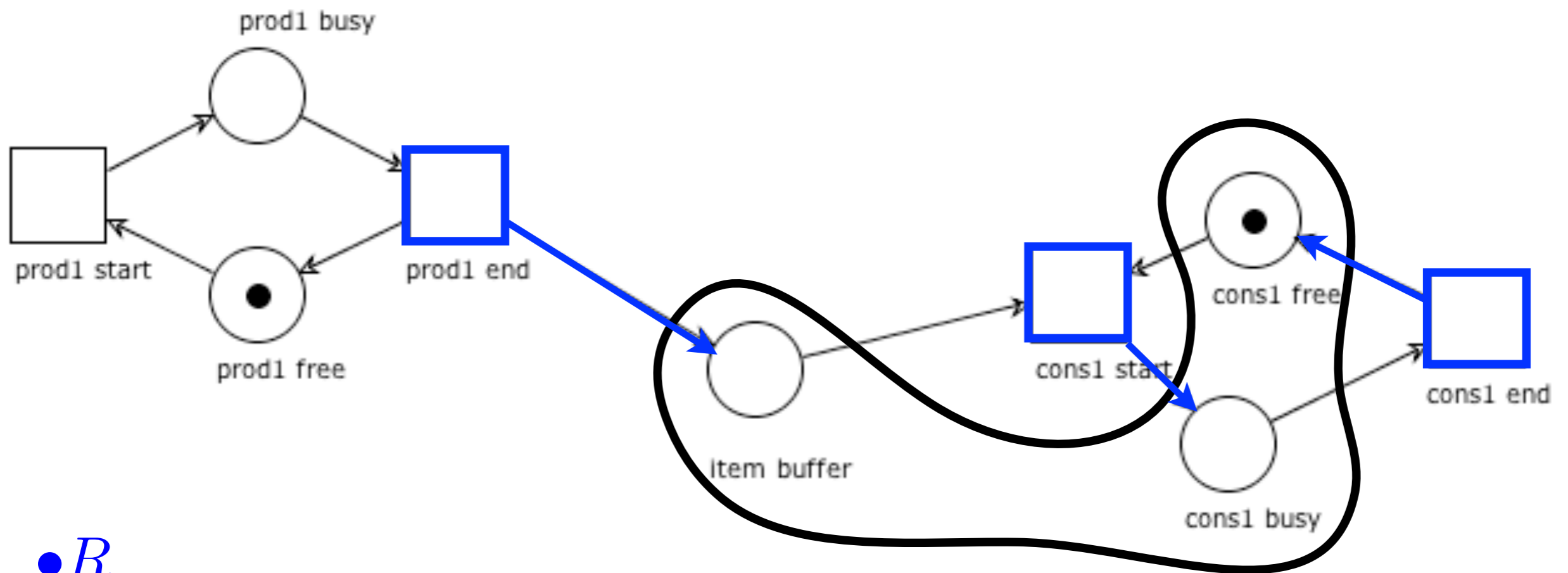
Is $R = \{ \text{itembuffer}, \text{cons1busy}, \text{cons1free} \}$ a trap?



• $R \supseteq R$ •

Trap check: example

Is $R = \{ \text{itembuffer}, \text{cons1busy}, \text{cons1free} \}$ a trap?

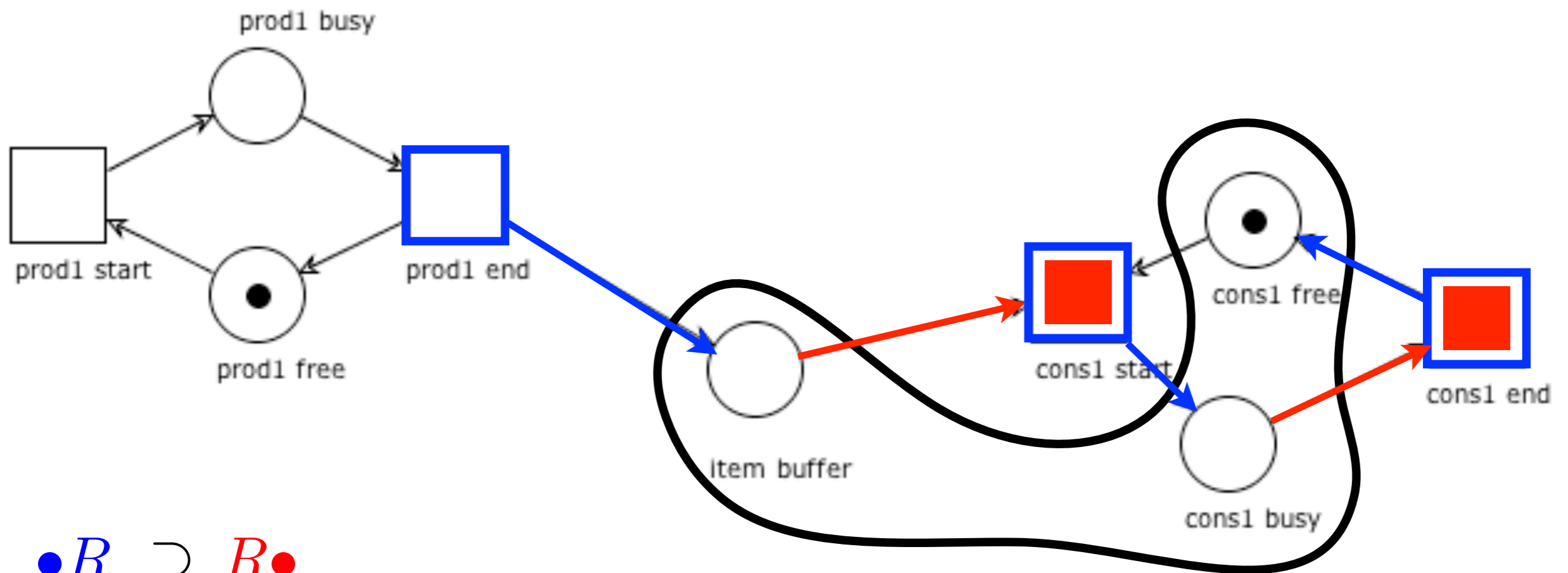


• R

$$\bullet R \supseteq R \bullet$$

Trap check: example

Is $R = \{ \text{itembuffer}, \text{cons1busy}, \text{cons1free} \}$ a trap?

