Business Processes Modelling

MPB (6 cfu, 295AA)

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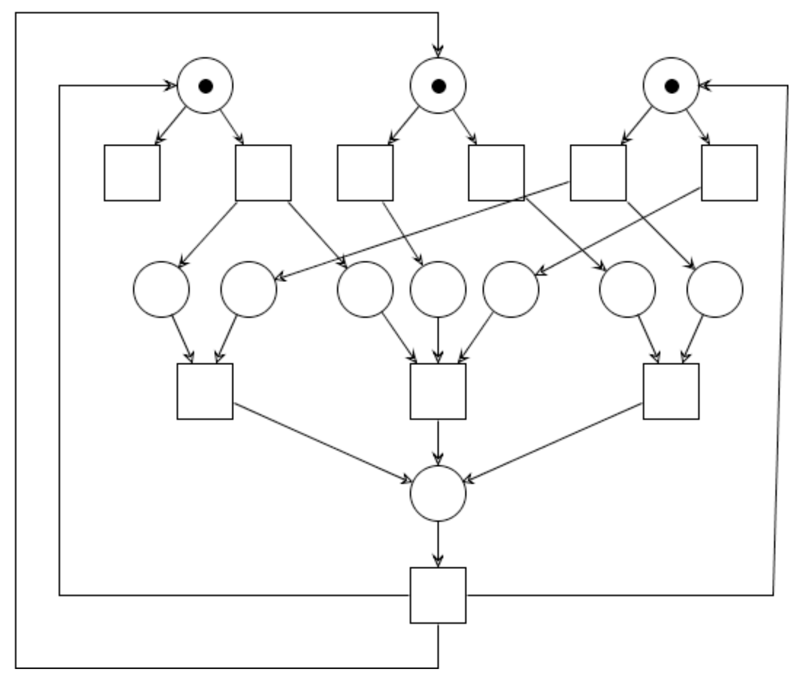
18 - Free-choice nets

Object

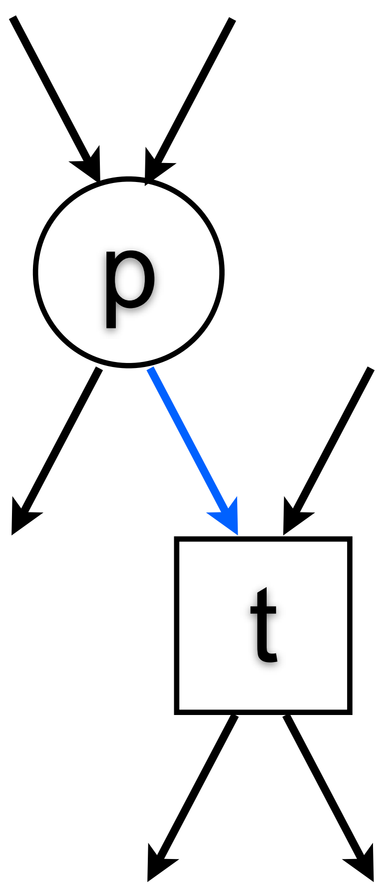


We study some “good” properties of free-choice nets

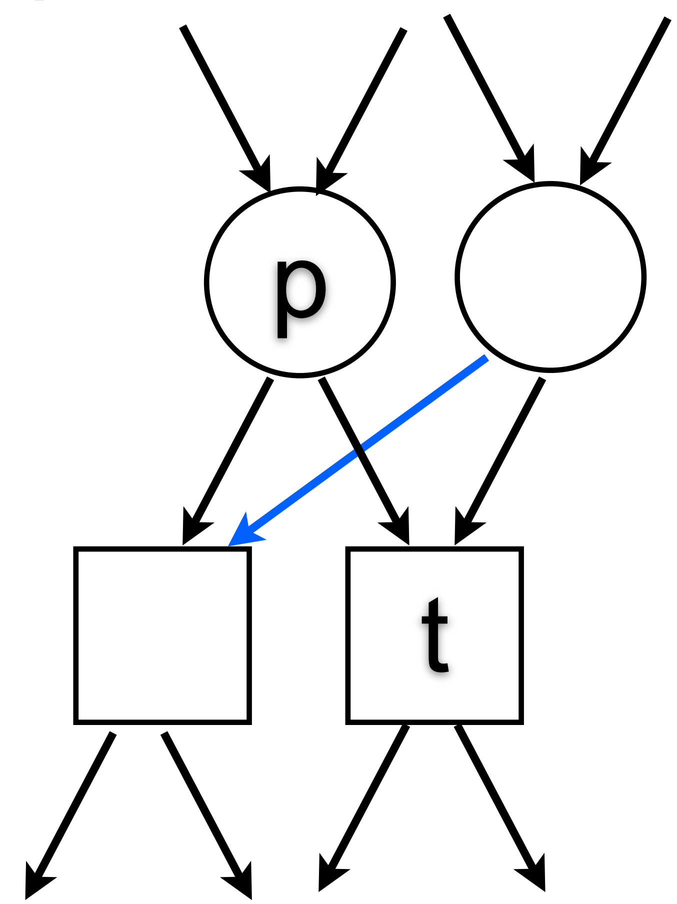
Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Free-choice net

Definition: We recall that a net N is **free-choice** if whenever there is an arc (p,t), then there is an arc from any input place of t to any output transition of p



implies



Free-choice net: alternative definitions

Proposition: All the following definitions of free-choice net are equivalent.

1) A net (P, T, F) is free-choice if:
$\forall p \in P, \forall t \in T$, $(p,t) \in F$ implies $\bullet t \times p \bullet \subseteq F$.

2) A net (P, T, F) is free-choice if:
$\forall p, q \in P, \forall t, u \in T$, $\{(p,t),(q,t),(p,u)\} \subseteq F$ implies $(q,u) \in F$.

3) A net (P, T, F) is free-choice if:
$\forall p, q \in P$, either $p\bullet = q\bullet$ or $p \bullet \cap q \bullet = \emptyset$.

4) A net (P, T, F) is free-choice if:
$\forall t, u \in T$, either $\bullet t = \bullet u$ or $\bullet t \cap \bullet u = \emptyset$.

Free-choice net: my favourite definition

4) A net (P, T, F) is free-choice if:
$\forall t, u \in T$, either $\bullet t = \bullet u$ or $\bullet t \cap \bullet u = \emptyset$.

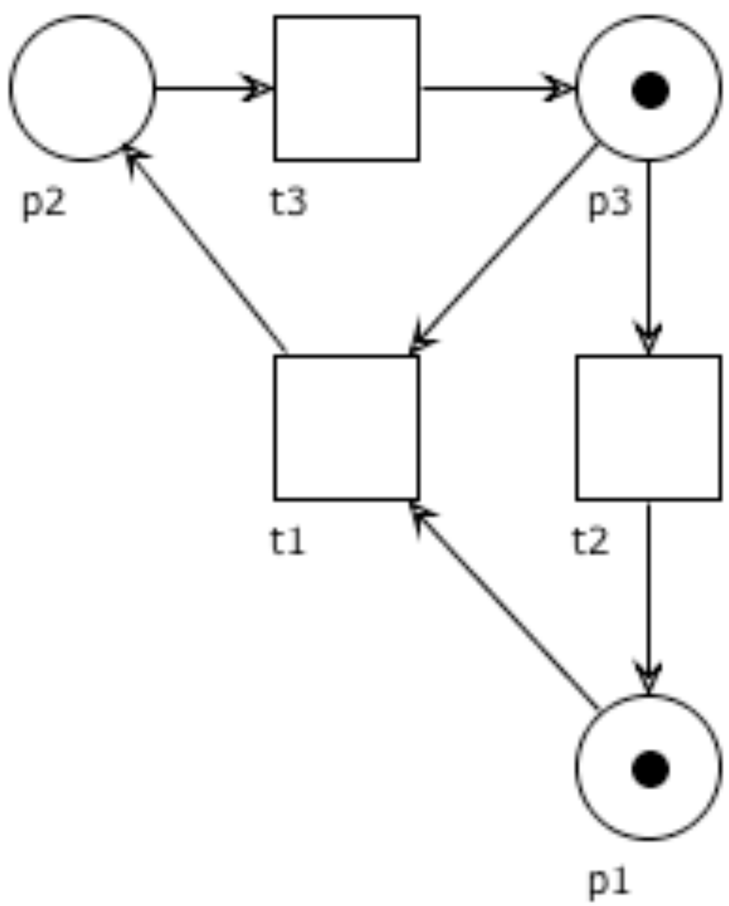
Free-choice system

Definition: A system (N,M_0) is **free-choice** if N is free-choice

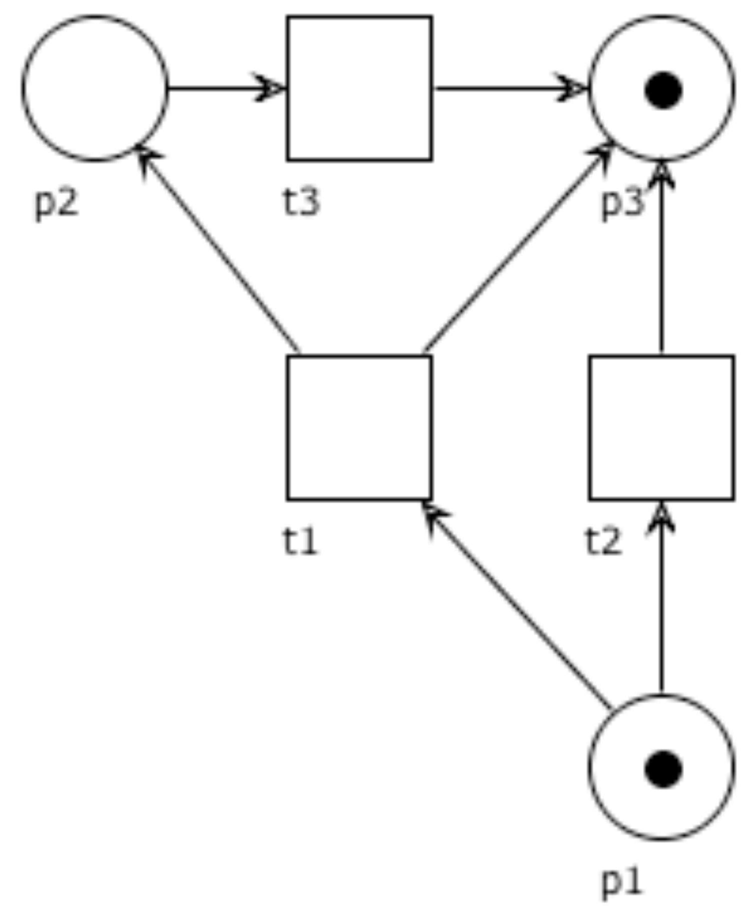
Example

$$\begin{aligned} \bullet t_1 &= \{ p_1, p_3 \} \\ \bullet t_2 &= \{ p_3 \} \\ \bullet t_1 &\neq \bullet t_2 \\ \bullet t_1 \cap \bullet t_2 &= \{ p_3 \} \neq \emptyset \end{aligned}$$

$$\begin{aligned} \bullet t_1 &= \bullet t_2 \\ \bullet t_1 \cap \bullet t_3 &= \emptyset \\ \bullet t_2 \cap \bullet t_3 &= \emptyset \end{aligned}$$



non free-choice



free-choice

Fundamental property of free-choice nets

Proposition: Let (P, T, F, M_0) be free-choice.
If $M \xrightarrow{t}$ and $t \in p\bullet$, then $M \xrightarrow{t'}$ for every $t' \in p\bullet$.

The proof is trivial, by definition of free-choice net

Free-choice N*

Proposition: A workflow net N is free-choice
iff N* is free-choice

N and N* differ only for the reset transition,
whose pre-set (o) is disjoint
from the pre-set of any other transition

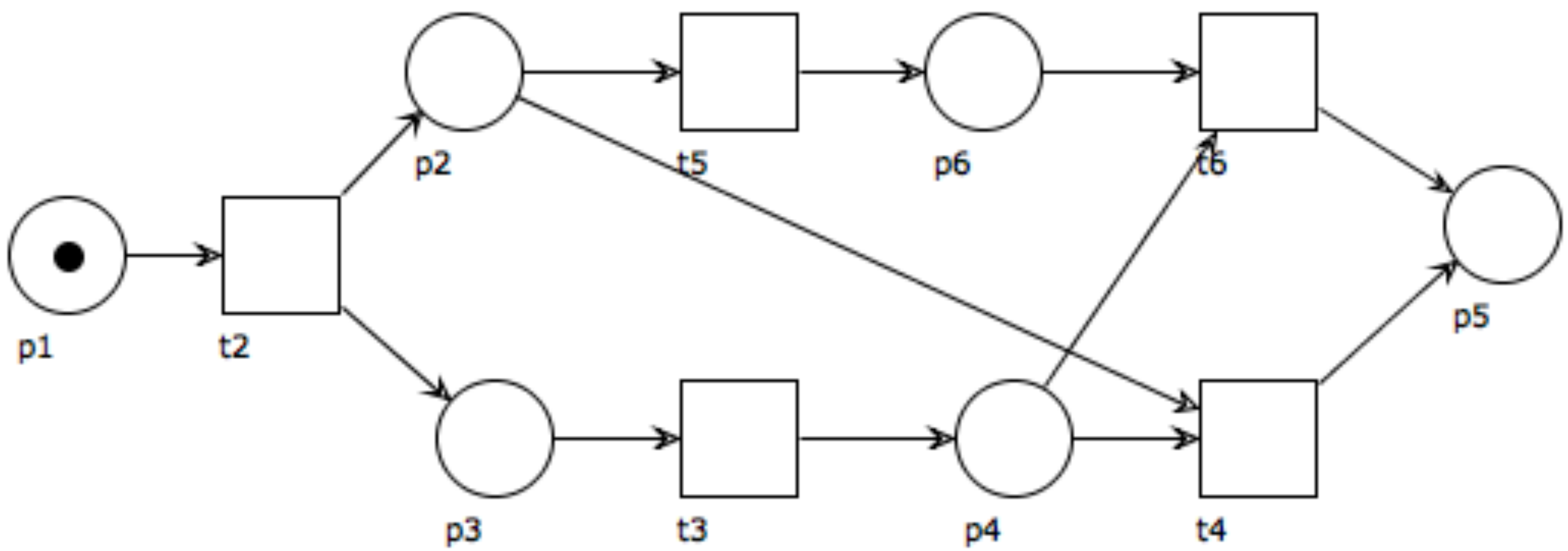
Free-Choice vs Soundness

Note that free-choice is orthogonal to soundness:

there exists WF-nets that are free-choice but not sound

there exists WF-nets that are sound but not free-choice

Example: sound but not free-choice



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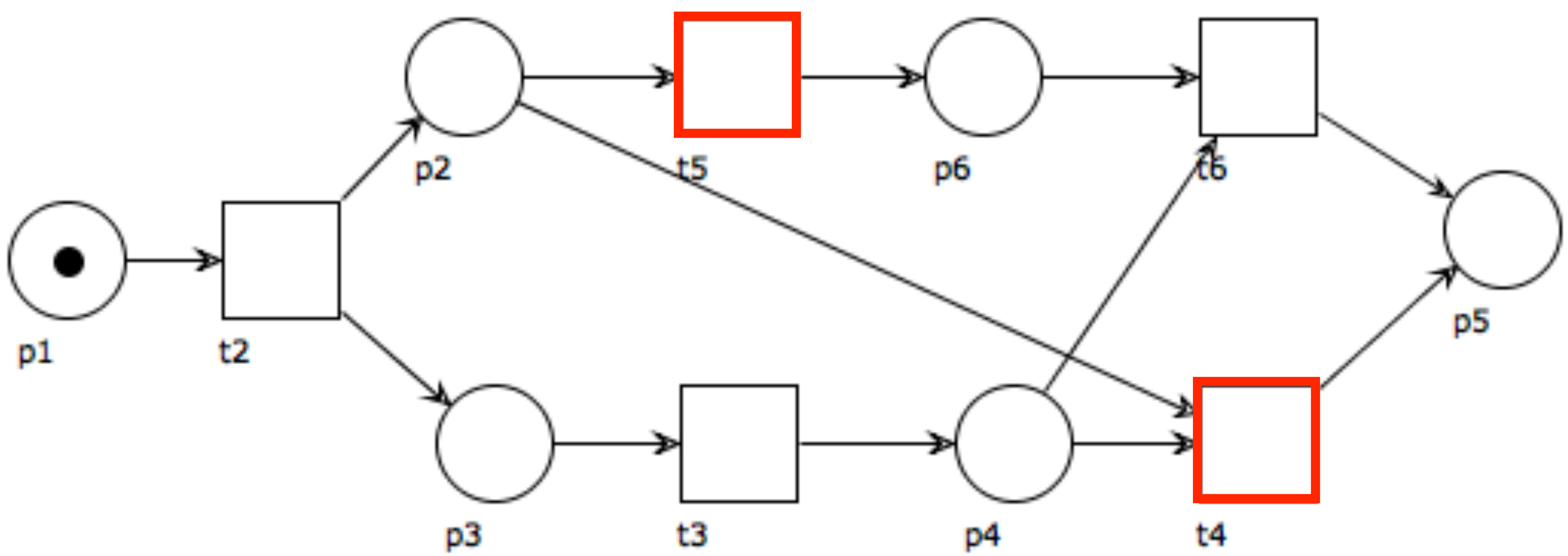

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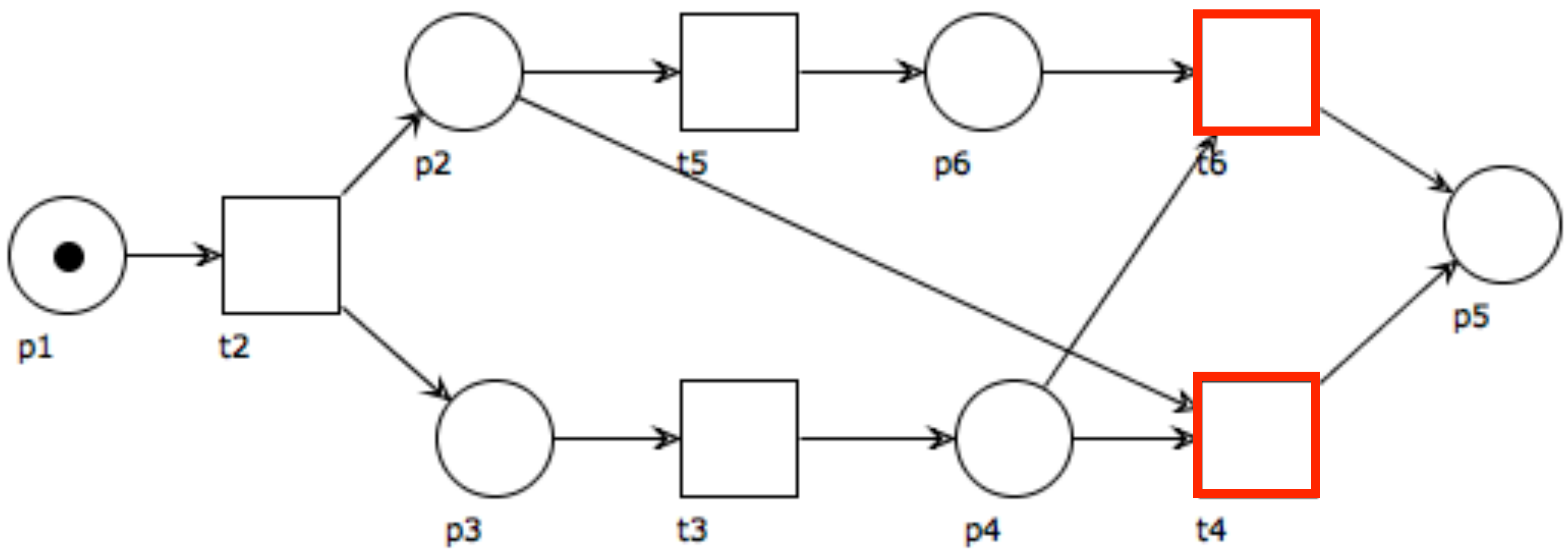

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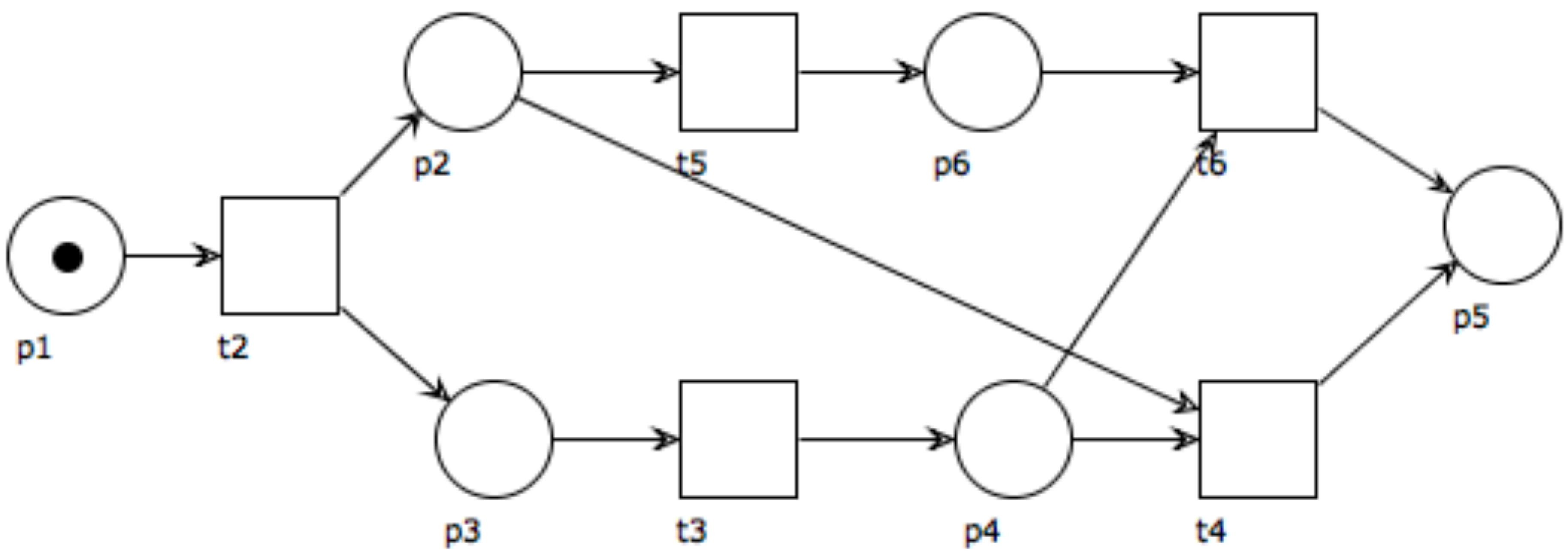
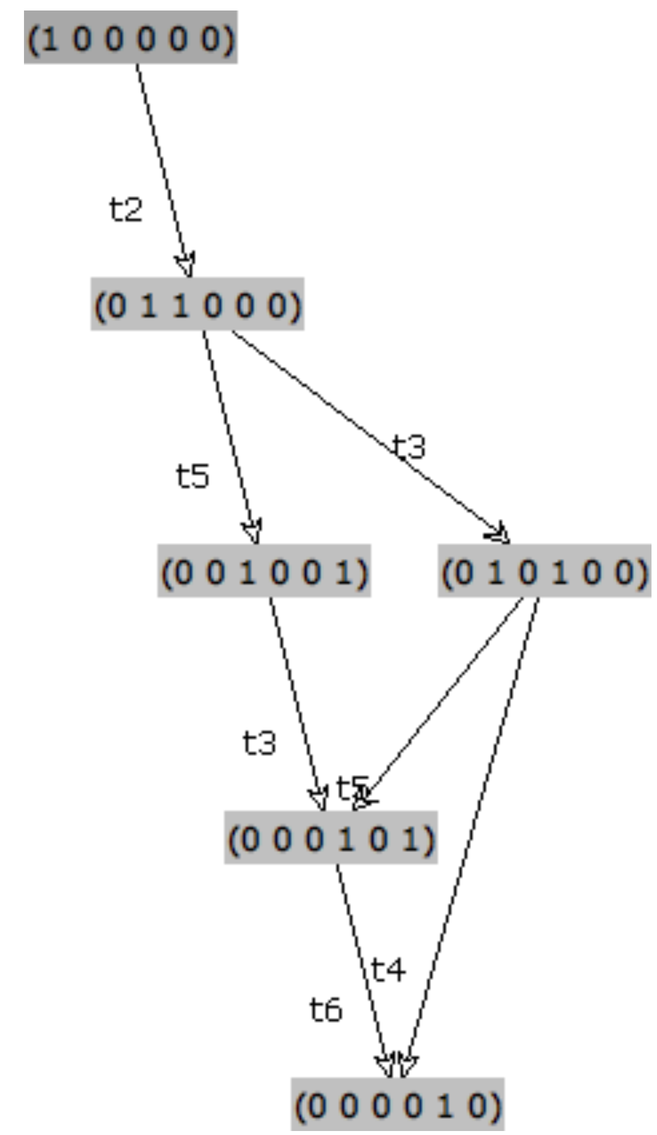




Exercise

Draw a workflow net that is free-choice but not sound

Rank Theorem (main result, proof omitted)

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |C_N| - 1$

(where C_N is the set of clusters)

Clusters

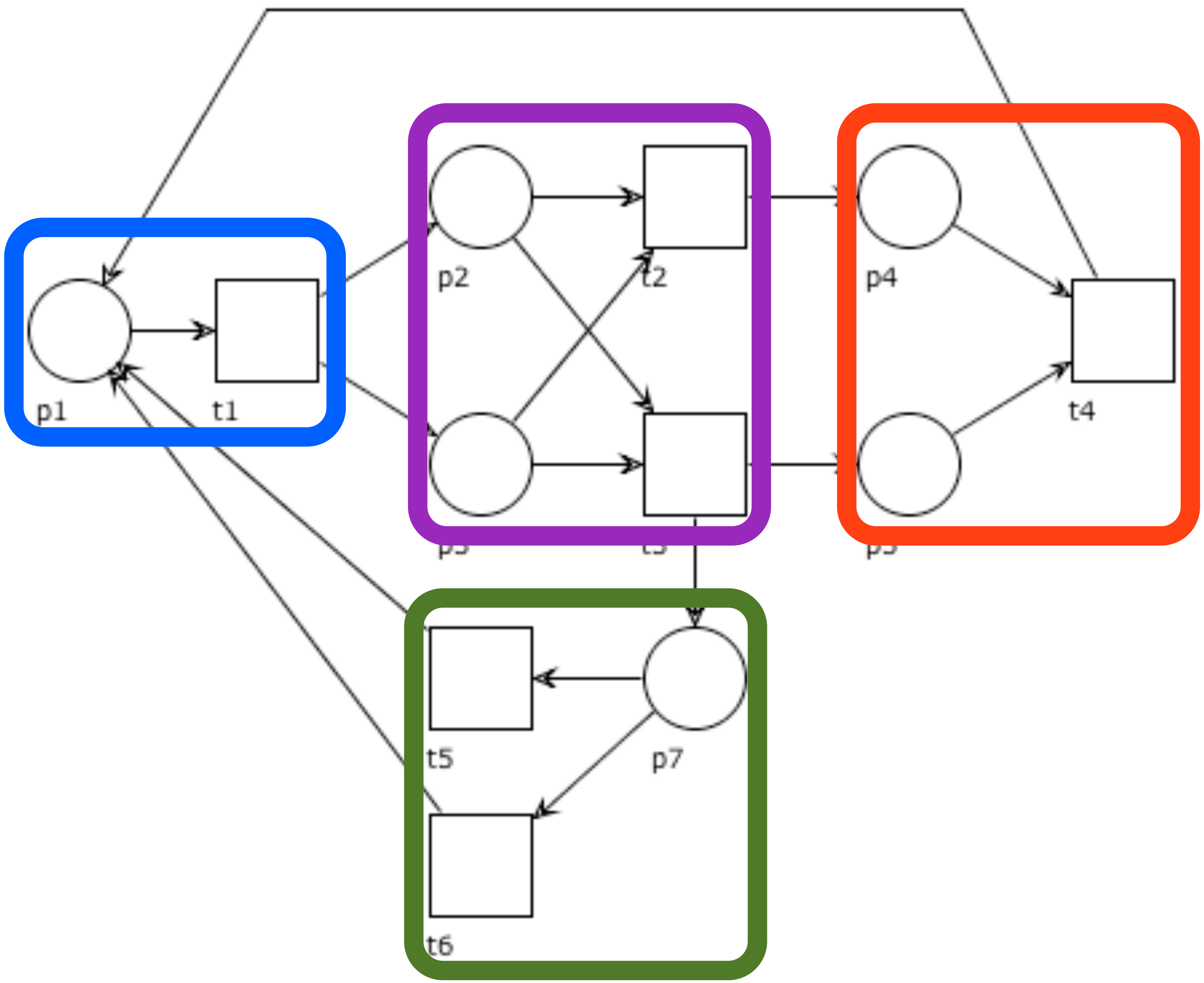
Cluster

Let x be the node of a net $N = (P, T, F)$
(not necessarily free-choice)

Definition:
The **cluster** of x, written $[x]$, is the least set s.t.

1. $x \in [x]$
2. if $p \in [x] \cap P$ then $p\bullet \subseteq [x]$ (if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)
3. if $t \in [x] \cap T$ then $\bullet t \subseteq [x]$ (if a transition t is in the cluster, then all places in the pre-set of t are in the cluster)

Cluster: example



Observation

Every place belongs to exactly one cluster

Every transition belongs to exactly one cluster

The set $\{ [x] \mid x \in P \cup T \}$ is a partition of $P \cup T$

Fundamental property of clusters in f.c. nets

Proposition:

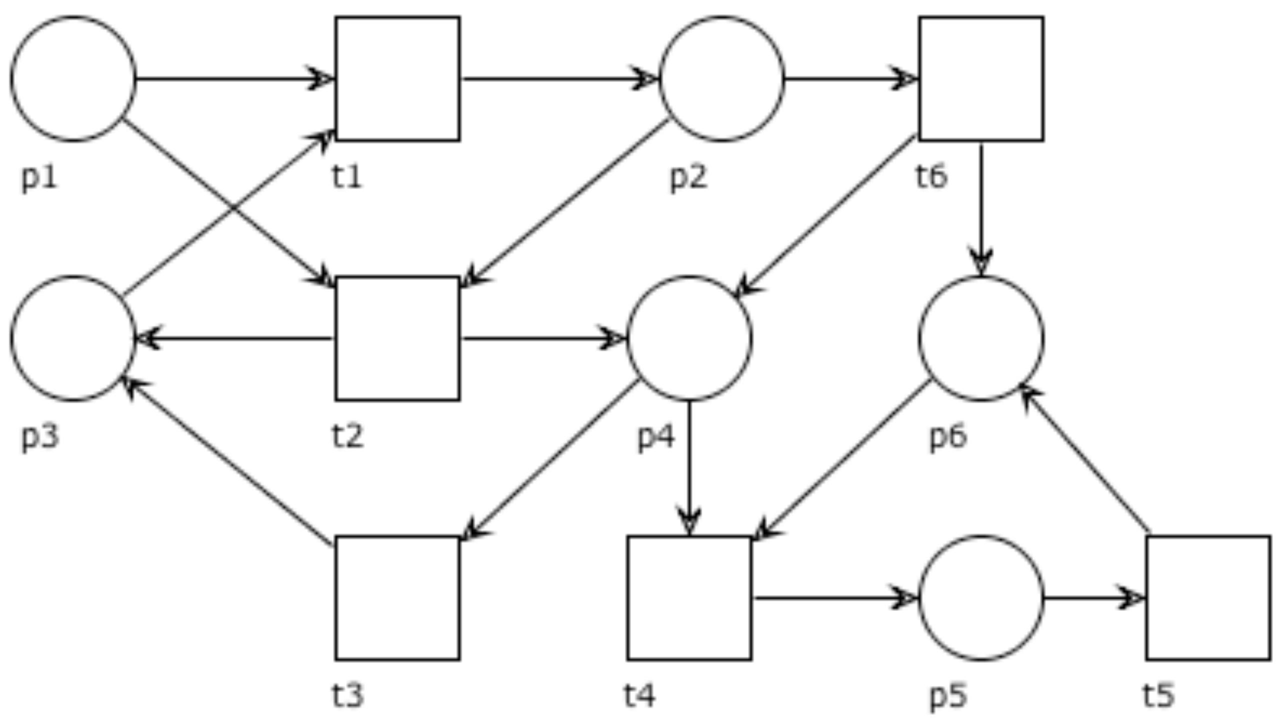
If $M \xrightarrow{t}$, then for any $t' \in [t]$ we have $M \xrightarrow{t'}$

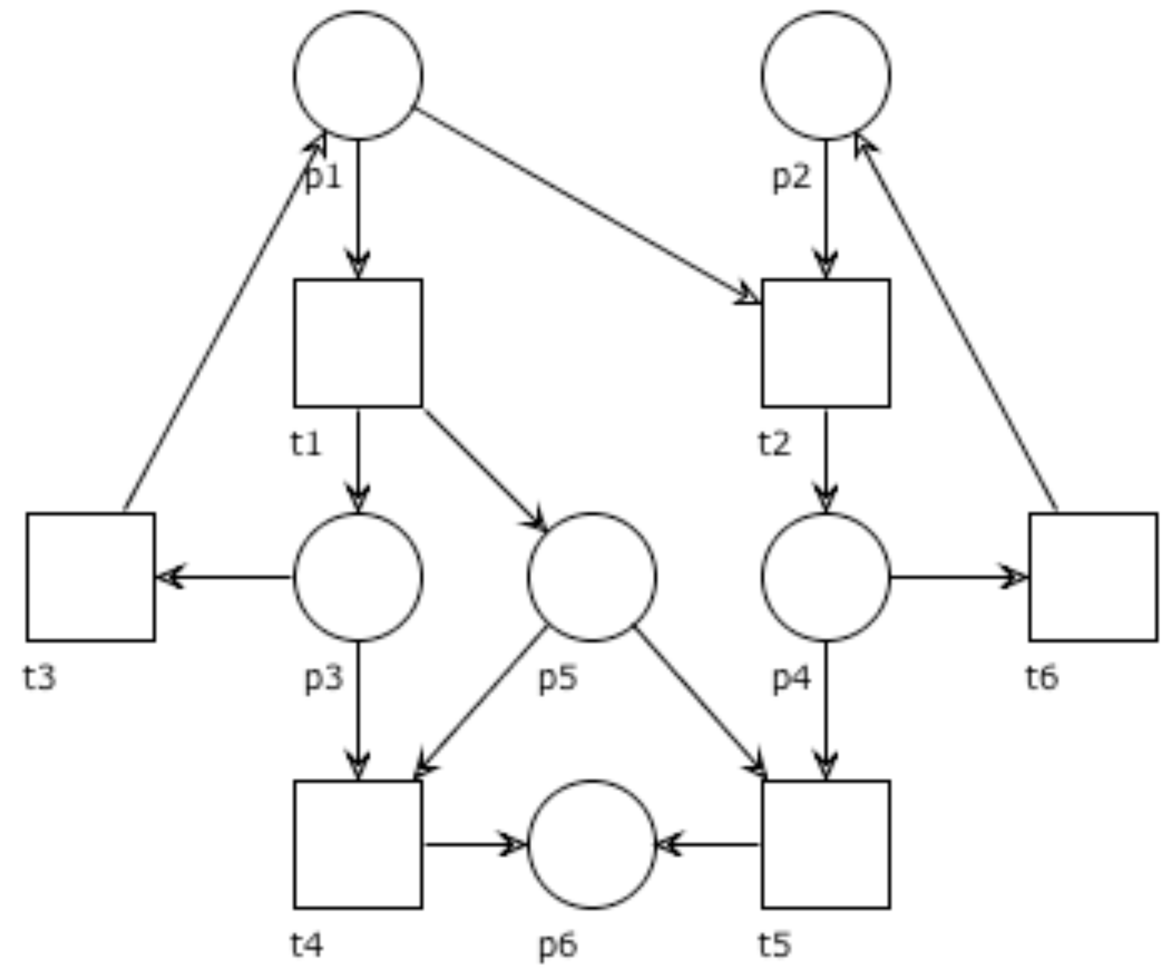
Immediate consequence of the fact that, for free-choice nets