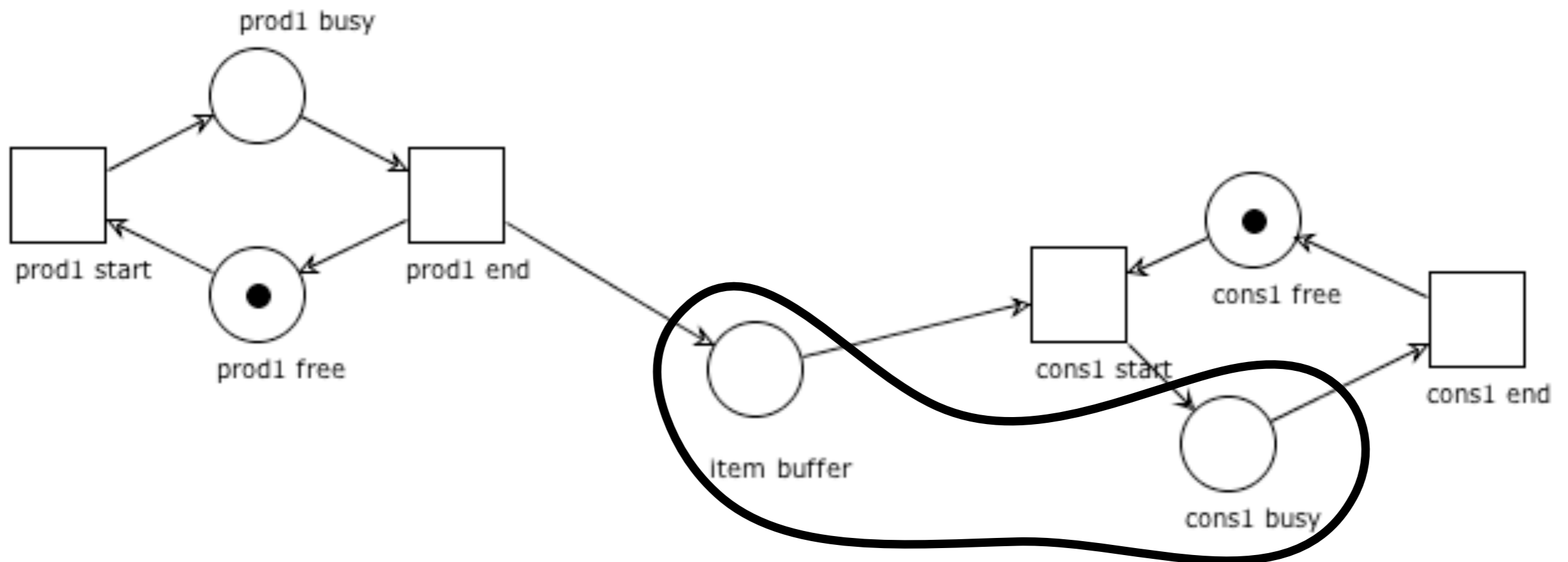


$$\bullet R \supseteq R \bullet$$

• $R \supseteq R$ •

Trap check: example

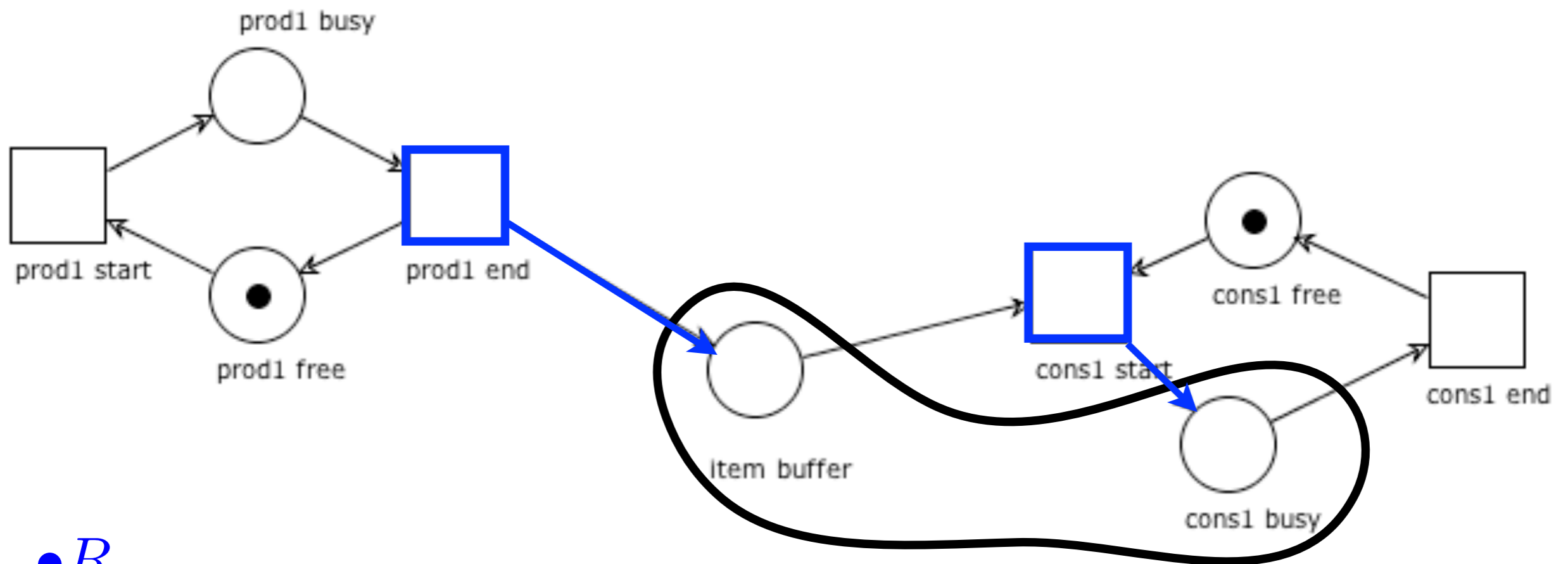
Is $R = \{ \text{itembuffer}, \text{cons1busy} \}$ a trap?



• $R \supseteq R$ •

Trap check: example

Is $R = \{ \text{itembuffer}, \text{cons1busy} \}$ a trap?

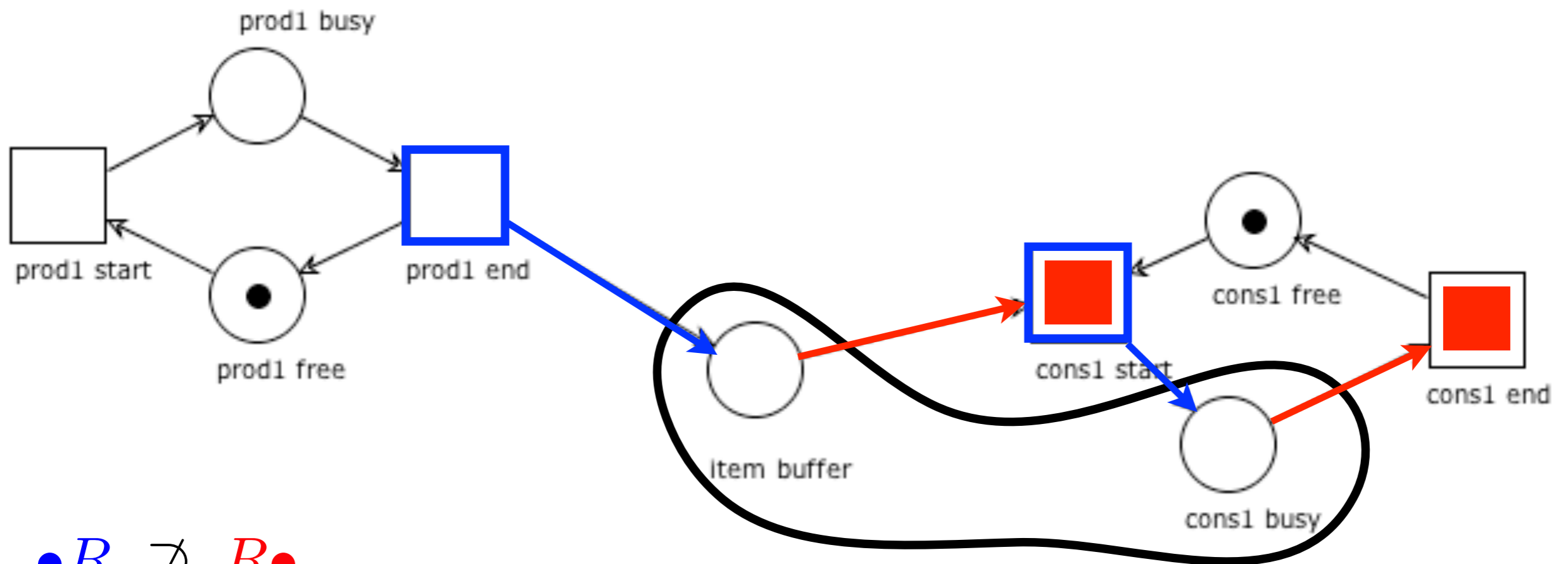


• R

• $R \subseteq R$ •

Trap check: example

Is $R = \{ \text{itembuffer}, \text{cons1busy} \}$ a trap?



• $R \not\subseteq R$ •

Fundamental property of traps

Proposition: Marked traps remain marked

Take a trap R .

We just need to prove that the set of markings

$$\mathbf{M} = \{ M \mid M(R) > 0 \}$$

is stable, which is immediate by definition of trap

Corollary:

If a trap R is unmarked at some reachable marking M ,
then it was initially unmarked at M_0

Complexity issues 1:
Is it hard to show that a
free-choice net is live?

Commoner's theorem

Theorem:

A free-choice system is live
iff

every proper siphon includes an initially marked trap

We show that

the non-liveness problem for free-choice systems



is NP-complete

Note

It is easy to observe that every siphon includes a
(possibly empty) unique maximal trap
with respect to set inclusion
(the union of traps is a trap)

Moreover, a siphon includes a marked trap
iff
its maximal trap is marked

Traps are closed under union

Lemma. The union of traps is a trap

Let X_1, X_2 be traps.

From $X_1 \bullet \subseteq \bullet X_1$ and $X_2 \bullet \subseteq \bullet X_2$ we have:

$$(X_1 \cup X_2) \bullet = X_1 \bullet \cup X_2 \bullet \subseteq \bullet X_1 \cup \bullet X_2 = \bullet (X_1 \cup X_2)$$

Commoner's theorem

Theorem:

A free-choice system is live
iff

every proper siphon includes an initially marked trap

Theorem:

A free-choice system is non-live
iff

there is a proper siphon that only includes initially
unmarked traps

A non-deterministic algorithm for non-liveness

1. guess a set of places R
[polynomial time, non-deterministic step]
2. check if R is a siphon ($\bullet R \subseteq R \bullet$)
[polynomial time]
3. if R is a siphon, compute the maximal trap $Q \subseteq R$
[complexity?]
4. if $M_0(Q)=0$, then answer "non-live", otherwise "live"
[polynomial time]

A polynomial algorithm for maximal trap in a siphon

$$\bullet R \subseteq R \bullet$$

$$\bullet Q \supseteq Q \bullet$$

3. if R is a siphon, compute the maximal trap $Q \subseteq R$

Input: A net $N = (P, T, F)$ and $R \subseteq P$

Output: $Q \subseteq R$ maximal trap in R

$Q := R$

while $(\exists p \in Q, \exists t \in p\bullet, t \notin \bullet Q)$

$Q := Q \setminus \{p\}$

return Q

A polynomial non-det. algorithm for non-liveness

1. guess a set of places R
[polynomial time, non-deterministic step]
2. check if R is a siphon ($\bullet R \subseteq R \bullet$)
[polynomial time]
3. if R is a siphon, compute the maximal trap $Q \subseteq R$
[polynomial time]
4. if $M_0(Q)=0$, then answer “non-live”, otherwise “live”
[polynomial time]

Non-liveness for f.c. nets is in NP

The non-liveness problem for free-choice systems is in NP

Is the same problem in P?

The corresponding deterministic algorithm cannot make
the guess in step 1

It has to explore all possible subsets of places
 $2^{|P|}$ cases!

NP-completeness

We next sketch the proof of the reduction to non-liveness
in a free-choice net of the CNF-SAT problem

(SATisfiability problem
for propositional formulas in Conjunctive Normal Form)

CNF-SAT is an NP-complete problem

CNF-SAT decision problem

Variables: x_1, x_2, \dots, x_n

Literals: $x_1, \bar{x}_1, x_2, \bar{x}_2, \dots, x_n, \bar{x}_n$

Clause: disjunction of literals

Formula: conjunction of clauses

Example: $\phi = (x_1 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3)$

Is there an assignment of boolean values to the variables such that $\phi = \text{true}$?