$$t, t' \in [x] \qquad \text{iff} \qquad \bullet\, t = \bullet t'$$

Exercise

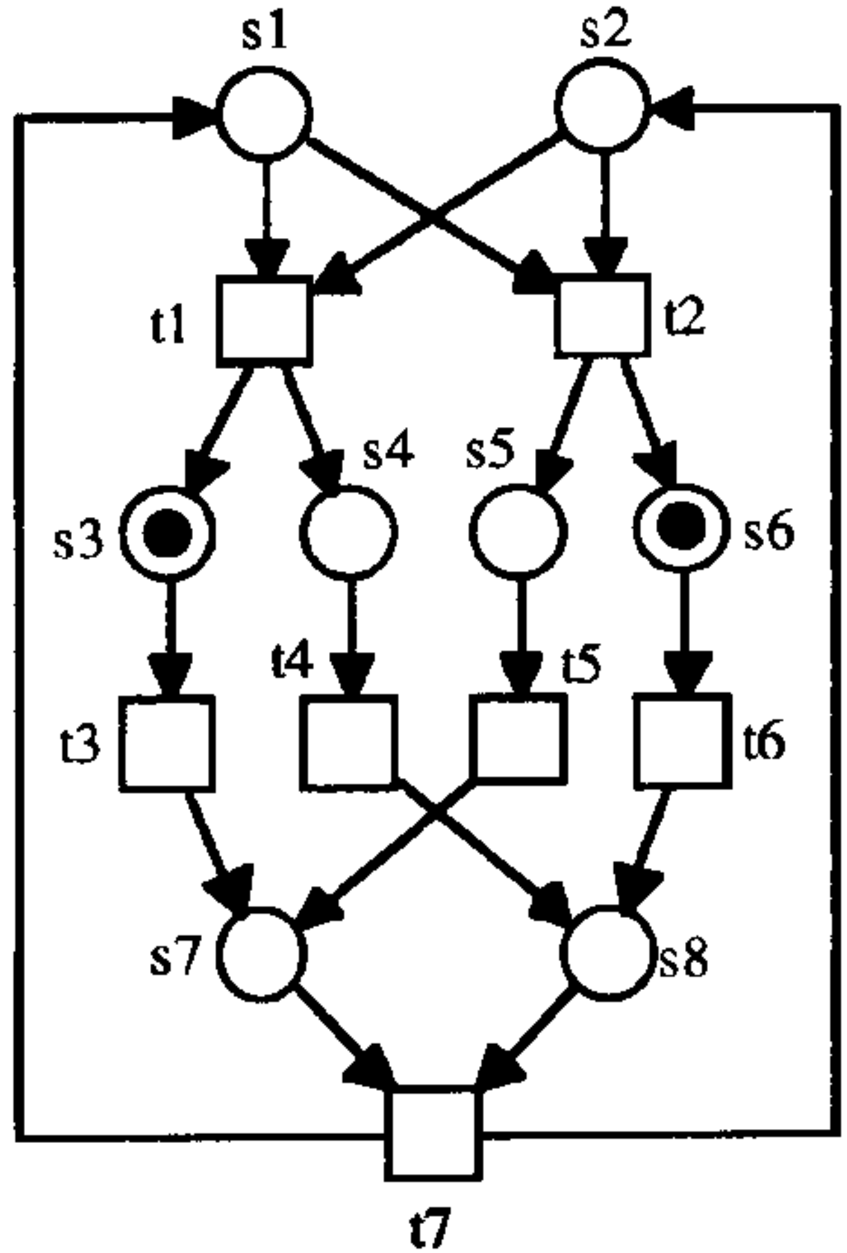
Draw all clusters in the nets below





Exercise

Draw all clusters in the free-choice net below



Stable markings

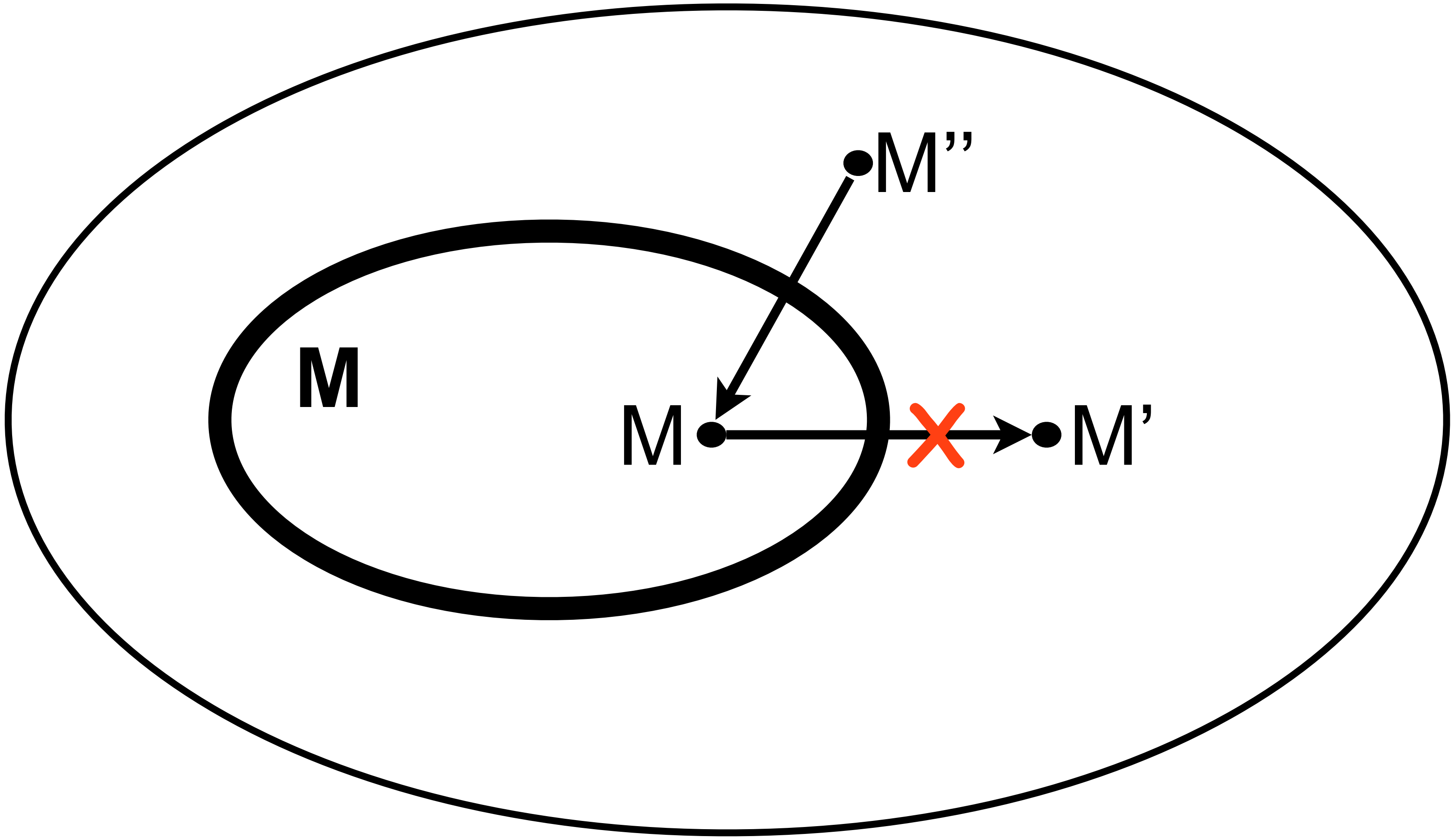
Stable set of markings

Definition: A set of markings $\mathbf{M}$ is called **stable** if

$$M \in \mathbf{M} \qquad \text{implies} \qquad [\, M \,\rangle \subseteq \mathbf{M}$$

(starting from any marking in the stable set **M**, no marking outside **M** is reachable)

Stable set of markings



(starting from any marking M in the stable set **M**, no marking M' outside **M** is reachable)

Stability check

$$\mathbf{M} \text{ is stable iff}$$
$$\forall M, t, M'.\, (M \in \mathbf{M} \;\wedge\; M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Question time

Given a net system:

Is the singleton set { 0 } a stable set?

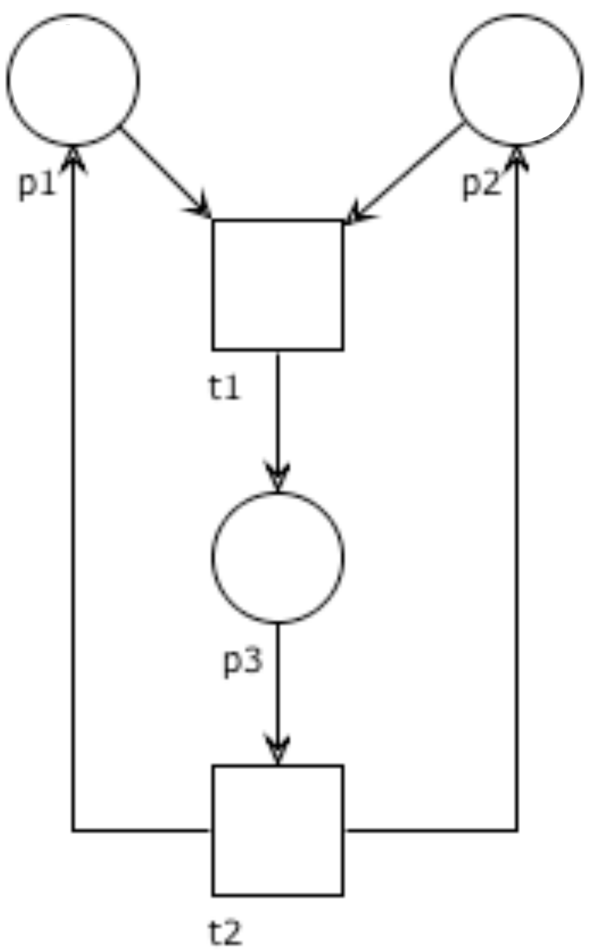
Is the set of all markings a stable set?

Is the set of live markings a stable set?

Is the set of deadlock markings a stable set?

Example

Which of the following is a stable set of markings?



$\{ 2p_1+p_2 \}$

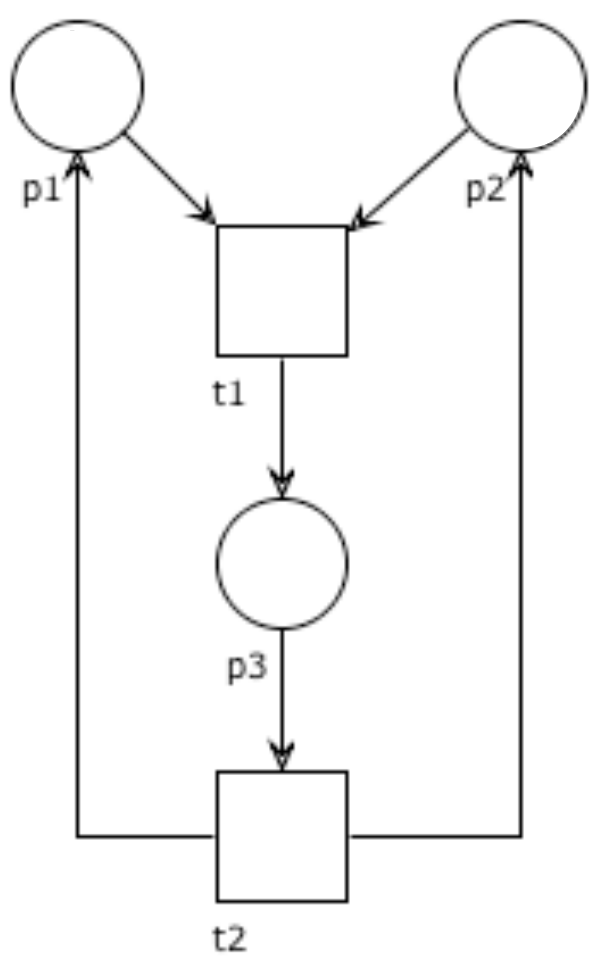
$\{ 2p_1+p_2 , p_1+2p_3 \}$

$\{ p_1 , p_2 \}$

$\{ p_1+p_2 , p_3 \}$

Exercises

Which of the following is a stable set of markings?



$\{ p_1 , \ p_3 \}$

$\{ 2p_1+2p_2 , 2p_3 \}$

$\{ 2p_1+2p_2 , p_1+p_2+p_3 , 2p_3 \}$

$\{ p_1, 2p_1+2p_2 , p_1+p_2+p_3 , 2p_3 \}$

Exercises

Given a net system (P,T,F,M_0):

Is the set $\{ M \mid M(P)=1 \}$ a stable set?

Is the set of markings reachable from M_0 a stable set?

Is the set $\{ M \mid M(P)<k \}$ a stable set?

Exercises

Let **I** be an S-invariant

Is the set $\{ M \mid \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M = 1 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M = 0 \}$ a stable set?

Exercises

Let **M** and **M'** be stable sets
Prove that their union is a stable set
Prove that their intersection is a stable set
Is their difference a stable set?

What is the least stable set that includes a marking M_0?

What is the largest stable set of a net?

Siphons

Proper siphon

Definition:

A set of places R is a **siphon** if $\bullet R \subseteq R \bullet$

It is a **proper siphon** if $R \neq \emptyset$

Siphons, intuitively

A set of places R is a siphon if

all transitions that can produce tokens in the places of R

require some place in R to be marked

Therefore:
if no token is present in R,
then no token will ever be produced in R

Siphon check

Let R be a set of places of a net

mark with √ all transitions that consume tokens from R

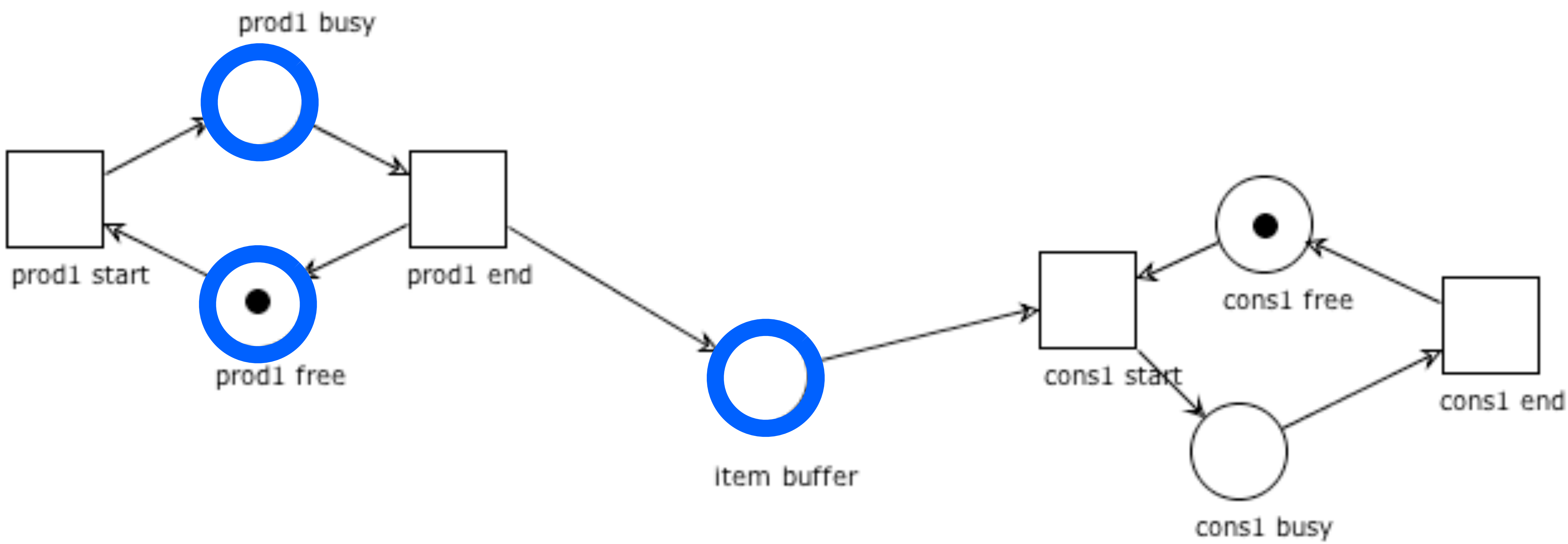
if there is a transition producing tokens in some place of R that is not marked by √, then R is not a siphon

Otherwise R is a siphon

$\bullet R \subseteq R \bullet$

Siphon check: example

Is R = { prod1busy, prod1free, itembuffer} a siphon?



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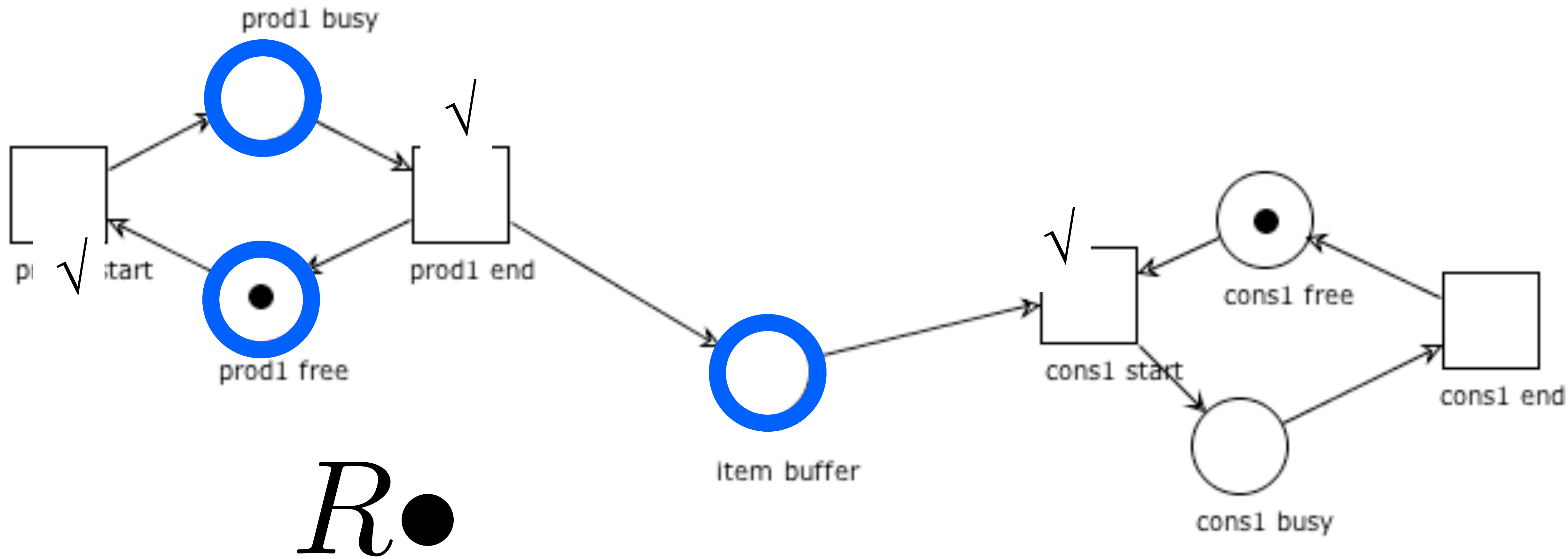

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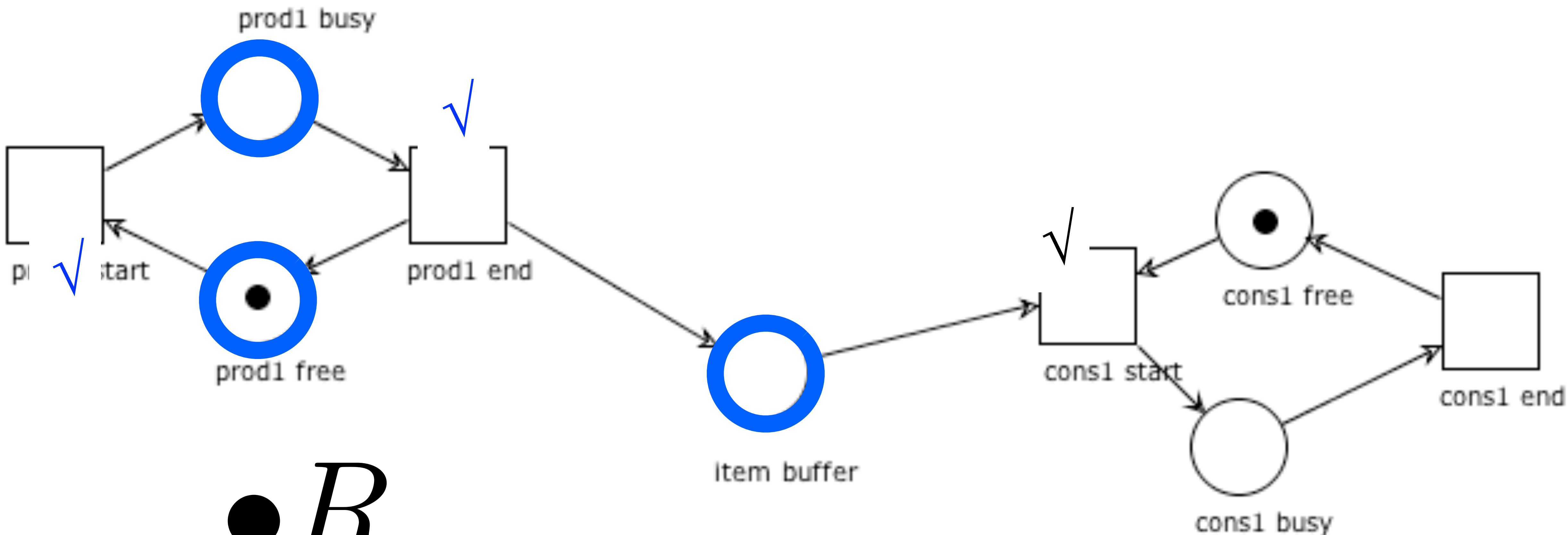