The free-choice net of a formula

Given a formula ϕ , the idea is to construct a free-choice system (P, T, F, M_0) and show that

the formula ϕ is satisfiable
iff
 (P, T, F, M_0) is not live

The free-choice net of a formula

Given a formula ϕ , the idea is to construct a free-choice system (P, T, F, M_0) and show that

the formula ϕ is not satisfiable
iff
 (P, T, F, M_0) is live

CNF-SAT formulas

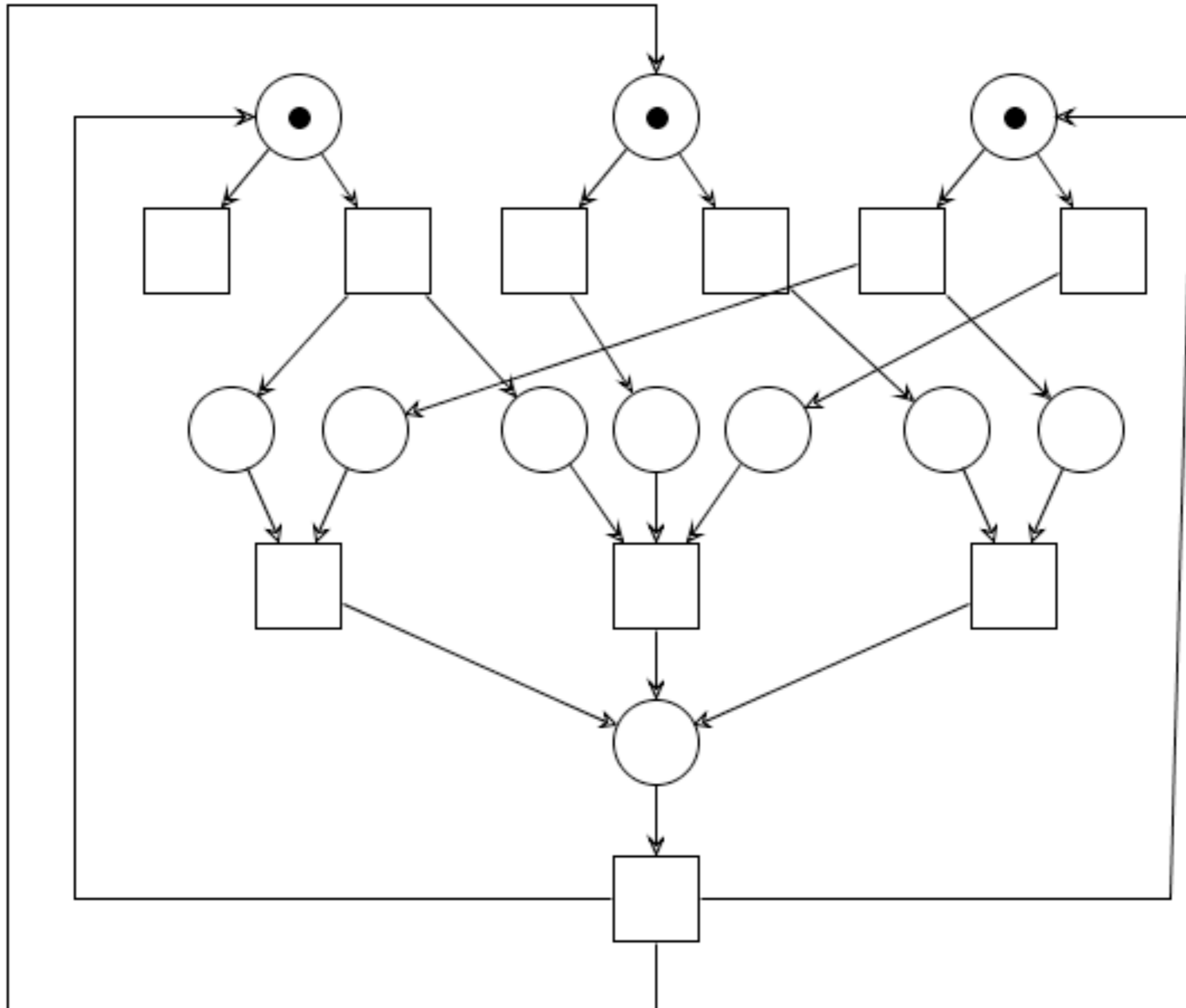
Is there an assignment of boolean values to the variables such that $\phi = \text{true}$?

Is there an assignment of boolean values to the variables such that $\neg\phi = \text{false}$?

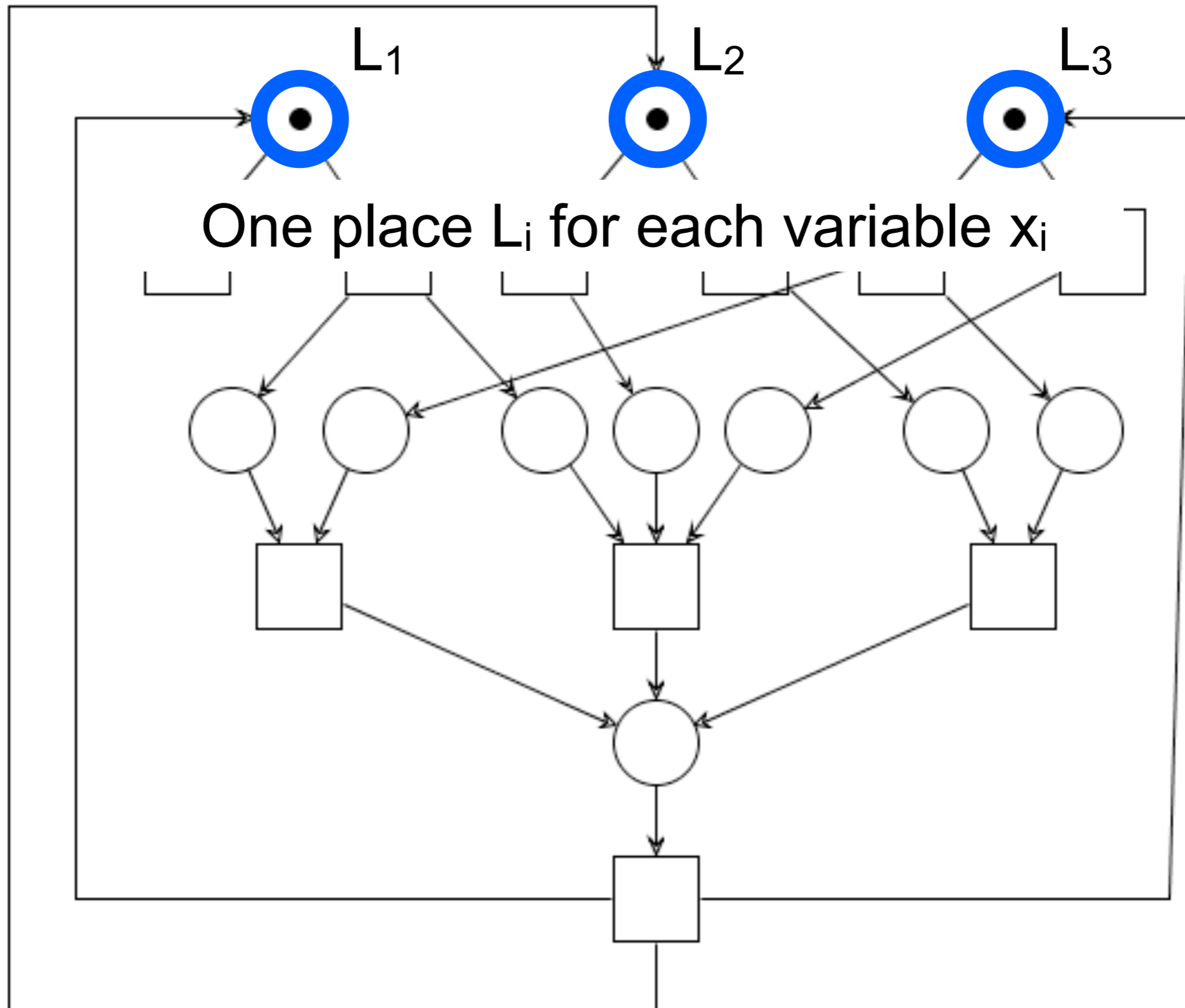
$$\phi = (x_1 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3)$$

$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$

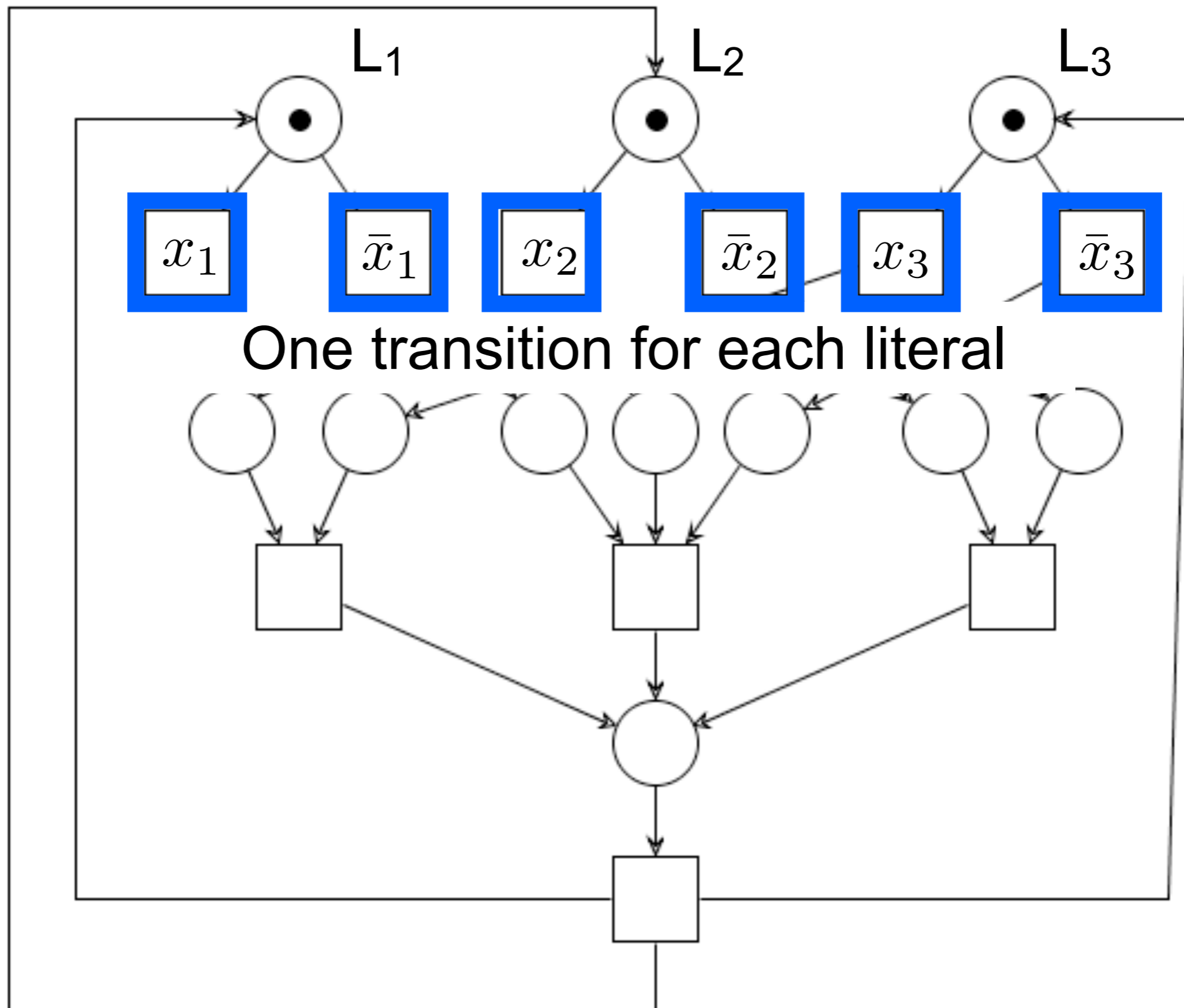
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



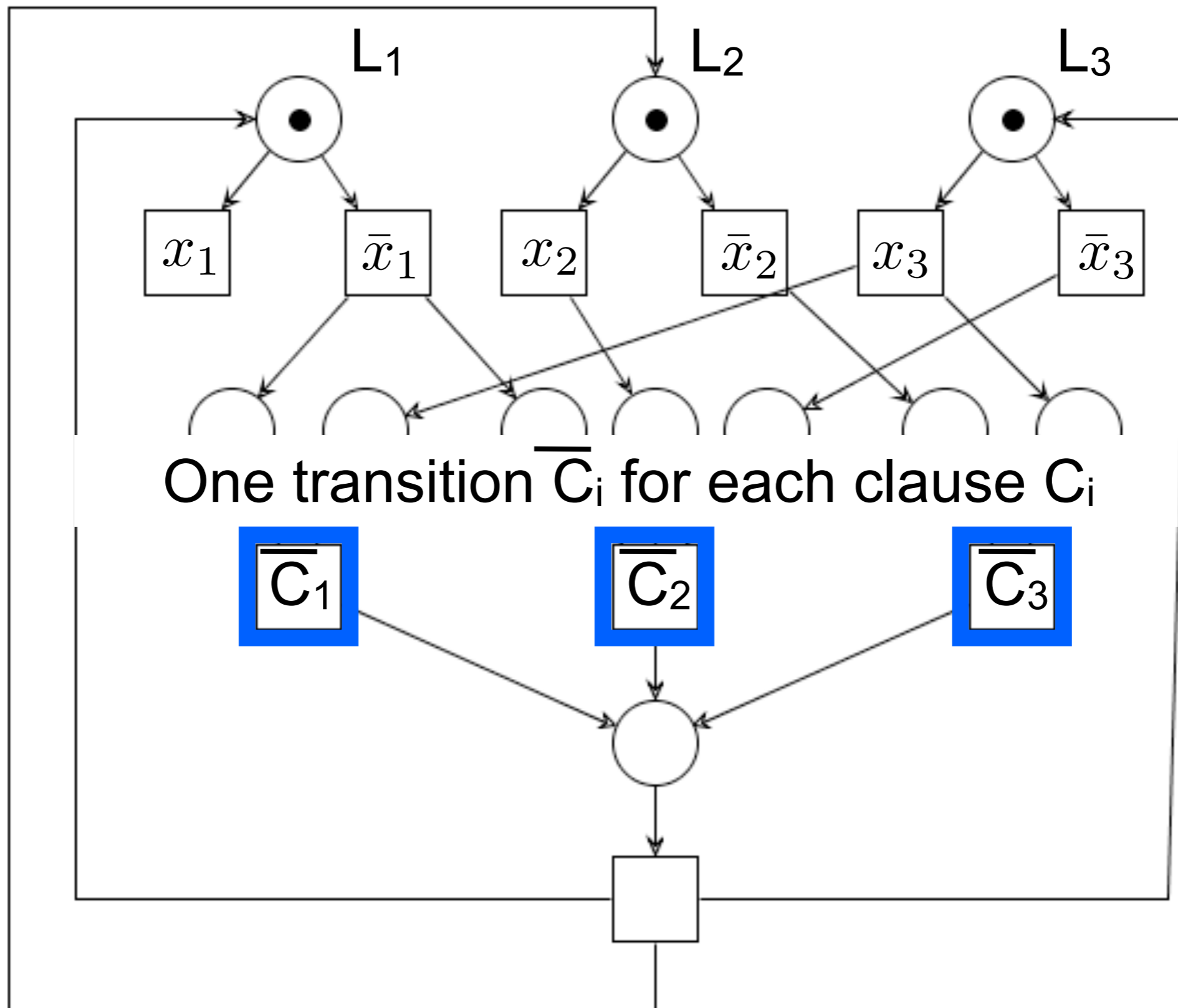
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



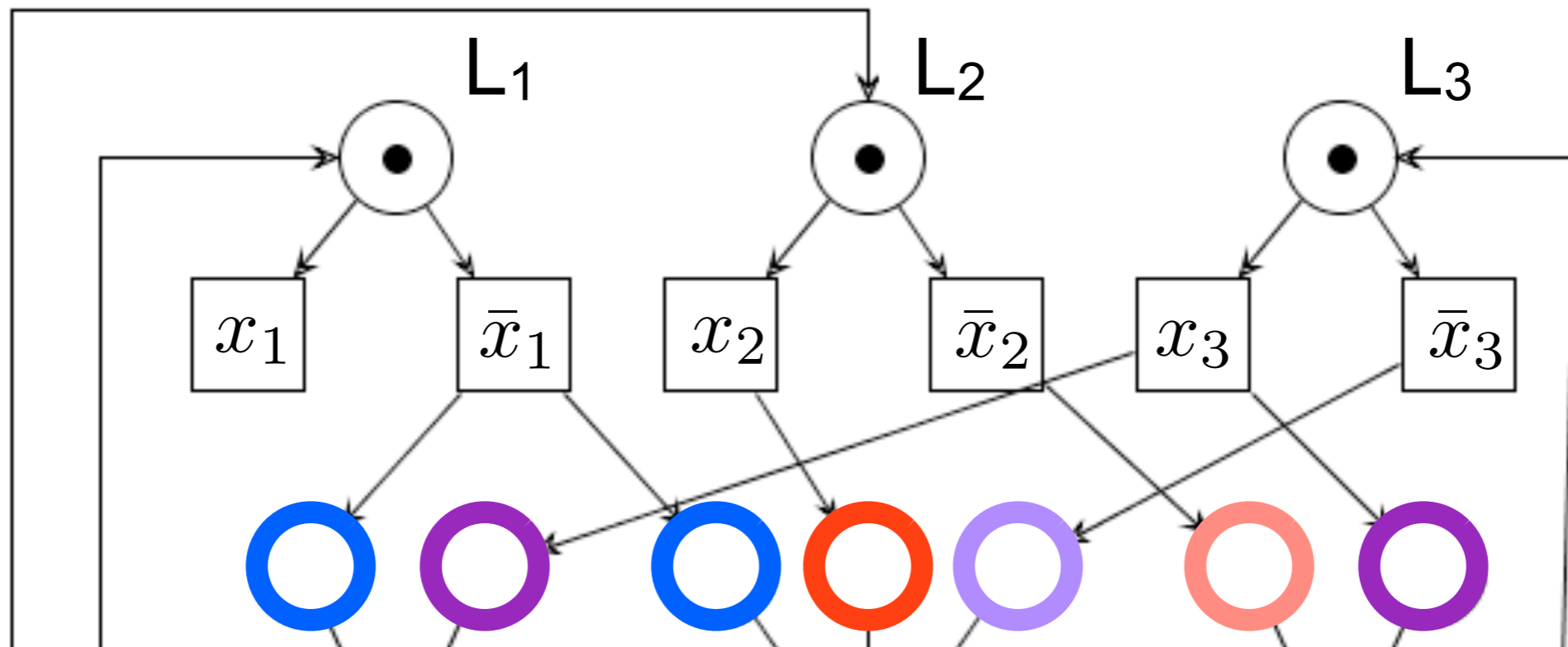
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