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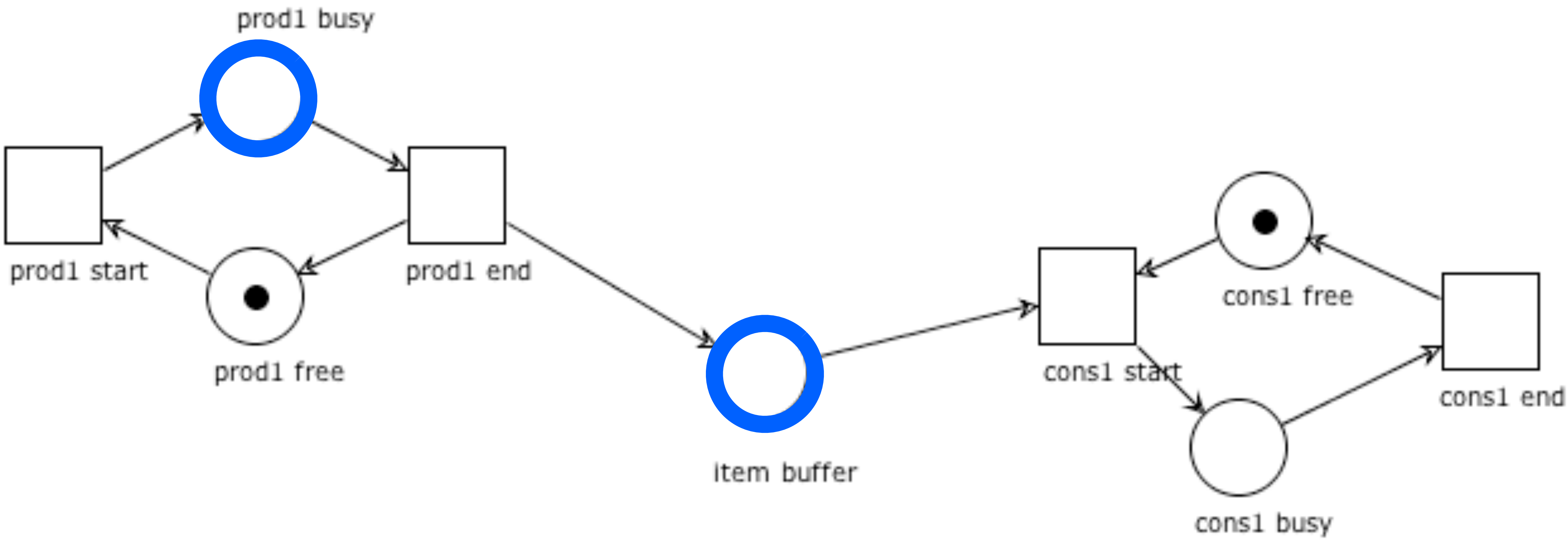

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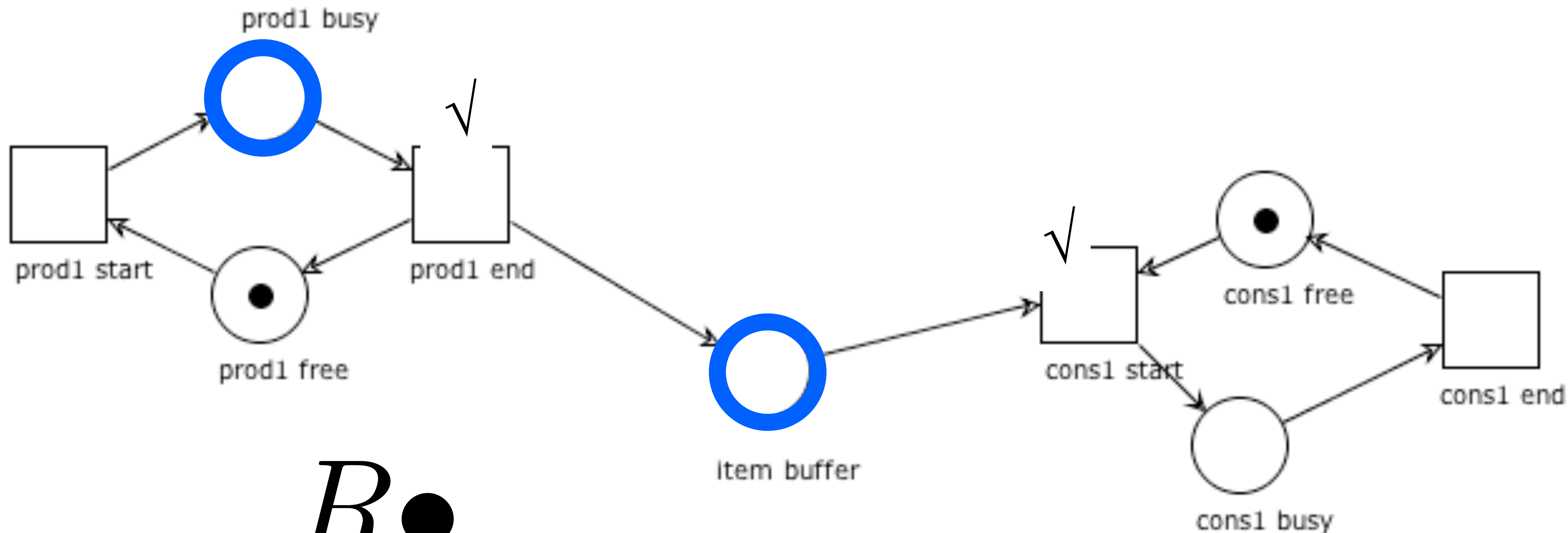

$R \bullet$

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Siphon check: example

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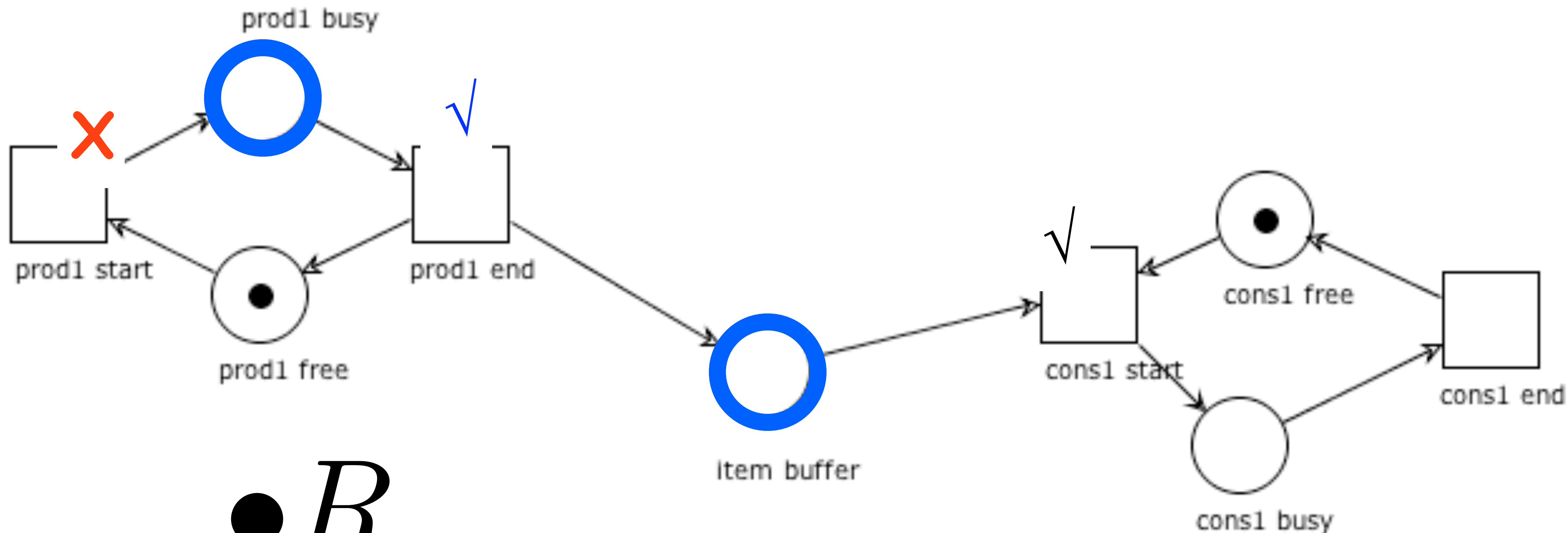


$\bullet R$

Fundamental property of siphons

Proposition: Unmarked siphons remain unmarked

Take a siphon R.
We just need to prove that the set of markings
M = { M | M(R)=0 }
is stable, which is immediate by definition of siphon

Corollary:
If a siphon R is marked at some reachable marking M, then it was initially marked at M_0

Siphons and liveness

Prop.: Live systems have no unmarked proper siphons

(We prove: $M_0(R)>0$ for every proper siphon R of a live system)

Take $p \in R$ and let $t \in \bullet p \cup p\bullet$

Since the system is live, then there are $M, M' \in [\, M_0 \,\rangle$ such that

$$M \xrightarrow{t} M'$$

Therefore p is marked at either M or M'
Therefore R is marked at either M or M'
Therefore R was initially marked (at M_0)

Siphons and liveness

Corollary: If a system has an unmarked proper siphon then it is not live

Siphons and liveness

Corollary: If a system has an unmarked proper siphon then it is not live

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |C_N| - 1$

(where C_N is the set of clusters)

Siphons and deadlock

Prop.: Deadlocked systems have an unmarked proper siphon

Let M be a deadlocked marking

Let $R = \{\, p \mid M(p) = 0 \,\}$

Since M is deadlock: $R\bullet = T$

Therefore $\bullet R \subseteq T = R\bullet$ and R is a siphon.
Since T cannot be empty, R is proper

A key observation

If we can guarantee that

all proper siphons are marked
at **every** reachable marking,

then the system is deadlock free

Exercise

Prove that the union of siphons is a siphon

Traps

Proper trap

Definition:

A set of places R is a **trap** if $\bullet R \supseteq R \bullet$

It is a **proper trap** if $R \neq \emptyset$

Traps, intuitively

A set of places R is a trap if

all transitions that can consume tokens from R

produce some token in some place of R

Therefore:
if some token is present in R,
then it is never possible for R to become empty

Trap check

Let R be a set of places of a net

mark with √ all transitions that produce tokens in R

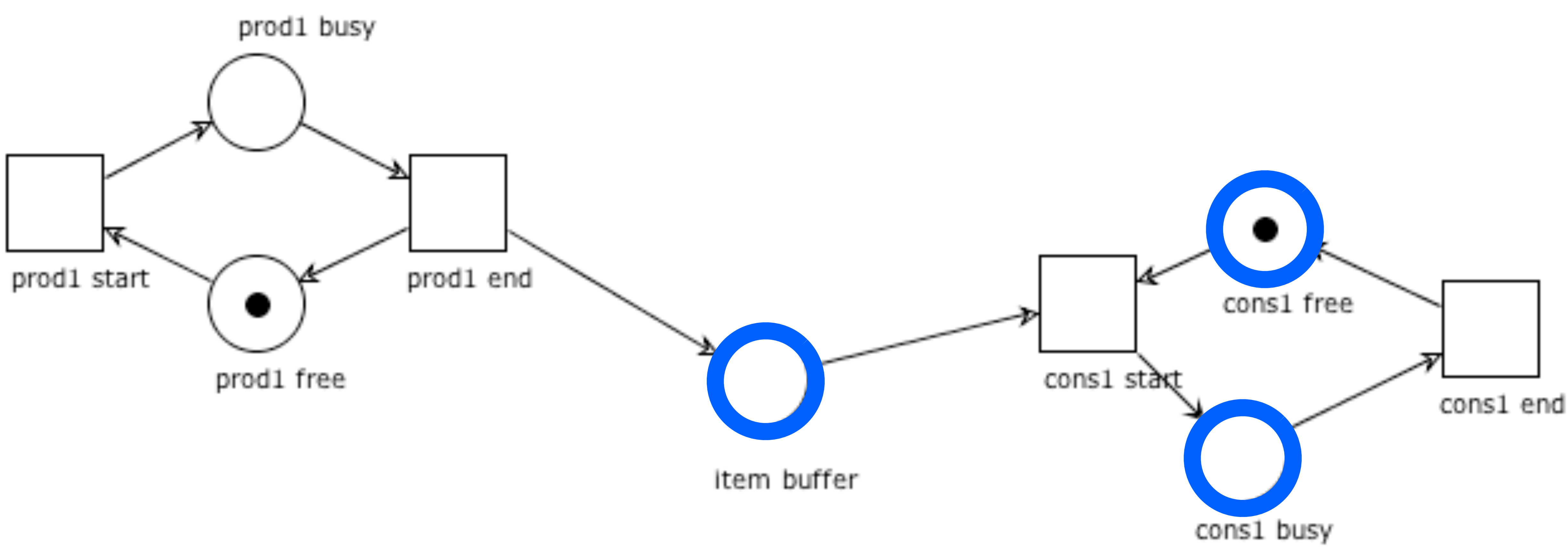
if there is a transition consuming tokens from some place in R that is not marked by √, then R is not a trap

Otherwise R is a trap

$\bullet R \supseteq R \bullet$

Trap check: example

Is R = { itembuffer, cons1busy, cons1free} a trap?



$\bullet R \supseteq R\bullet$

Trap check: example

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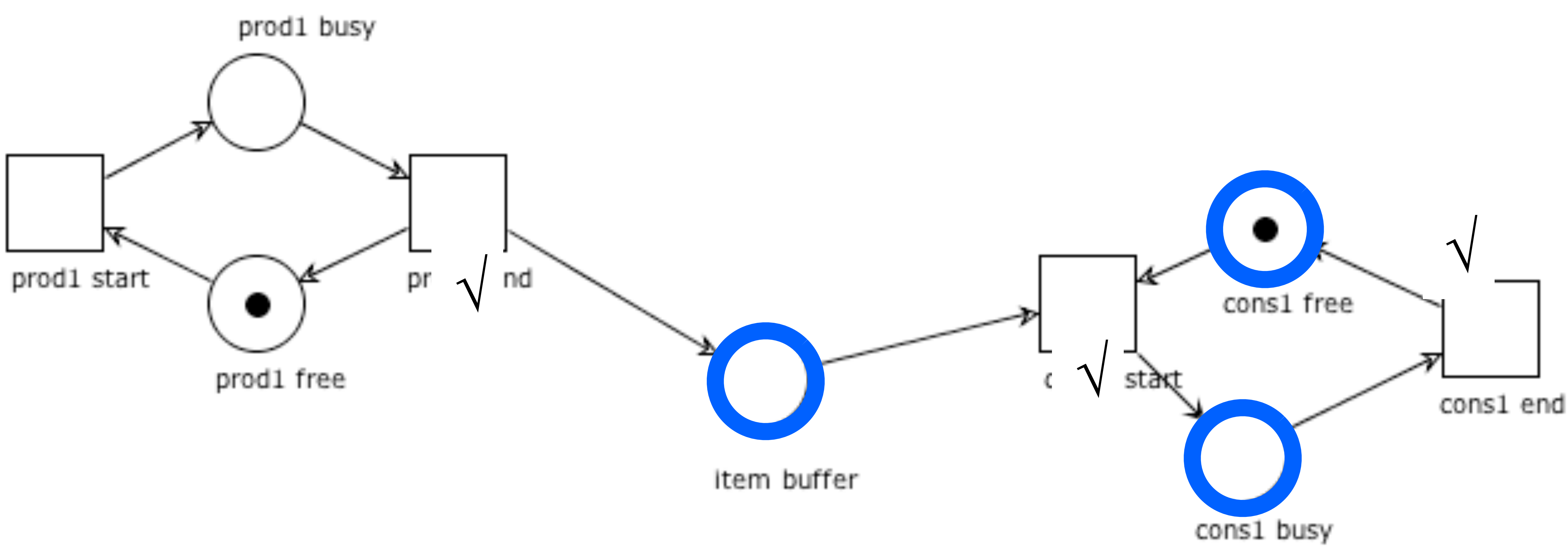

$\bullet R$

$\bullet R \supseteq R \bullet$

Trap check: example

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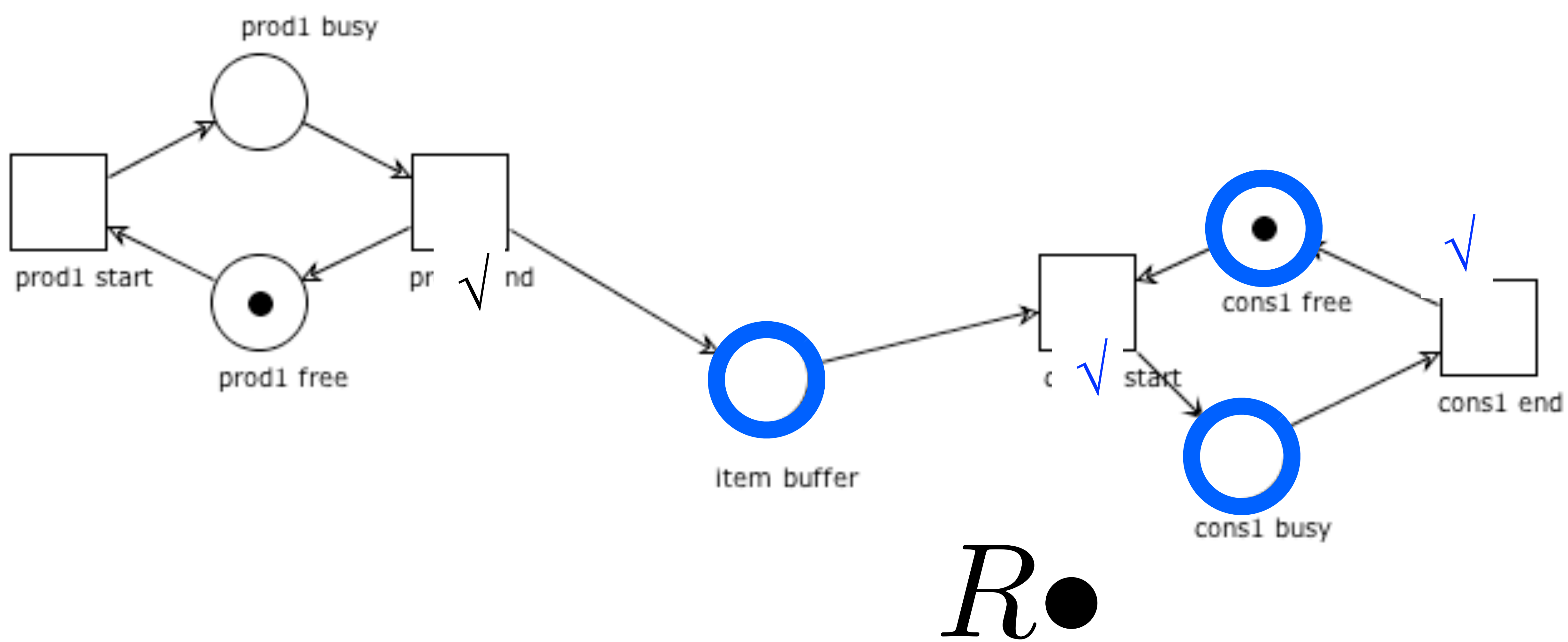