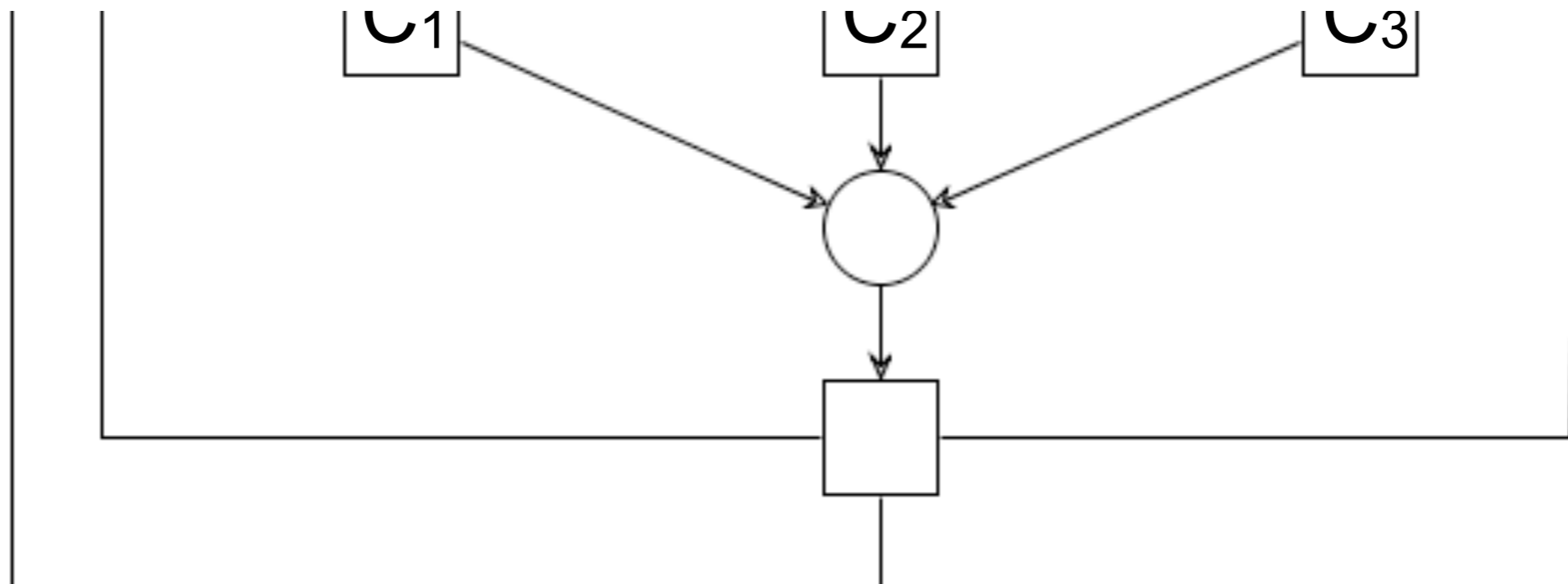
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



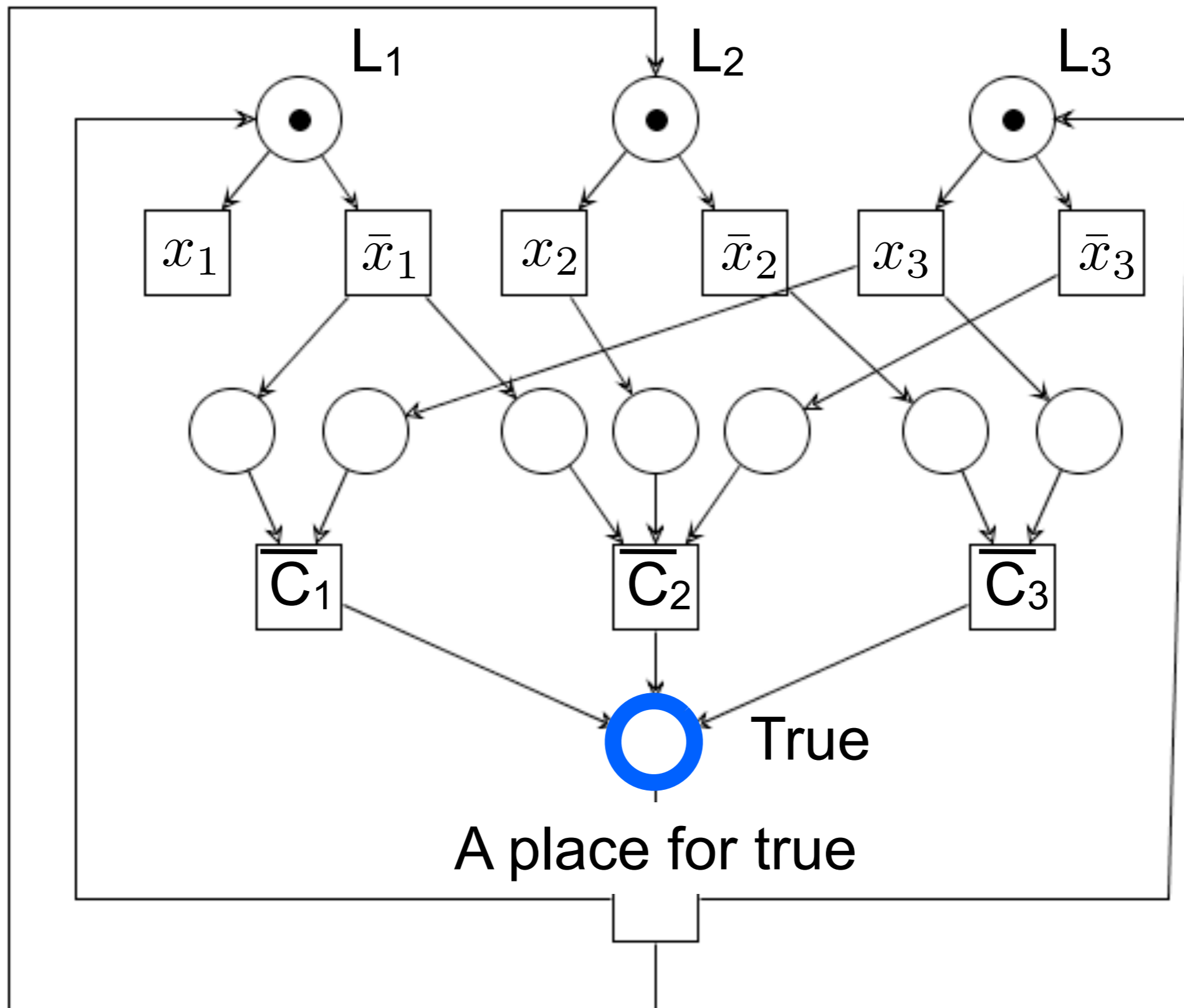
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



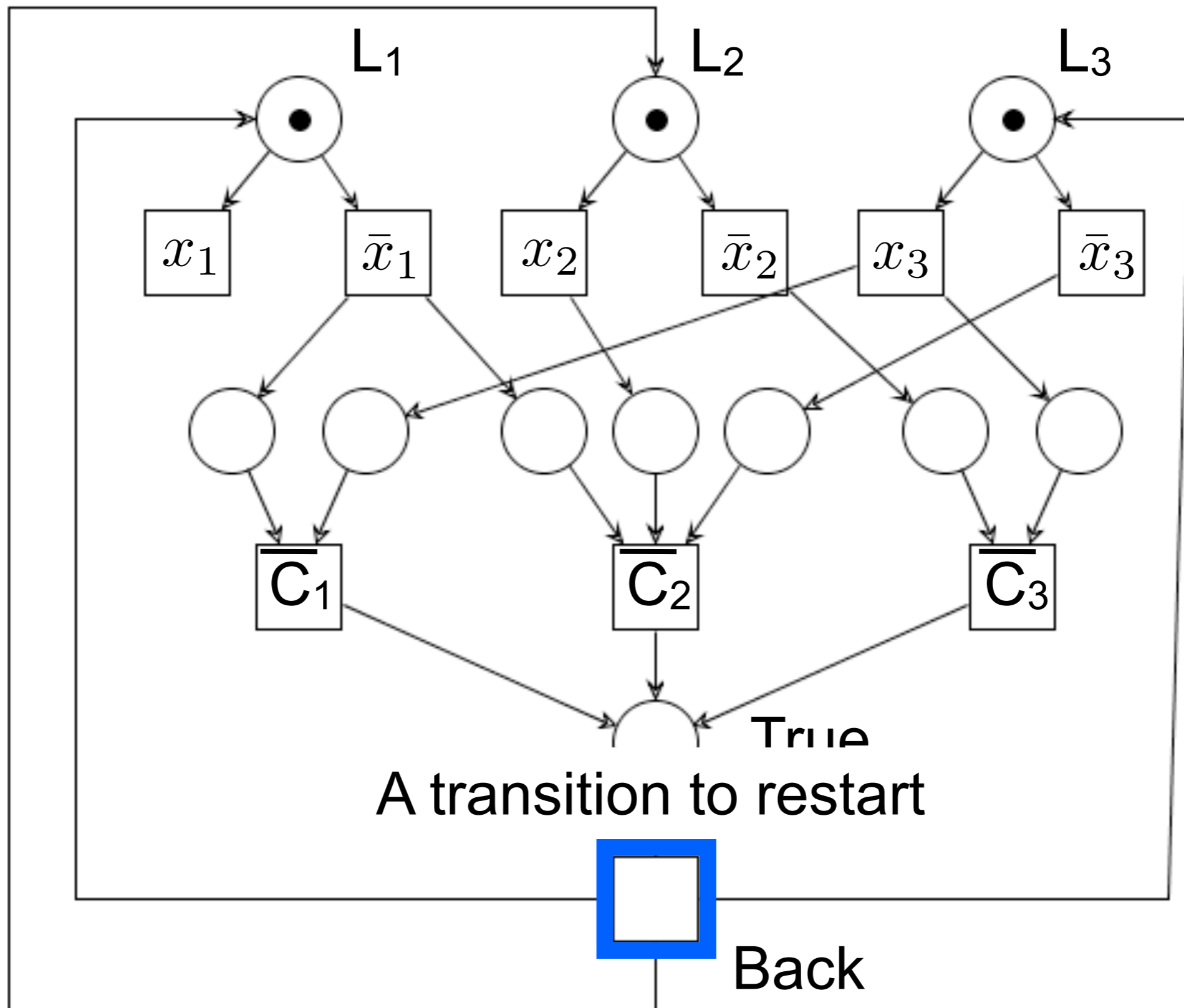
A place for each occurrence of a literal