$\bullet R \supseteq R \bullet$

Trap check: example

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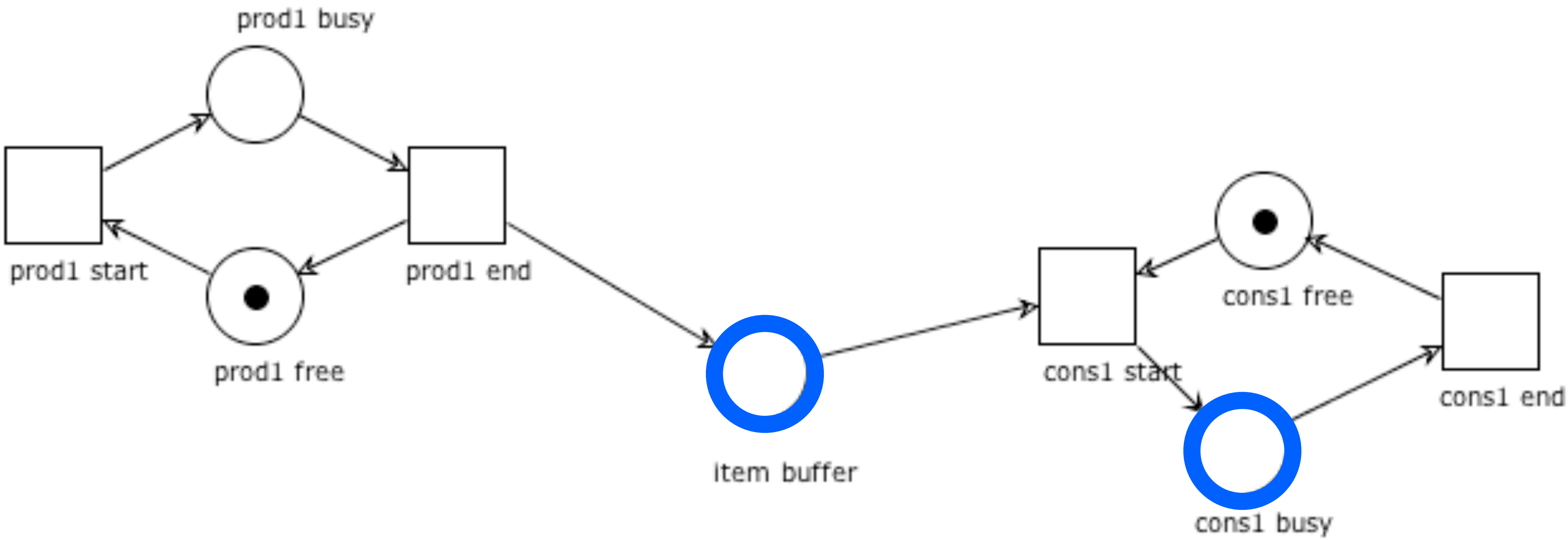

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Trap check: example

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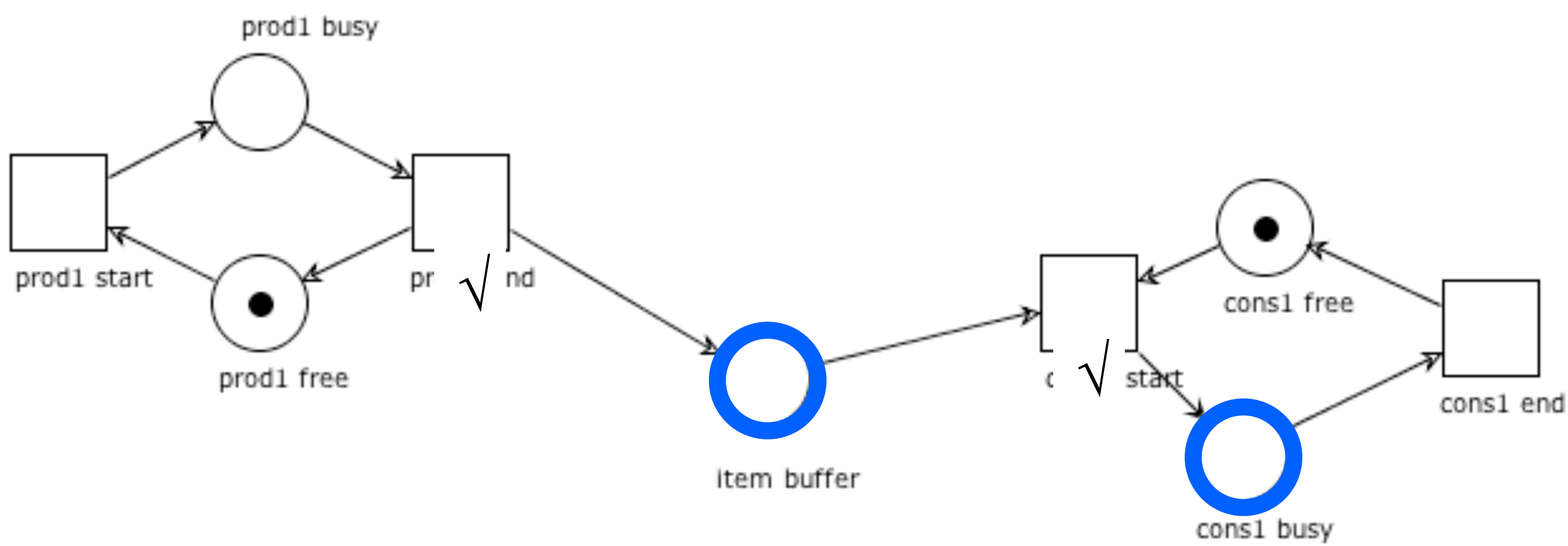

$\bullet R$

$\bullet R \supseteq R \bullet$

Trap check: example

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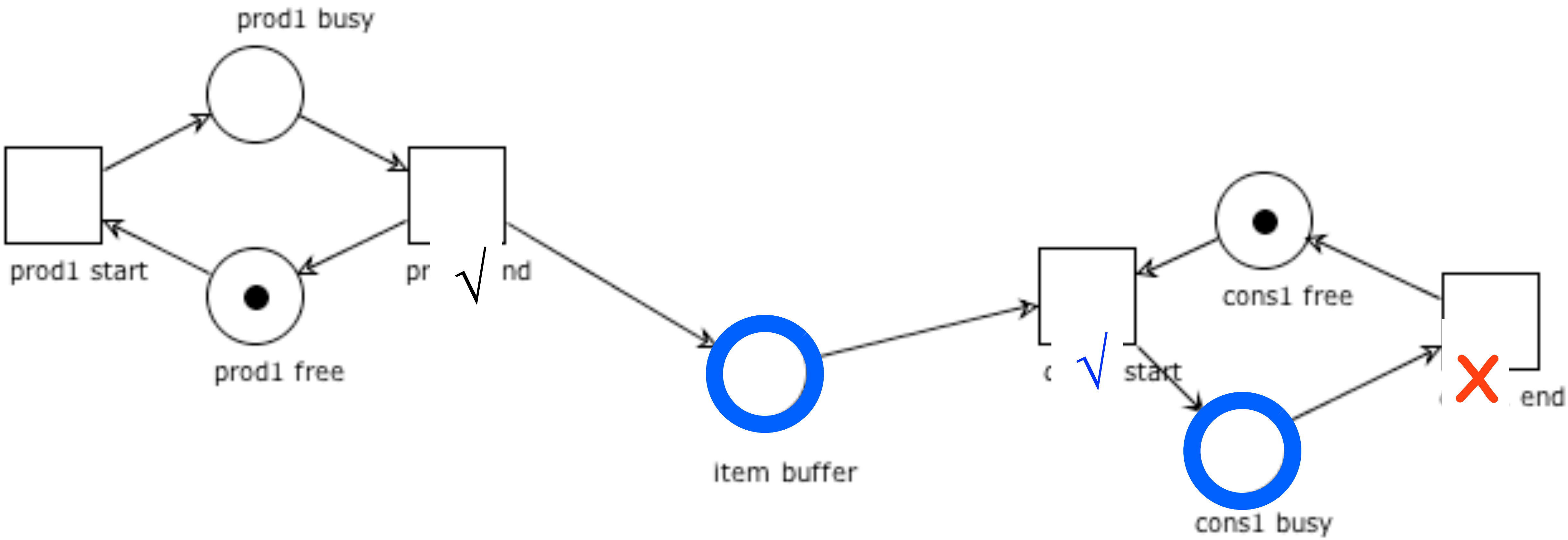


$R \bullet$

Fundamental property of traps

Proposition: Marked traps remain marked

Take a trap R.
We just need to prove that the set of markings
M = { M | M(R)>0 }
is stable, which is immediate by definition of trap

Corollary:
If a trap R is unmarked at some reachable marking M,
then it was initially unmarked at M_0

Exercise

Prove that the union of traps is a trap

Putting pieces together

unmarked siphons stay unmarked
(marked siphons can become unmarked)

if a siphon is marked at M, it was marked at M_0

if all proper siphons always stay marked => deadlock-free

Putting pieces together

if all proper siphons always stay marked => deadlock-free

marked traps stay marked
(unmarked traps can become marked)

if a siphon contains a marked trap, it stays marked

if all siphons contain marked traps, they stay marked
=> deadlock-free

A sufficient condition for deadlock-freedom

Proposition:
If every proper siphon of a system contains a marked trap, then the system is deadlock-free

We show that if the system is not deadlock free, then there is a siphon that does not include any marked trap.

Assume some reachable M is dead.
Let R be the set of unmarked places at M.
Then, we have seen that R is a proper siphon.
Since M(R)=0, then R includes no trap marked at M.
Therefore, R includes no trap marked at M_0

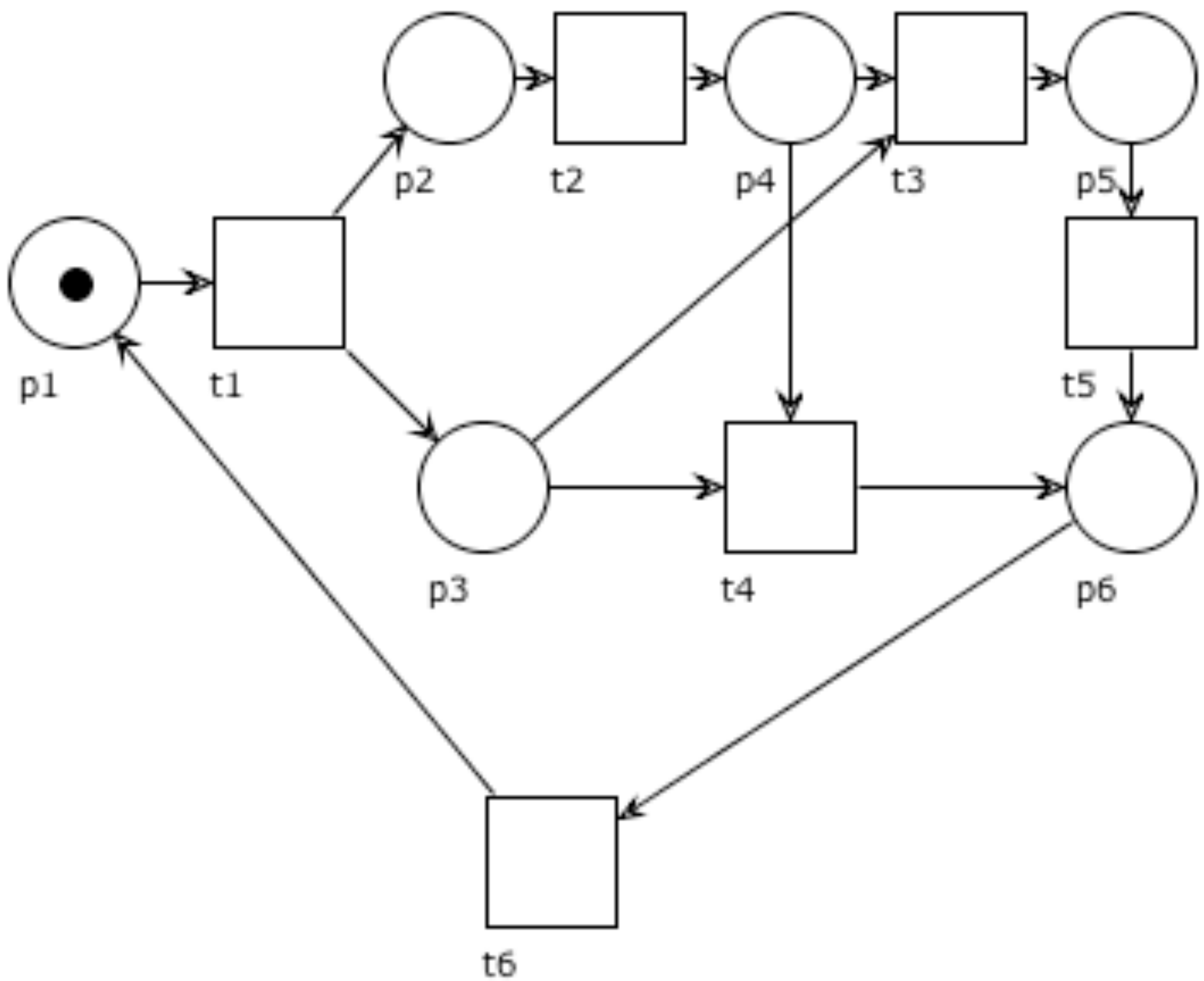
Note

It is easy to observe that every siphon includes a (possibly empty) unique maximal trap with respect to set inclusion

Moreover, a siphon includes a marked trap
iff
its maximal trap is marked

Exercise

Find all proper siphons and traps in the net below (at most 2^6 sets to consider)



Liveness = Place-liveness (in free-choice systems)

Place liveness (reminder)

Definition: Let (P, T, F, M_0) be a net system.

A place $p \in P$ is **live** if $\forall M \in [\, M_0 \,\rangle .\, \exists M' \in [\, M \,\rangle .\, M'(p) > 0$

A place p is live
if every time it becomes unmarked
there is still the possibility to be marked in the future
(or if it is always marked)

Definition:
A net system (P, T, F, M_0) is **place-live** if every place $p \in P$ is live

liveness implies place-liveness

Dead nodes (reminder)

Definition: Let (P, T, F) be a net system.

A transition $t \in T$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M' \stackrel{t}{\not\longrightarrow}$

A place $p \in P$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M'(p) = 0$

Some obvious facts

If a system is not live,
it has a transition dead at some reachable marking M

If a system is not place-live,
it has a place dead at some reachable marking M

If a place / transition is dead at M, then it remains dead at any marking M’ reachable from M
(the set of dead nodes can only increase during a run)

Every transition in the pre- or post-set of a dead place is also dead

An obvious facts in free-choice systems

In a free-choice system:

if an output transition t of a place p is dead at M

then any output transition t’ of p is dead at M

(because t and t’ must have the same pre-set)

Dead t, dead p

Lemma: If the transition t is dead at M in a free-choice system, then there is a non-live place p in the pre-set of t

By contraposition, we prove: if all input places of t are live then t is not dead
Let $\bullet t = [t] \cap P = \{p_1, ..., p_n\}$

Since all places $p_1, ..., p_n$ are live at M, there exists
$M \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} ... \xrightarrow{\sigma_n} M_n$
such that $M_i(p_i) > 0$ for all i

If the sequence contains $u \in [t]$ then t is not dead at M

If no transition in $[t]$ appears in the sequence, then no token in $\bullet t$ is consumed
Hence $M_n(p_i) > 0$ for all i, and $M_n \xrightarrow{t}$ and t is not dead at M

Place-liveness implies liveness in f.c. systems

Proposition: If a free-choice system is place-live, then it is live

Place-liveness implies liveness in f.c. systems

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By contraposition, we prove:
non-liveness implies non-place-liveness

Place-liveness implies liveness in f.c. systems

Proposition: If a free-choice system is place-live, then it is live

By contraposition, we prove:
non-liveness implies non-place-liveness

If a free-choice system is not live then there is a transition t dead at some reachable marking M

But then some input place of t must be non-live at M, so the system is not place-live

Consequence in f.c. nets: place-liveness = liveness

If a free-choice system is place-live, then it is live

In any system, liveness implies place-liveness

Corollary:
A free-choice system is live **iff** it is place-live

Commoner's theorem

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

We show just the “if” direction, which is simpler

We need a technical lemma

Commoner's theorem: "if" direction

A free-choice system is live
if
every proper siphon includes an initially marked trap

By contraposition, we prove:
if a free-choice is non-live, then
a proper siphon exists whose traps are all unmarked

Commoner's theorem: "if" direction

If a free-choice is non-live, **then**
a proper siphon exists whose traps are all unmarked

A non-live free-choice system contains a proper siphon
R such that M(R)=0 at some reachable M
(see next lemma)

So every trap included in R is unmarked at M

(since marked traps remain marked)
every trap included in R must be unmarked initially

Non-liveness and unmarked siphons

Lemma: Every non-live free-choice system has a proper siphon R and a reachable marking M such that M(R)=0

By non-liveness: the system is not place-live,
i.e., some p is dead at some $L \in [M_0\rangle$

Take $M \in [\, L \,\rangle$ such that every place not dead at M
is not dead at any marking in $[\, M \,\rangle$
i.e. all markings in $[\, M \,\rangle$ have the same set R of dead places
(dead places remain dead)

Next we prove that R is a proper siphon and $M(R) = 0$

Non-liveness and unmarked siphons

Lemma: Every non-live free-choice system has a proper siphon R and a reachable marking M such that M(R)=0

1. R is a siphon $\bullet R \subseteq R \bullet$
   - any $t \in \bullet R$ is dead at M
   - every t dead at M has an input place in R
     (t has some input place dead at some marking reachable from M)
2. R is proper

   p is dead at L, hence it is dead at M, hence $p \in R$, hence $R \neq \emptyset$
3. $M(R) = 0$ because it contains dead places

Complexity of the non-liveness problem in free-choice systems

Commoner's theorem

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

A non-deterministic algorithm for non-liveness

1. guess a set of places R
   (polynomial time)

2. check if R is a siphon (•R ⊆ R•)
   (polynomial time)

3. if R is a siphon, compute the maximal trap Q ⊆ R

4. if $M_0(Q)=0$, then answer “non-live”, otherwise "live"
   (polynomial time)

A polynomial algorithm for maximal trap in a siphon

$\bullet R \subseteq R \bullet$ $\bullet Q \supseteq Q \bullet$

3. if R is a siphon, compute the maximal trap Q ⊆ R

Input: A net $N = (P, T, F)$ and $R \subseteq P$
Output: $Q \subseteq R$

$Q := R$
while $(\exists p \in Q,\ \exists t \in p\bullet,\ t \notin \bullet Q)$
$\quad Q := Q \setminus \{p\}$
return Q

Non-liveness for f.c. nets is in NP

The non-liveness problem for free-choice systems is in NP

Is the same problem in P?

The corresponding deterministic algorithm cannot make the guess in step 1

It has to explore all possible subsets of places
$2^{|P|}$ cases!

NP-completeness

We next sketch the proof of the reduction to non-liveness in a free-choice net of the CNF-SAT problem

(SATisfiability problem
for propositional formulas in Conjunctive Normal Form)

CNF-SAT is an NP-complete problem

CNF-SAT decision problem

Variables: $x_1, x_2, ..., x_n$

Literals: $x_1, \bar{x}_1, x_2, \bar{x}_2, ..., x_n, \bar{x}_n$

Clause: disjunction of literals

Formula: conjunction of clauses

Example: $\phi = (x_1 \vee \bar{x_3}) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3)$

Is there an assignment of boolean values to the variables such that $\phi = true$?

The free-choice net of a formula

Given a formula ϕ, the idea is

to construct a free-choice system (P,T,F,M_0)

and show that

the formula ϕ is satisfiable

iff

(P,T,F,M_0) is not live

The free-choice net of a formula

Given a formula ϕ, the idea is
to construct a free-choice system (P,T,F,M_0)
and show that

the formula ϕ is not satisfiable
iff
(P,T,F,M_0) is live

CNF-SAT formulas

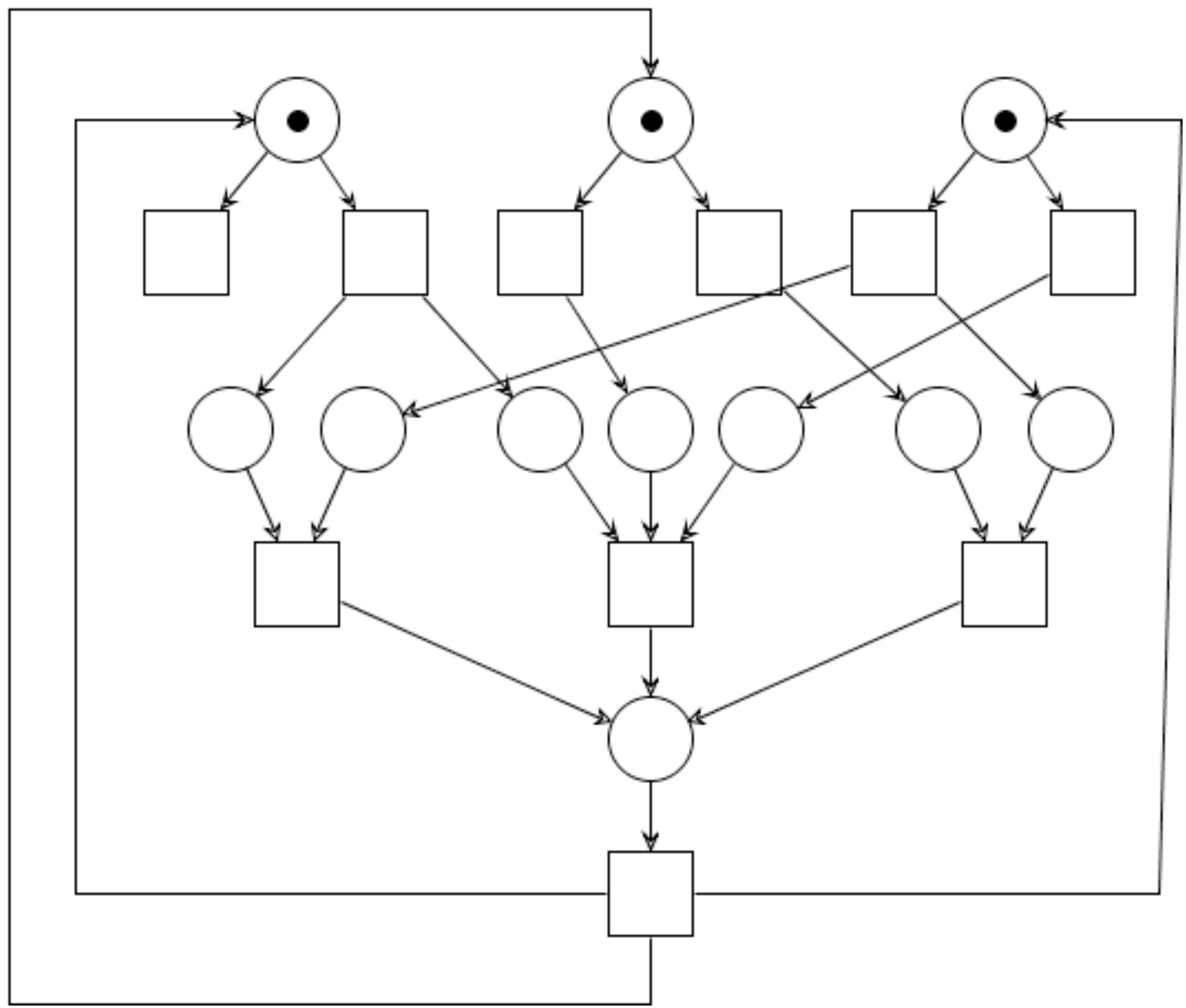
Is there an assignment of boolean values to the variables such that $\phi = true$?

Is there an assignment of boolean values to the variables such that $\neg\phi = false$?

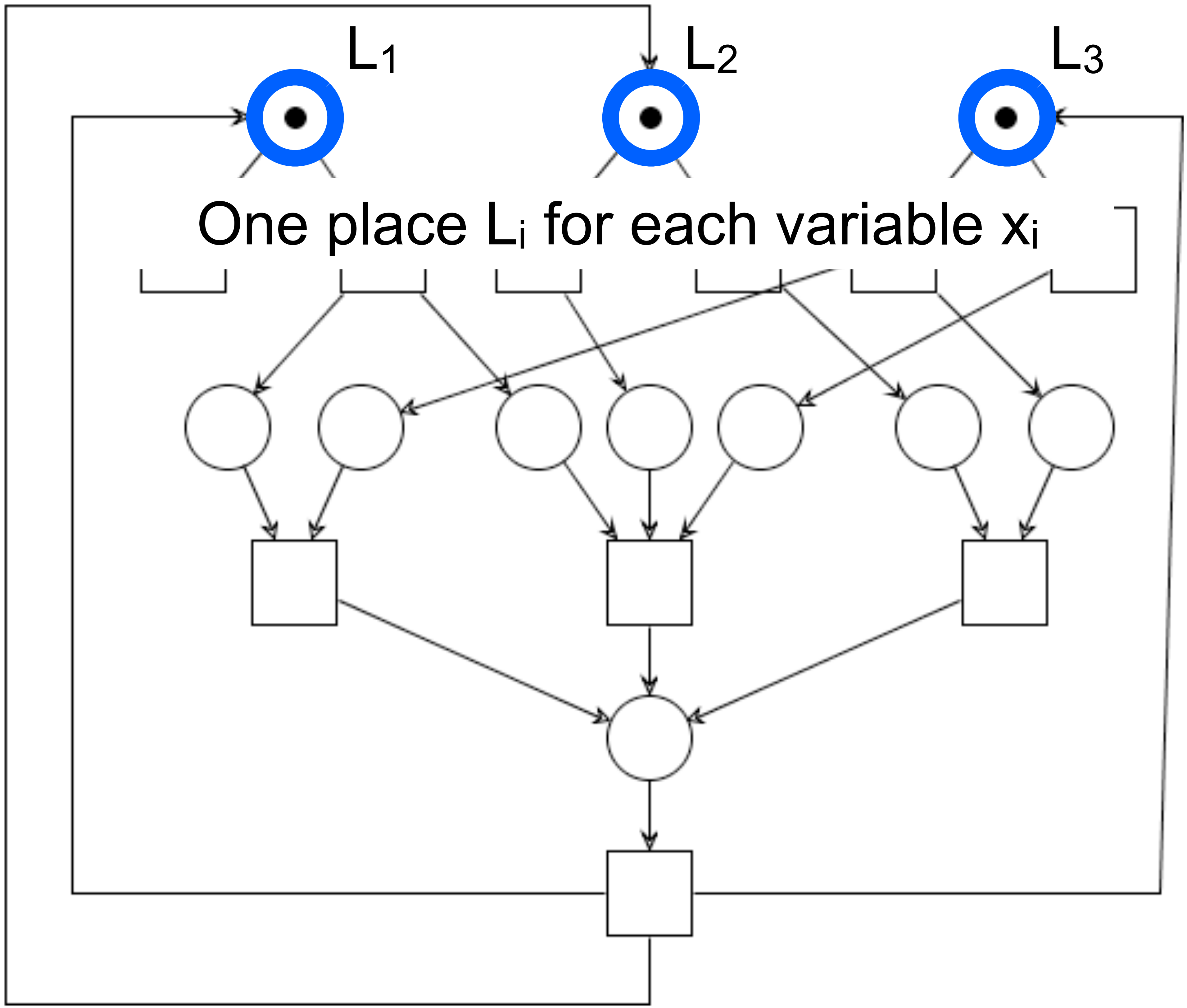
$$\phi = (x_1 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (x_2 \vee \overline{x}_3)$$

$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$

$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



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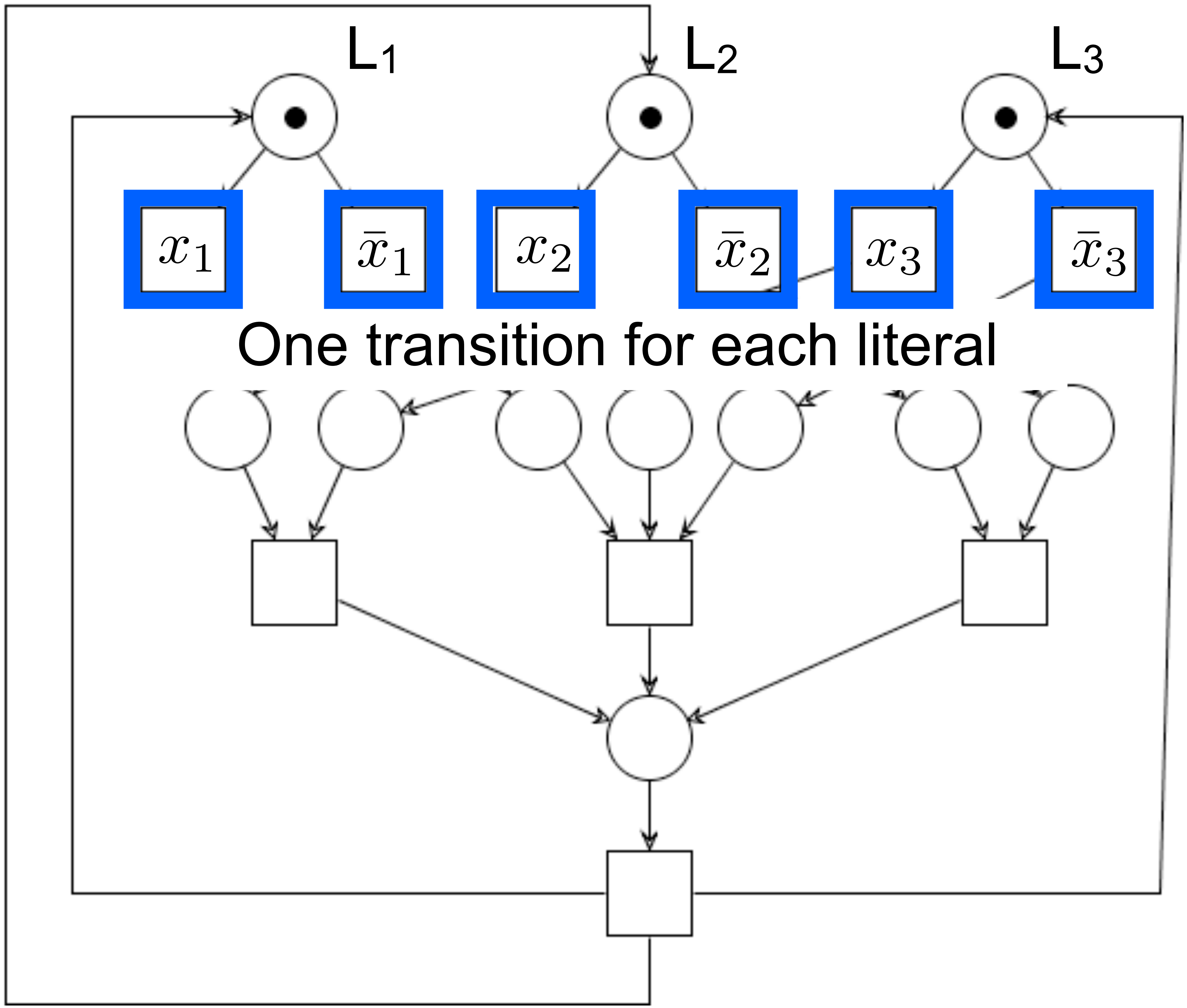

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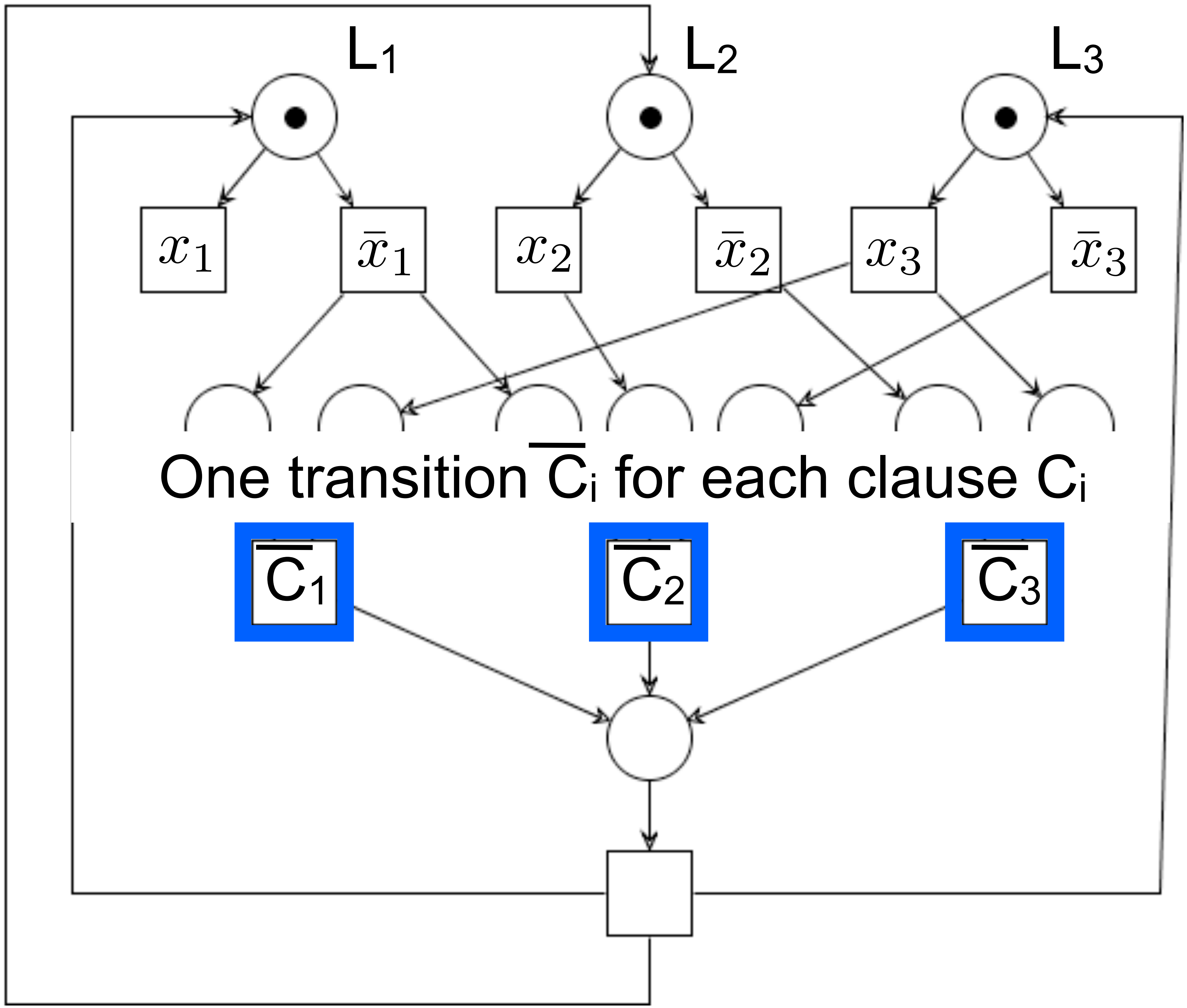