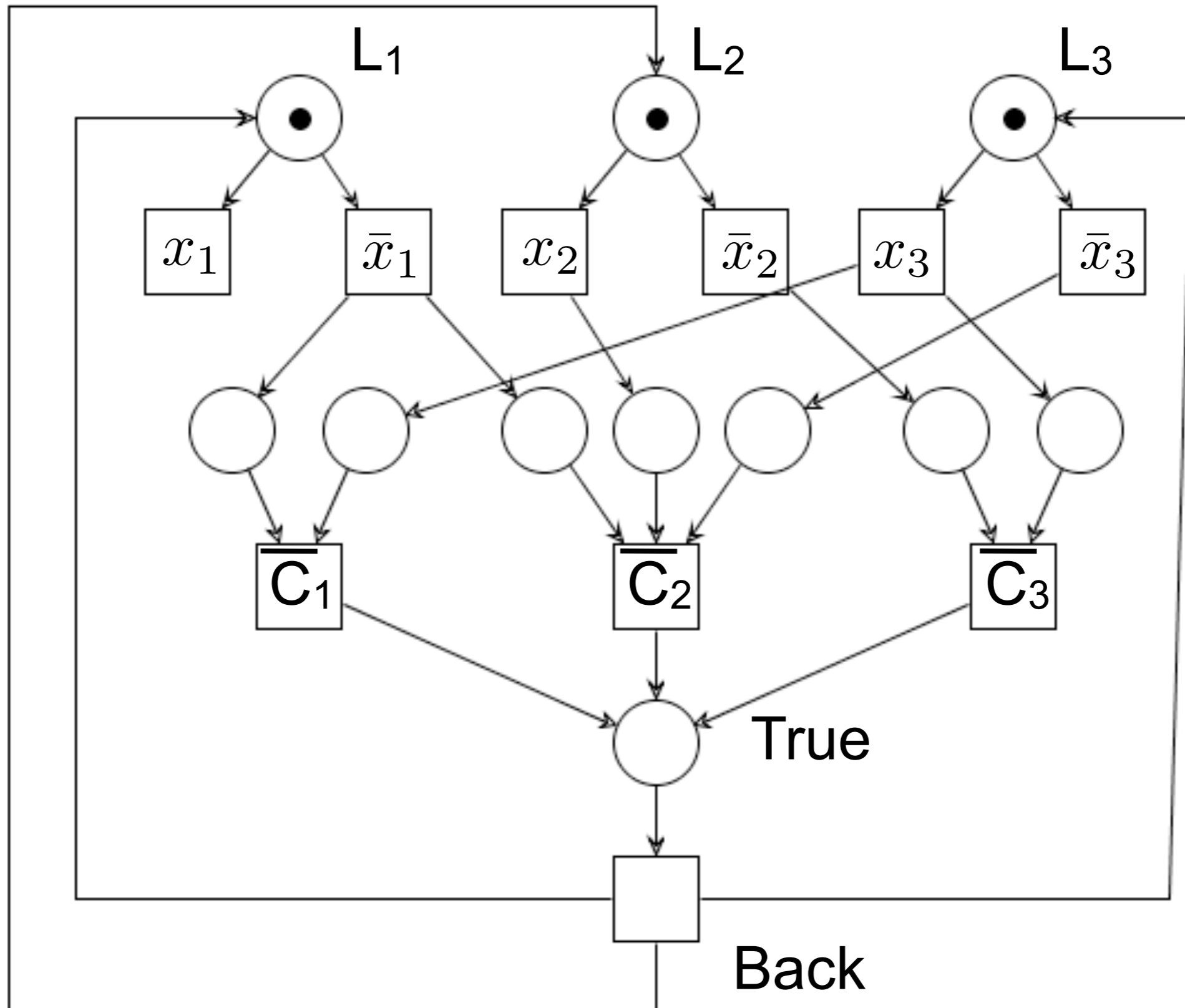
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



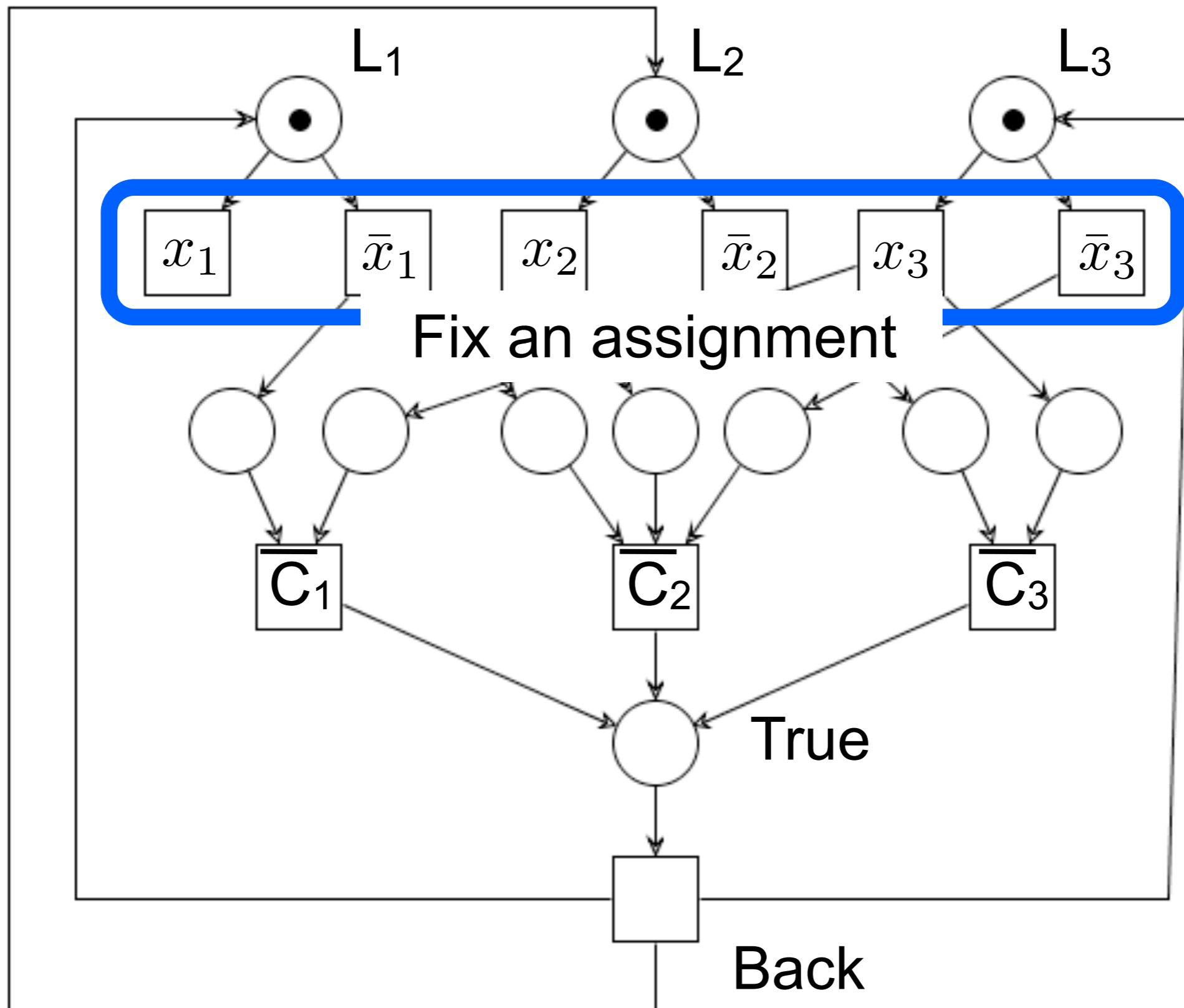
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