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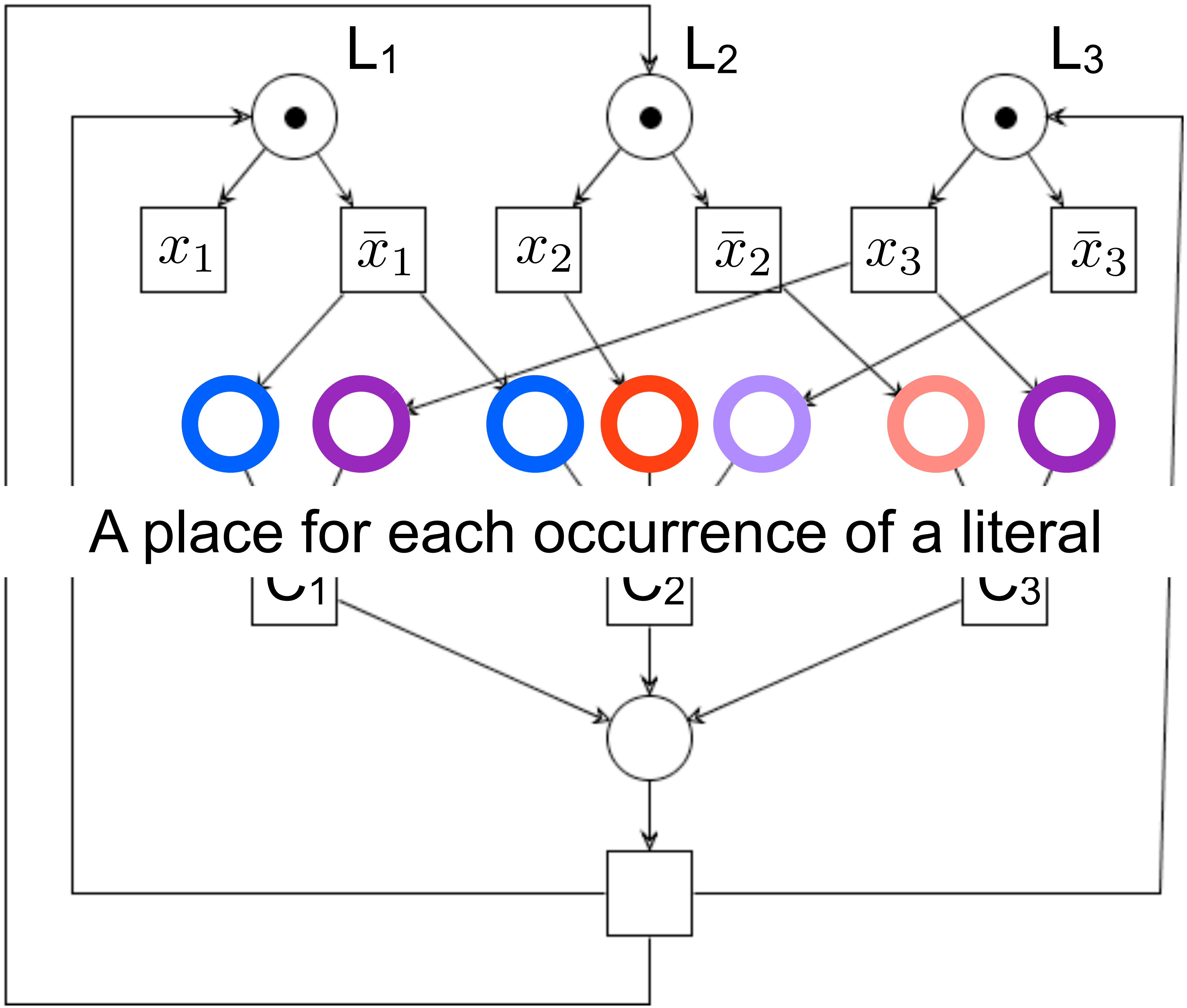
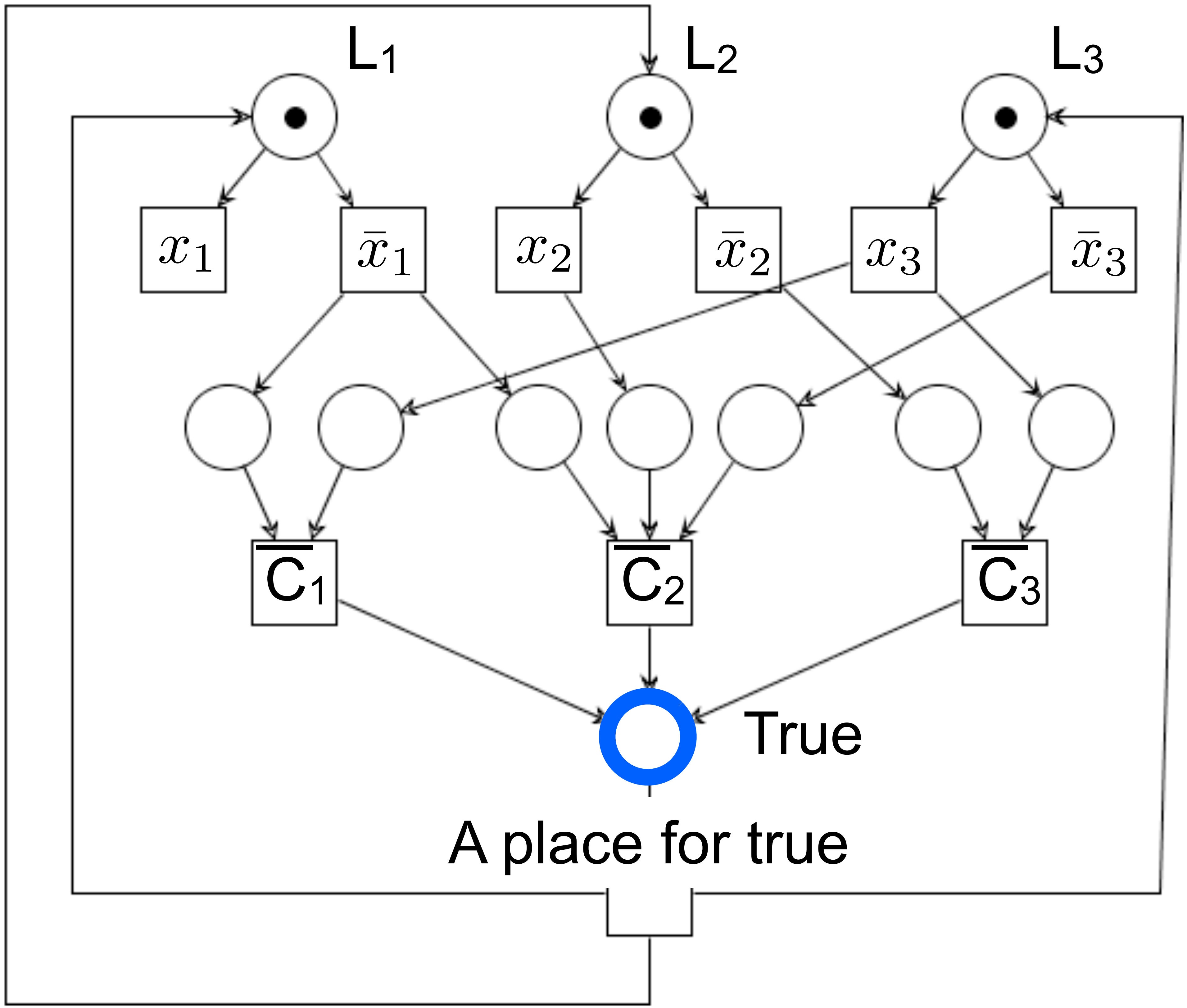
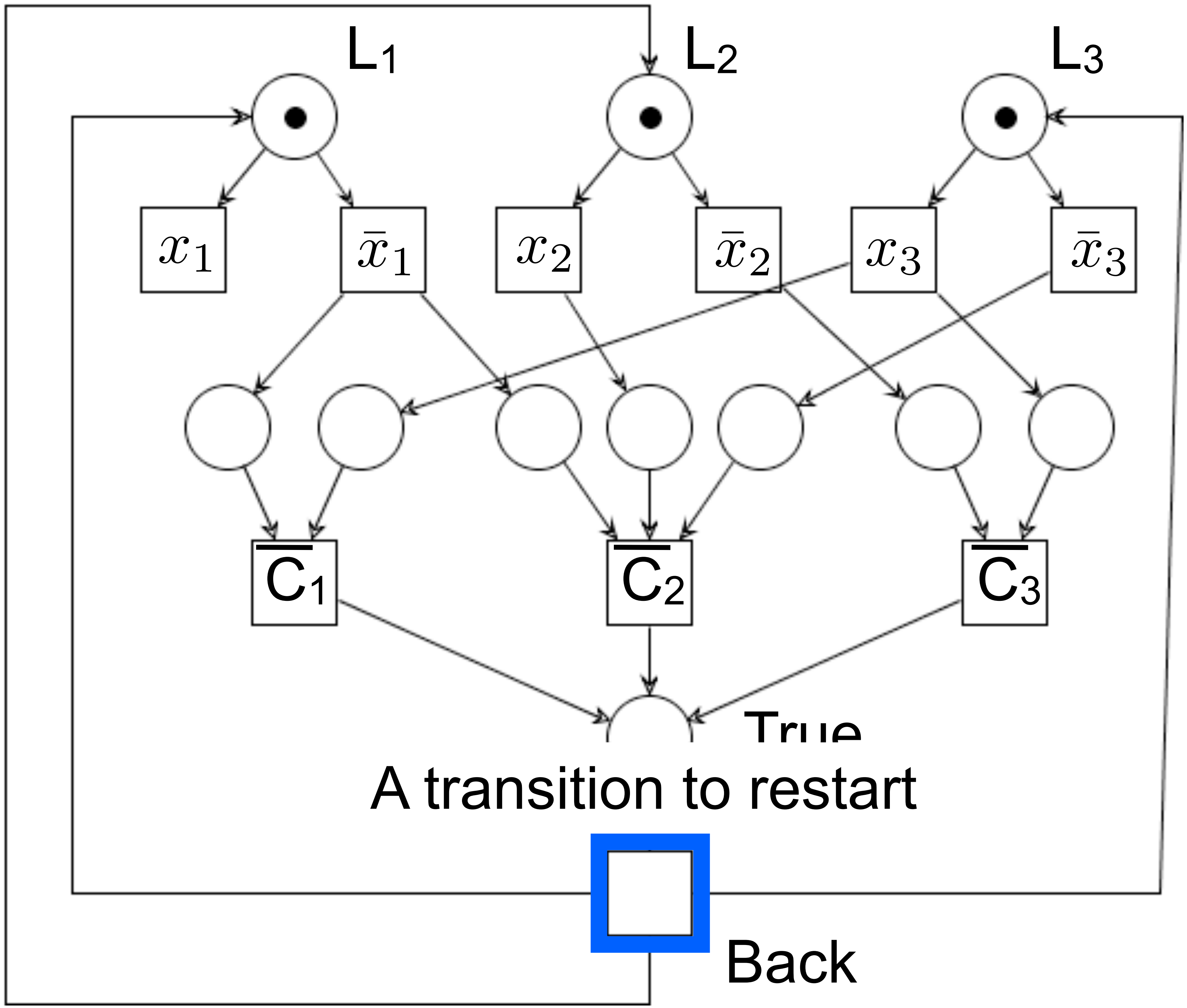


A place for each occurrence of a literal

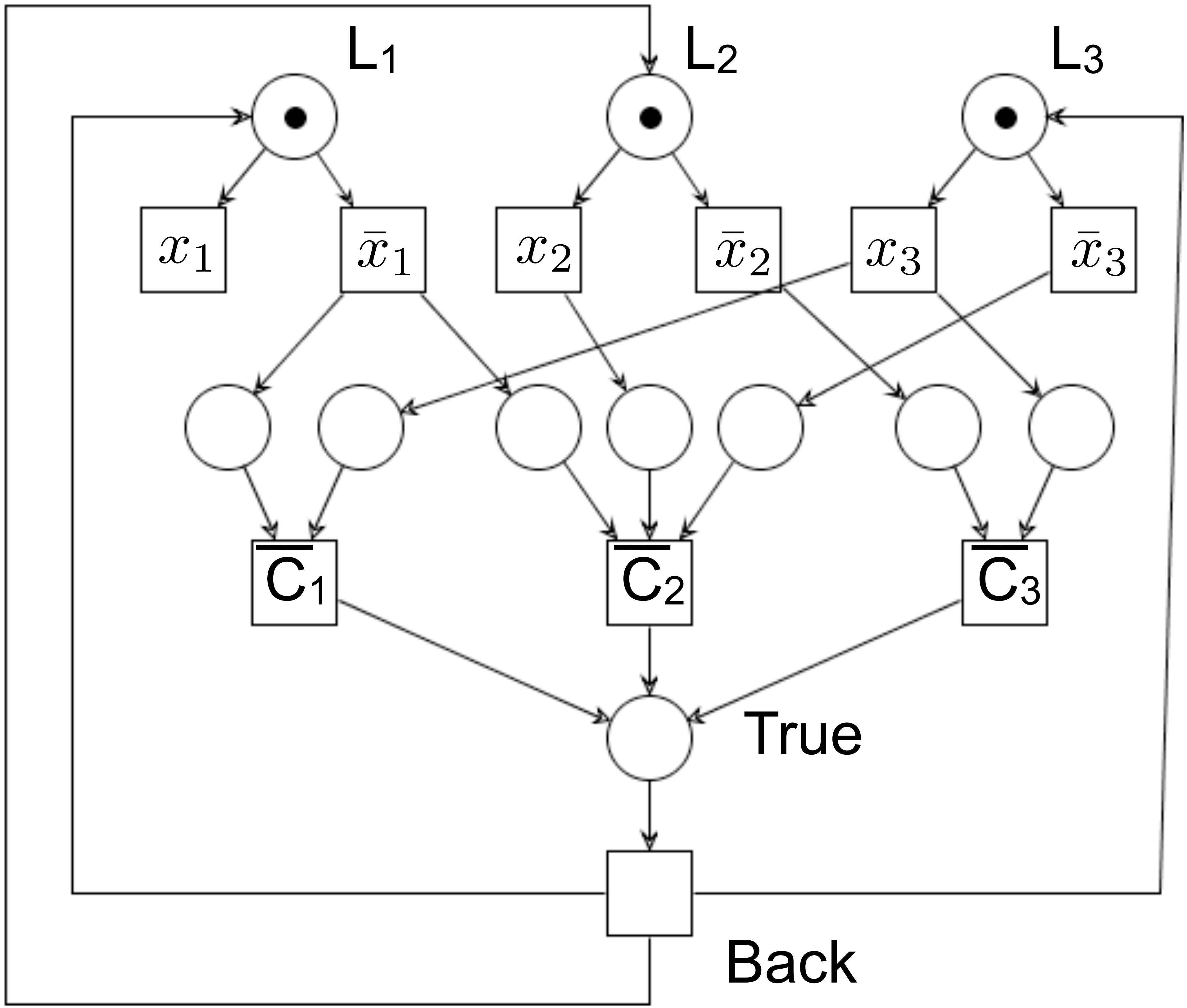
$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



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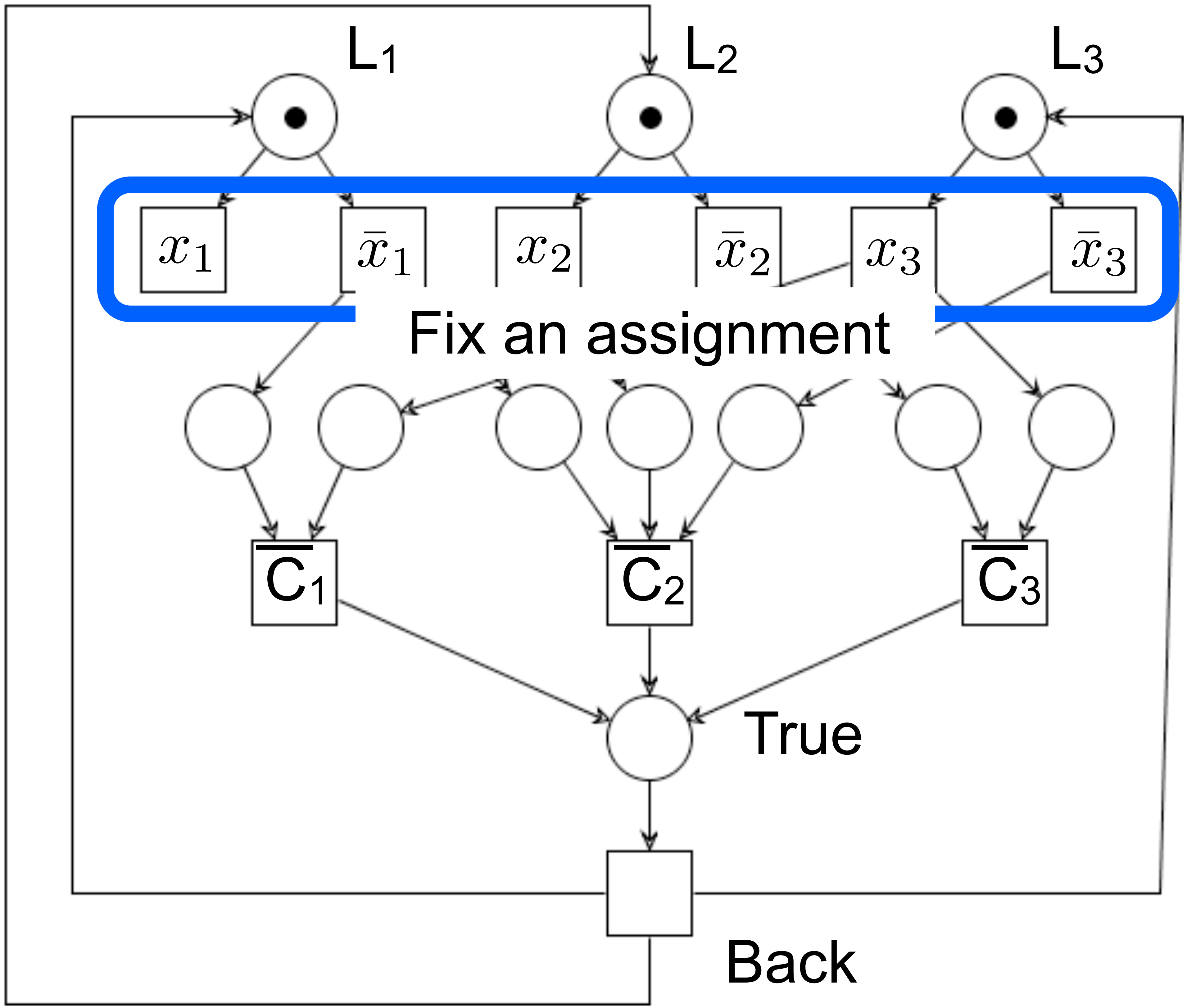

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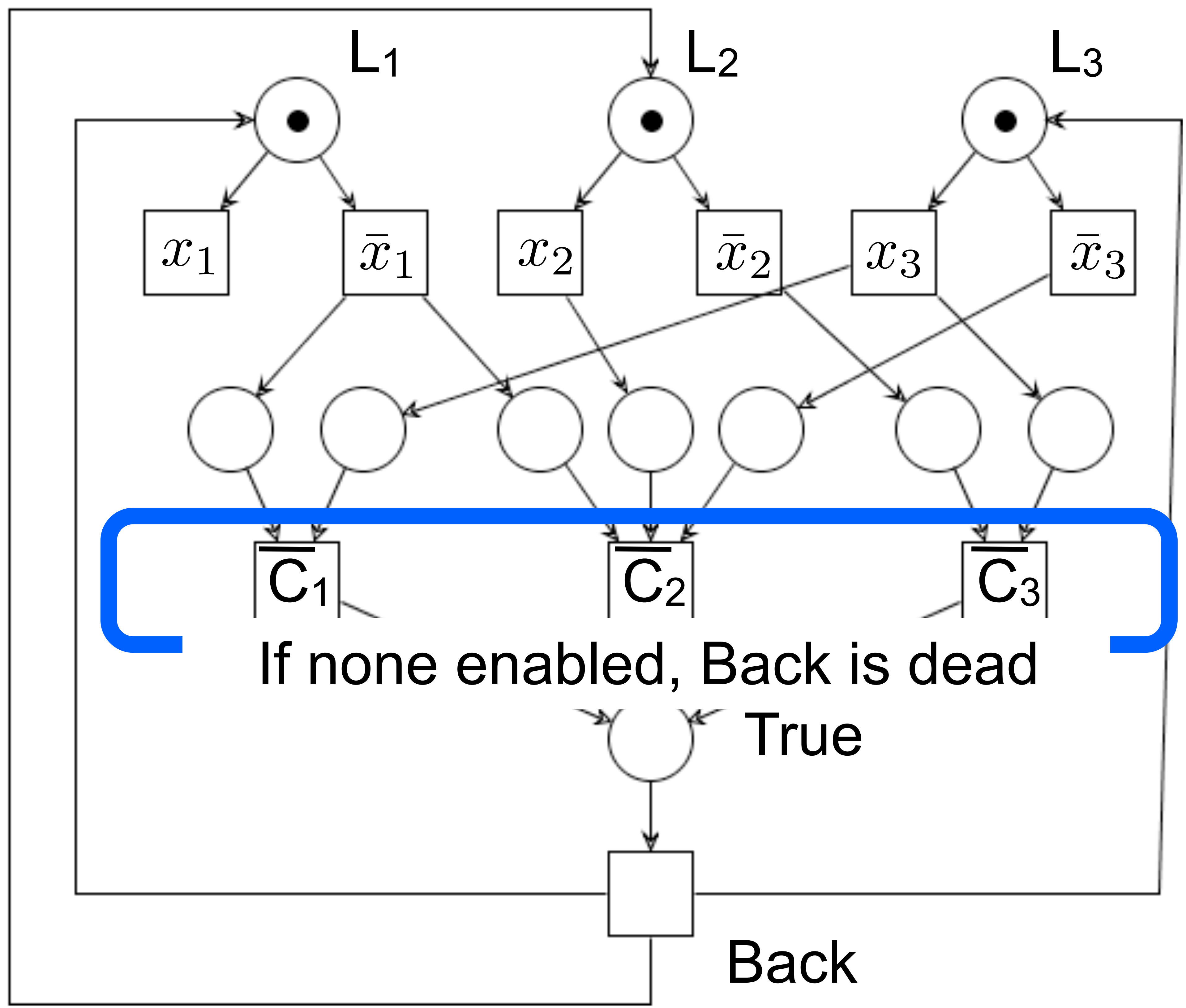