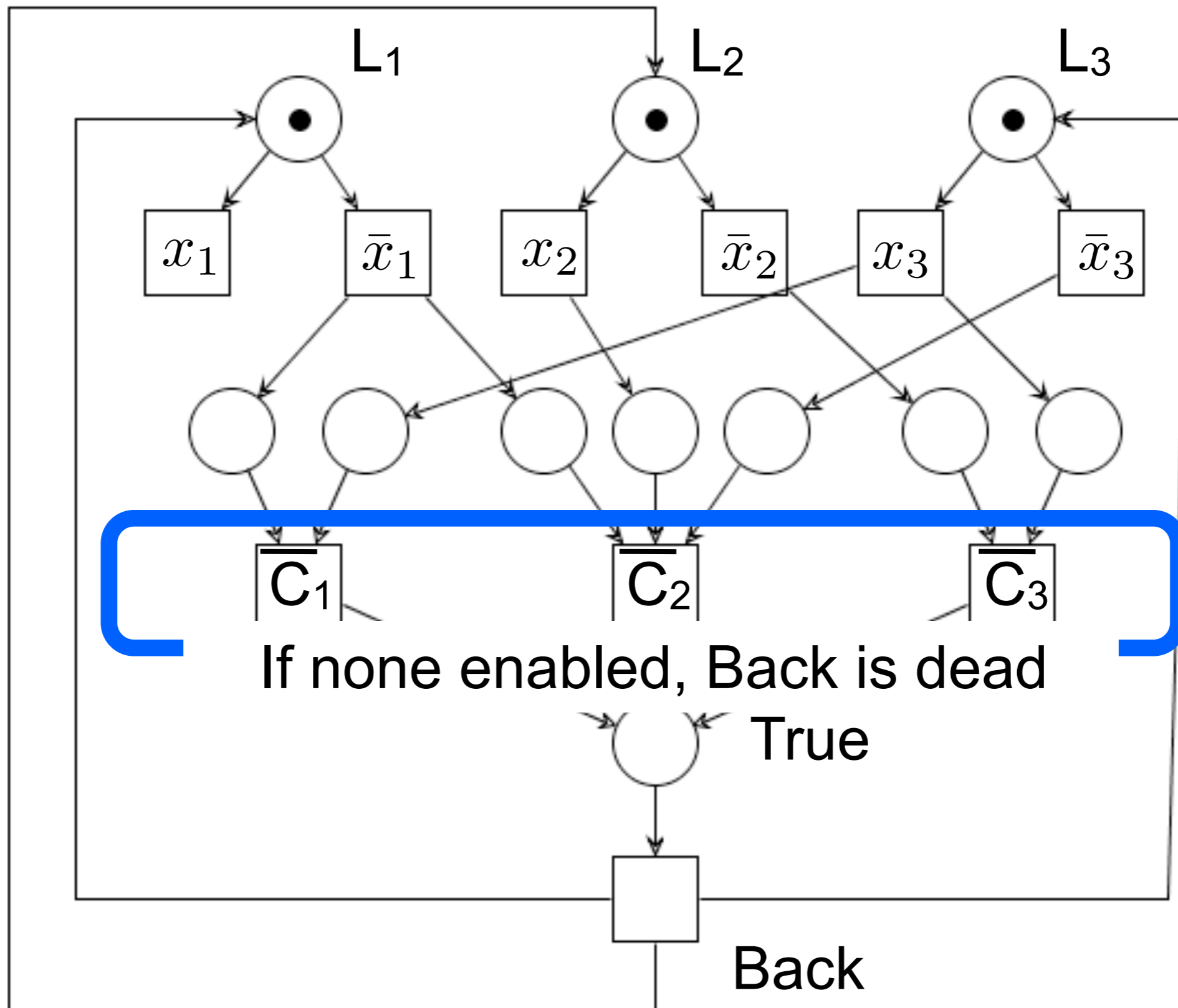
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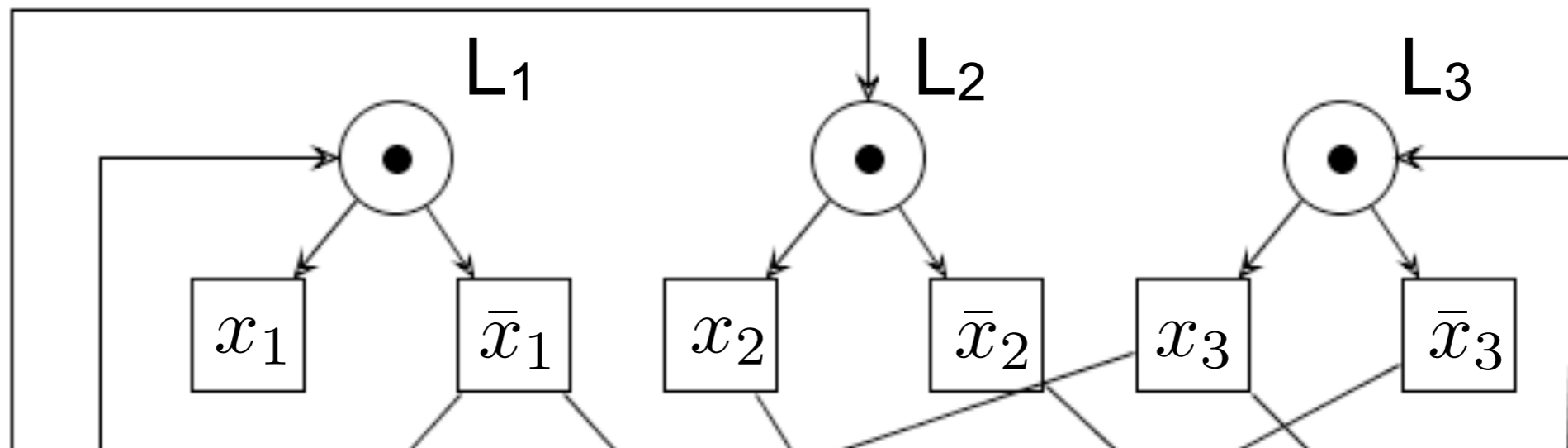
$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$

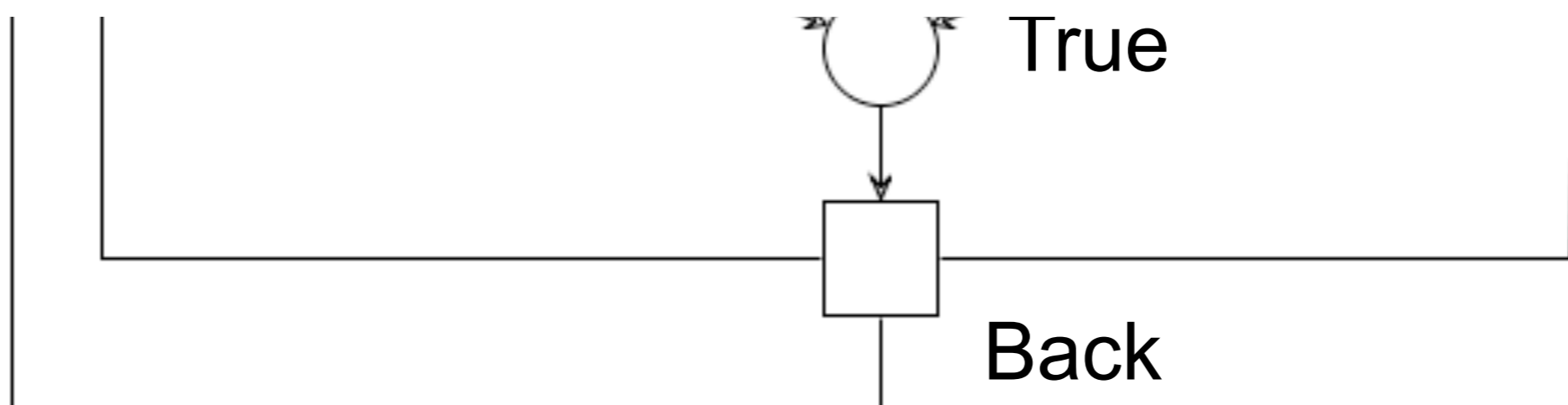


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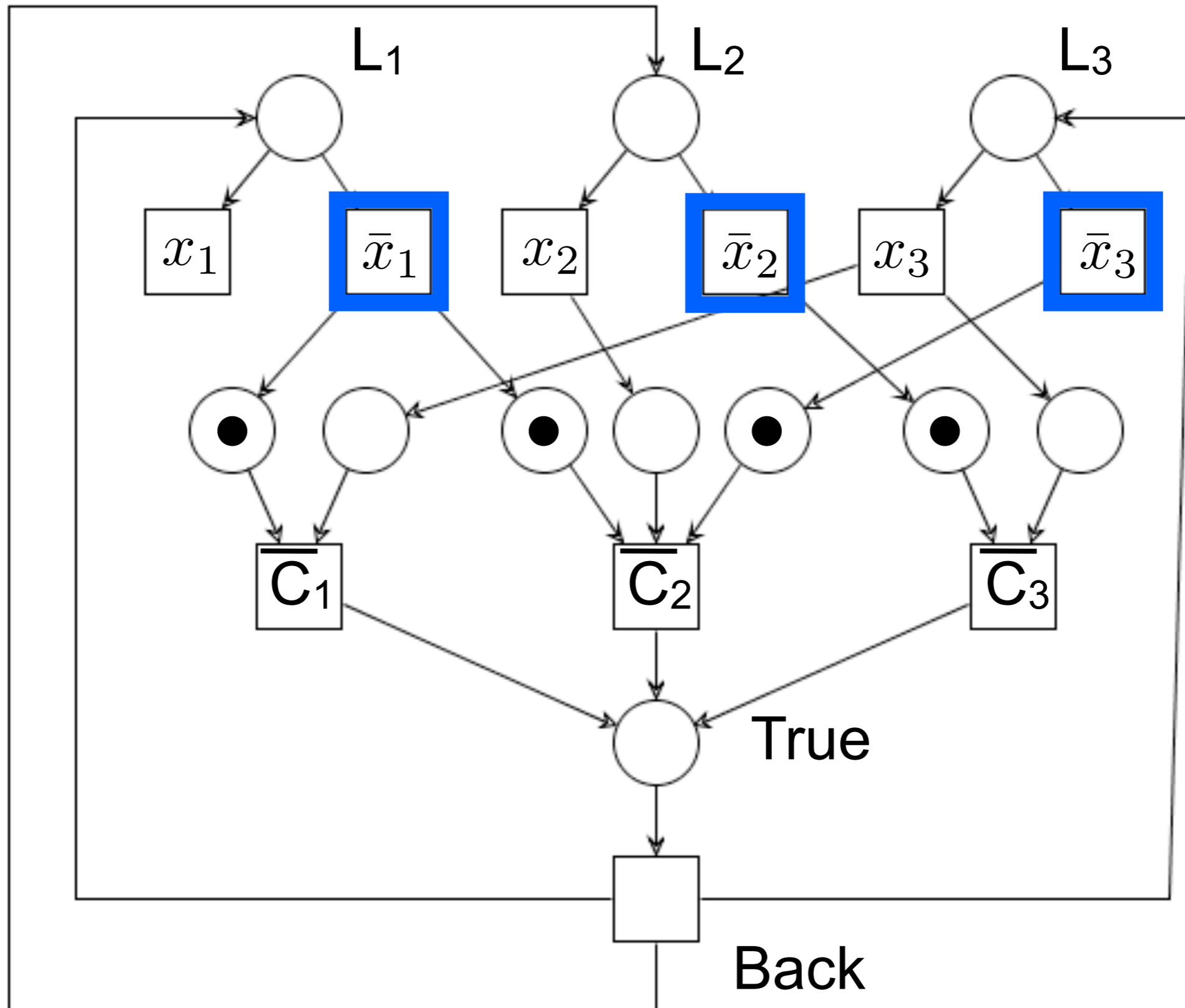


If ϕ is satisfiable, then the net is not live

If the net is not live, then ϕ is satisfiable



$$\neg\phi = (\bar{x}_1 \wedge x_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_3)$$



Main consequence

No deterministic polynomial algorithm to decide liveness of a free-choice system is currently available

(unless $P=NP$)



Complexity issues 2:
Is it hard to show that
a free-choice net is
live and bounded?

Rank Theorem: a polynomial decision algorithm

Theorem:

A free-choice system (P, T, F, M_0) is live and bounded
iff

1. it has at least one place and one transition polynomial
2. it is connected polynomial
3. M_0 marks every proper siphon
4. it has a positive S-invariant polynomial
5. it has a positive T-invariant polynomial
6. $\text{rank}(N) = |C_N| - 1$ polynomial

(where C_N is the set of clusters)

A polynomial algorithm for maximal unmarked siphon

3. M_0 marks every proper siphon **polynomial**

Input: A net $N = (P, T, F, M_0)$, $R = \{p \mid M_0(p) = 0\}$

Output: $Q \subseteq R$ maximal unmarked siphon
($\bullet Q \subseteq Q \bullet$)

$Q := R$

while ($\exists p \in Q, \exists t \in \bullet p, t \notin Q \bullet$)

$Q := Q \setminus \{p\}$

return Q If Q is empty then M_0 marks every proper siphon

Main consequence

**The problem to decide
if a free-choice system is live and bounded
can be solved in polynomial time
(thanks to the Rank Theorem)**



Recap: free-choice nets

f.c. net: place liveness \Leftrightarrow liveness

f.c. net: non-live \Rightarrow exists a proper siphon R and $M \in [M_0]$
such that $M(R)=0$

f.c. net: every siphon contains a marked trap \Leftrightarrow live

f.c. net: bounded and live \Leftrightarrow 6 conditions in Rank Theorem

• $R \subseteq R$ •

siphon

empty siphons
remain empty

Exercise

• $Q \supseteq Q$ •

trap

marked traps
remain marked

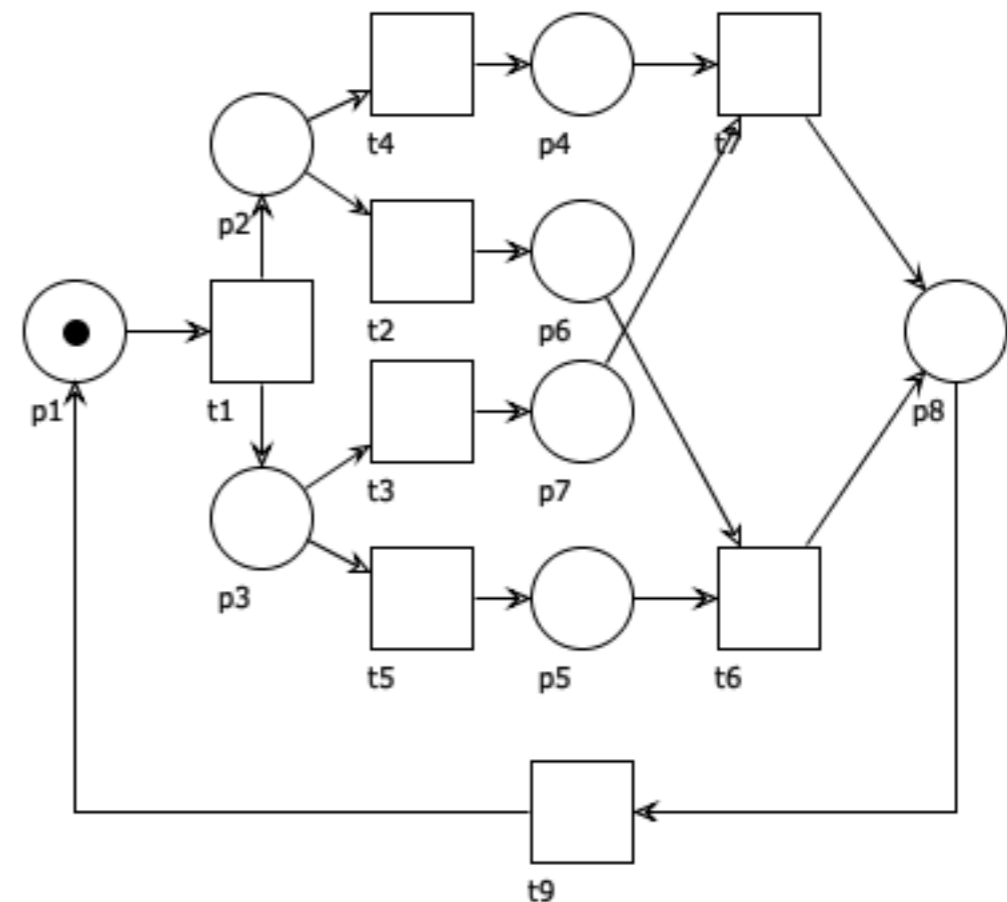
The system below is free-choice and non-live:
find a proper siphon that does not include a marked trap

Hint: take

$R = \{p_1, p_2, p_3, p_4, p_5, p_8\}$

and show that:

it is a siphon and
it contains no trap



• $R \subseteq R$ •

siphon

empty siphons
remain empty

Exercise

• $Q \supseteq Q$ •

trap

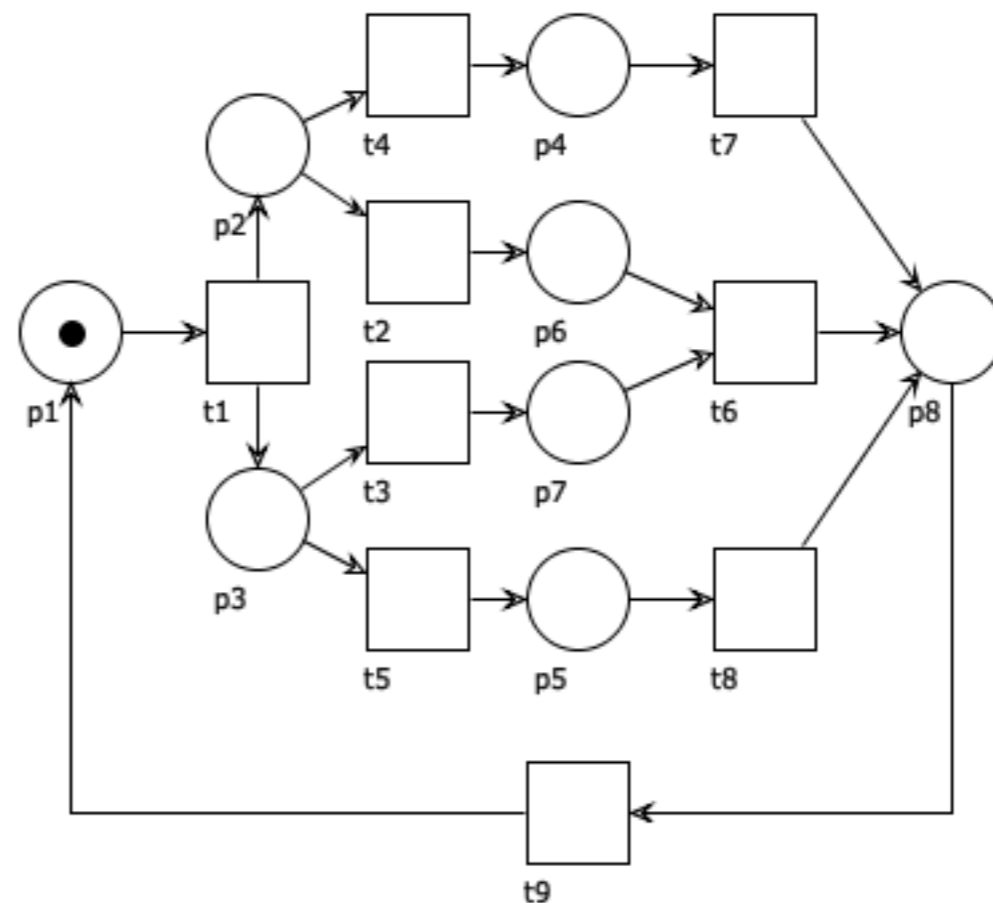
marked traps
remain marked

The system below is free-choice and live:
show that every proper siphon includes a marked trap

Hint: the only proper siphons are

$R_1 = \{p_1, p_2, p_3, p_4, p_5, p_7, p_8\}$
and

$R_2 = \{p_1, p_2, p_3, p_4, p_5, p_6, p_8\}$



Exercise

Draw the net corresponding to the formula

$$x_2 \wedge (x_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_4)$$

Is it satisfiable?