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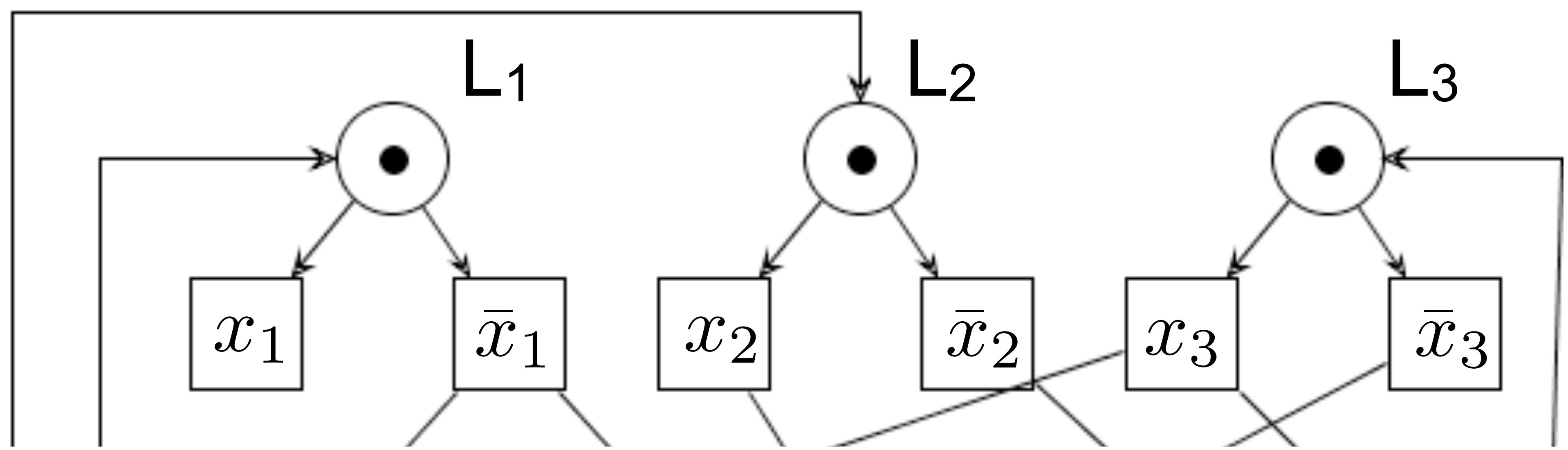
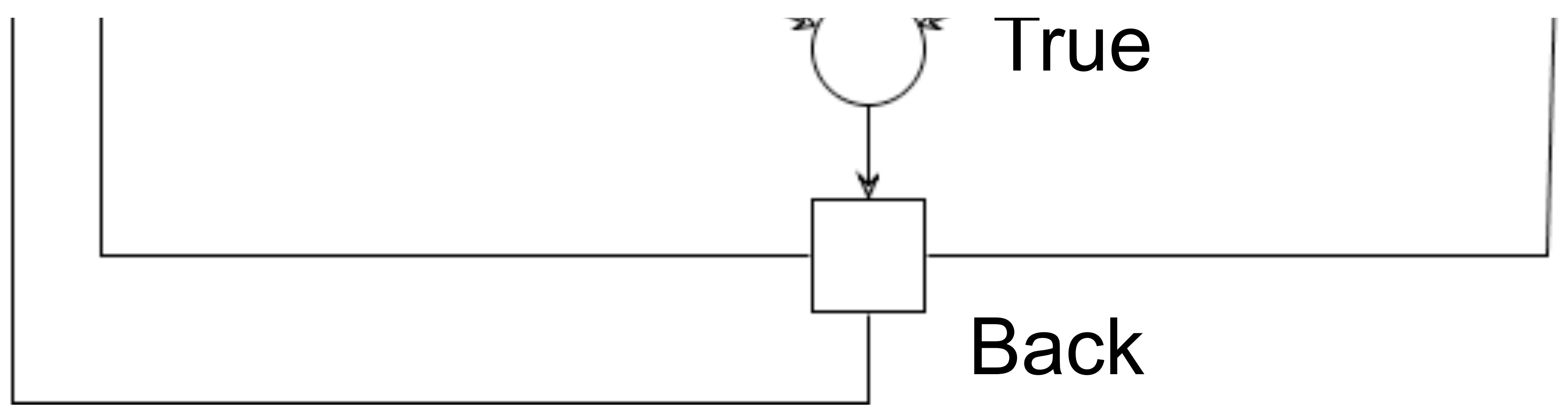
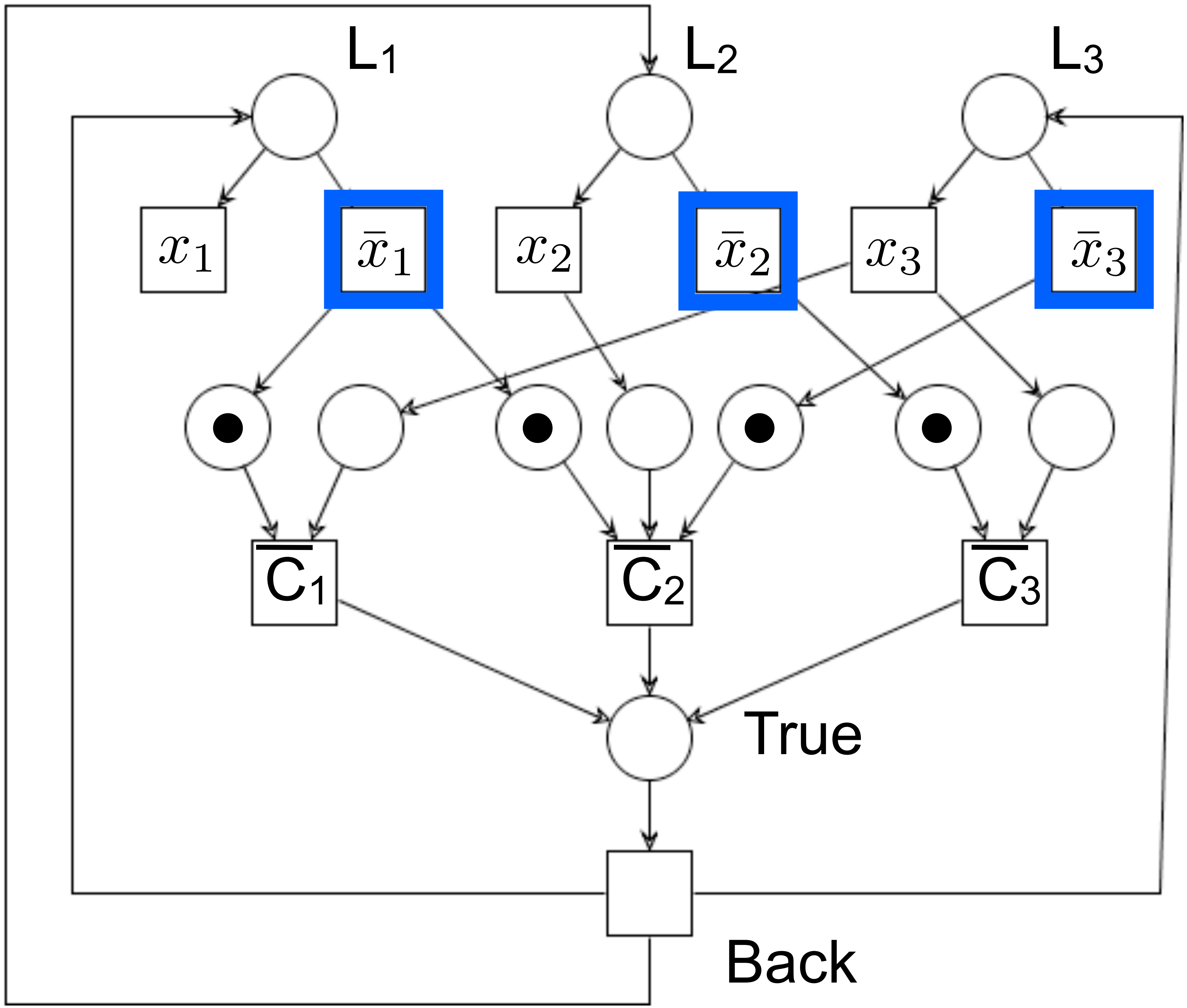


If ϕ is satisfiable, then the net is not live

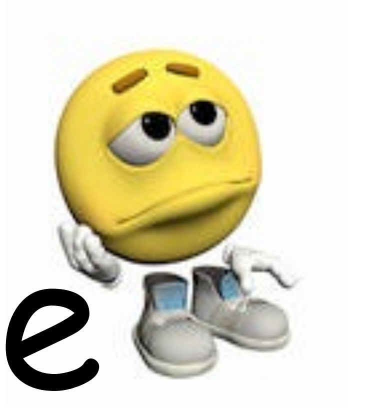
If the net is not live, then ϕ is satisfiable



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



Main consequence



No polynomial algorithm to decide liveness of a free-choice system is available

(unless P=NP)

Exercise

Draw the net corresponding to the formula

$$x_2 \;\wedge\; (x_1 \vee \overline{x}_3 \vee \overline{x}_4) \;\wedge\; (x_1 \vee \overline{x}_2) \;\wedge\; (\overline{x}_1 \vee x_4) \;\wedge\; (\overline{x}_2 \vee \overline{x}_4)$$

Is it satisfiable?

Live and bounded free-choice nets

Rank Theorem (main result, proof omitted)

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition polynomial
2. it is connected polynomial
3. M_0 marks every proper siphon
4. it has a positive S-invariant polynomial
5. it has a positive T-invariant polynomial
6. $rank(N) = |C_N| - 1$ polynomial

(where C_N is the set of clusters)

A polynomial algorithm for maximal siphon in R

Input: A net $N = (P, T, F, M_0)$, $R \subseteq P$
Output: $Q \subseteq R$ maximal siphon in R

$Q := R$
while $(\exists p \in Q,\ \exists t \in \bullet p,\ t \notin Q\bullet)$
 $Q := Q \setminus \{p\}$
return Q

Q is a **siphon** if $\bullet Q \subseteq Q\bullet$

A polynomial algorithm for maximal unmarked siphon

3. M_0 marks every proper siphon polynomial

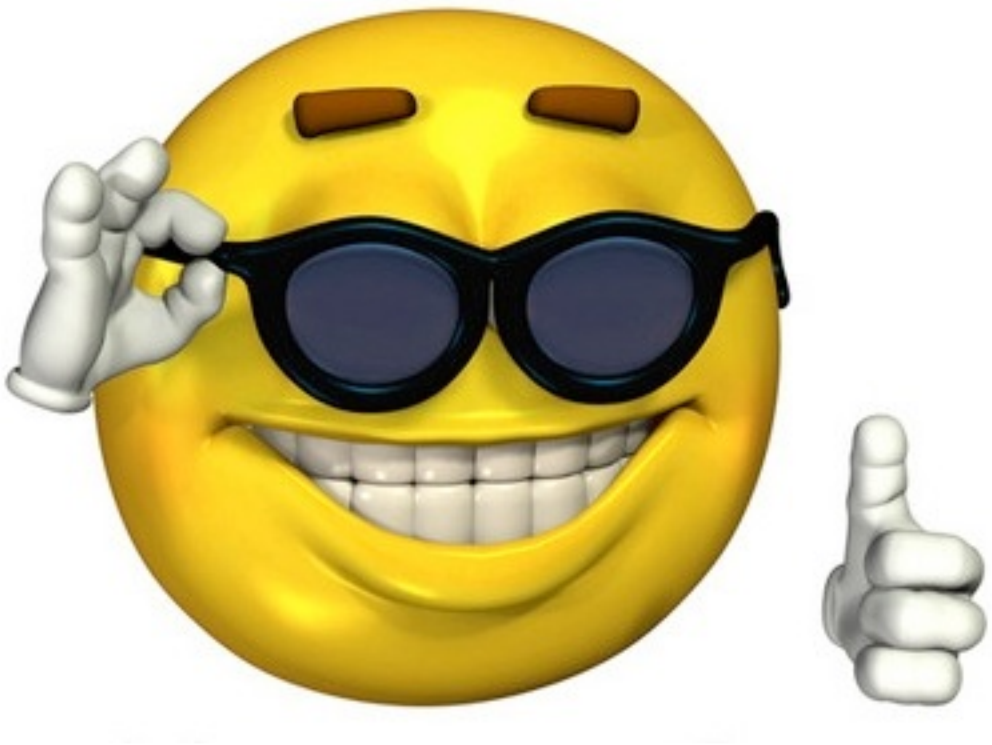
Input: A net $N = (P, T, F, M_0)$, $R = \{ p \mid M_0(p) = 0 \}$
Output: $Q \subseteq R$ maximal unmarked siphon

$Q := R$
while $(\exists p \in Q,\ \exists t \in \bullet p,\ t \notin Q\bullet)$
$\quad Q := Q \setminus \{p\}$
return Q

If Q is empty then M_0 marks every proper siphon

Main consequence

The problem to decide
if a free-choice system is live and bounded
can be solved in polynomial time
(using the Rank Theorem)



Coverability

Rank Theorem (main result, proof omitted)

Theorem:

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4. it has a positive S-invariant
5. it has a positive T-invariant
6. $\text{rank}(N) = |C_N| - 1$

(where C_N is the set of clusters)

A technique to find a positive S-invariant

Decompose the free-choice net N in suitable S-nets so that any place of N belongs to an S-net (the same place can appear in more S-nets)

Each S-net induces a uniform S-invariant

A positive S-invariant is obtained as the sum of the S-invariants of each subnet

S-Coverability analysis

A case is often composed by parallel threads of control
(each thread imposing some order over its tasks)

The notion of S-coverability allows to reveal such threads

S-component

take a set of nodes

Definition: Let $N = (P, T, F)$ and $\emptyset \subset X \subseteq P \cup T$
Let $N' = (P \cap X, T \cap X, F \cap (X \times X))$ be a subnet of N.
N' is an **S-component** if

forget the arcs to other nodes

1. it is a strongly connected S-net
2. for every place $p \in X \cap P$, we have $\bullet p \cup p \bullet \subseteq X$

if a place is selected
then all the attached transitions must be selected

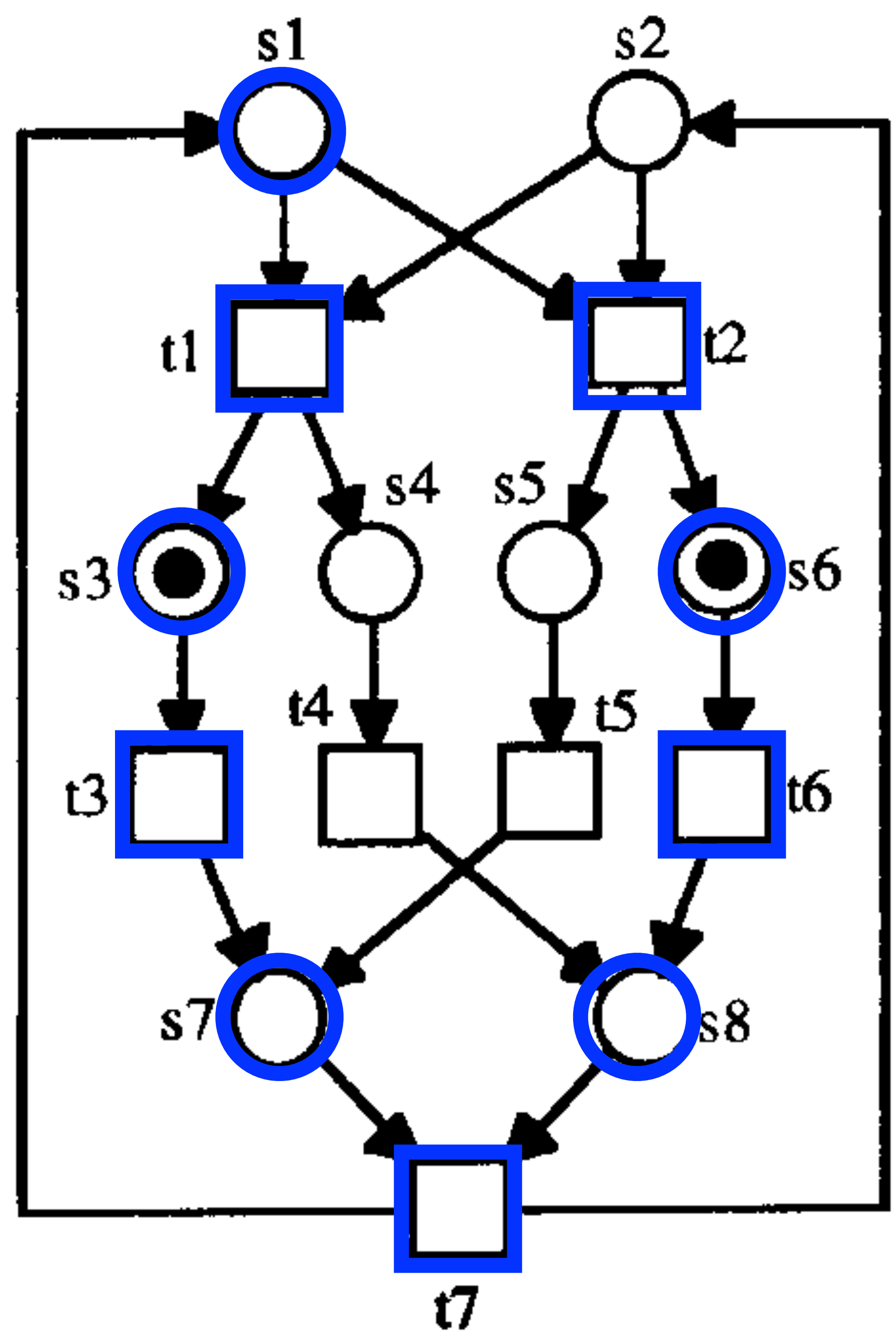
S-cover

Definition: Let **C** be a set of S-components of a net N

C is an **S-cover** if every place p of N belongs to one or more S-components in **C**

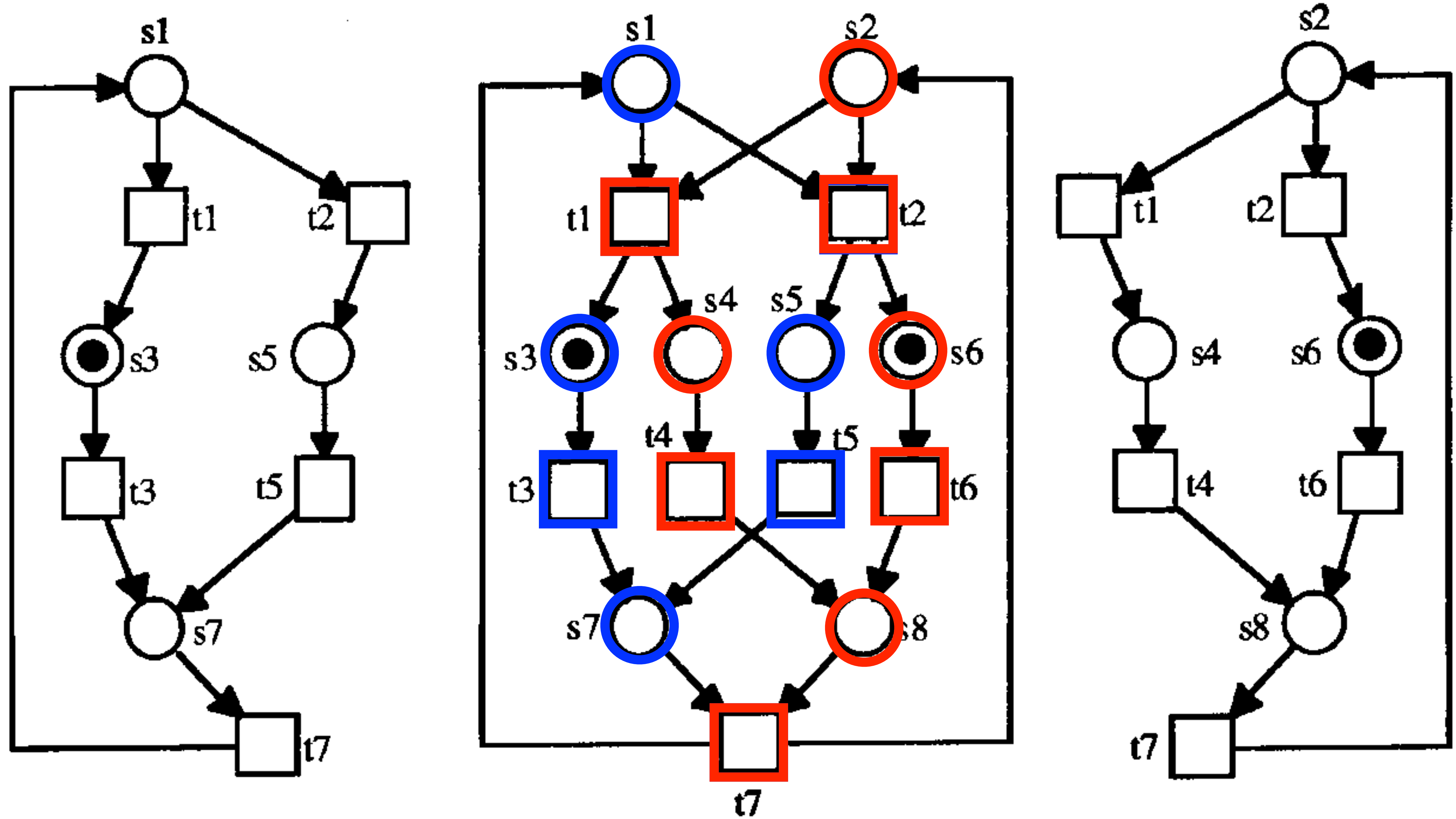
We say that N is **covered by S-components** if it has an S-cover

S-cover: example



not an S-net

S-cover: example



S-coverability theorem

Theorem: If a free-choice system is live and bounded then it is S-coverable

(proof omitted)

Consequence:
free-choice + not S-coverable => not (live and bounded)

Rank Theorem (main result, proof omitted)

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
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(where C_N is the set of clusters)

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N' is a **T-component** if

forget the arcs to other nodes

1. it is a strongly connected T-net
2. for every transition $t \in X \cap T$, we have $\bullet t \cup t \bullet \subseteq X$

if a transition is selected
then all the attached places must be selected

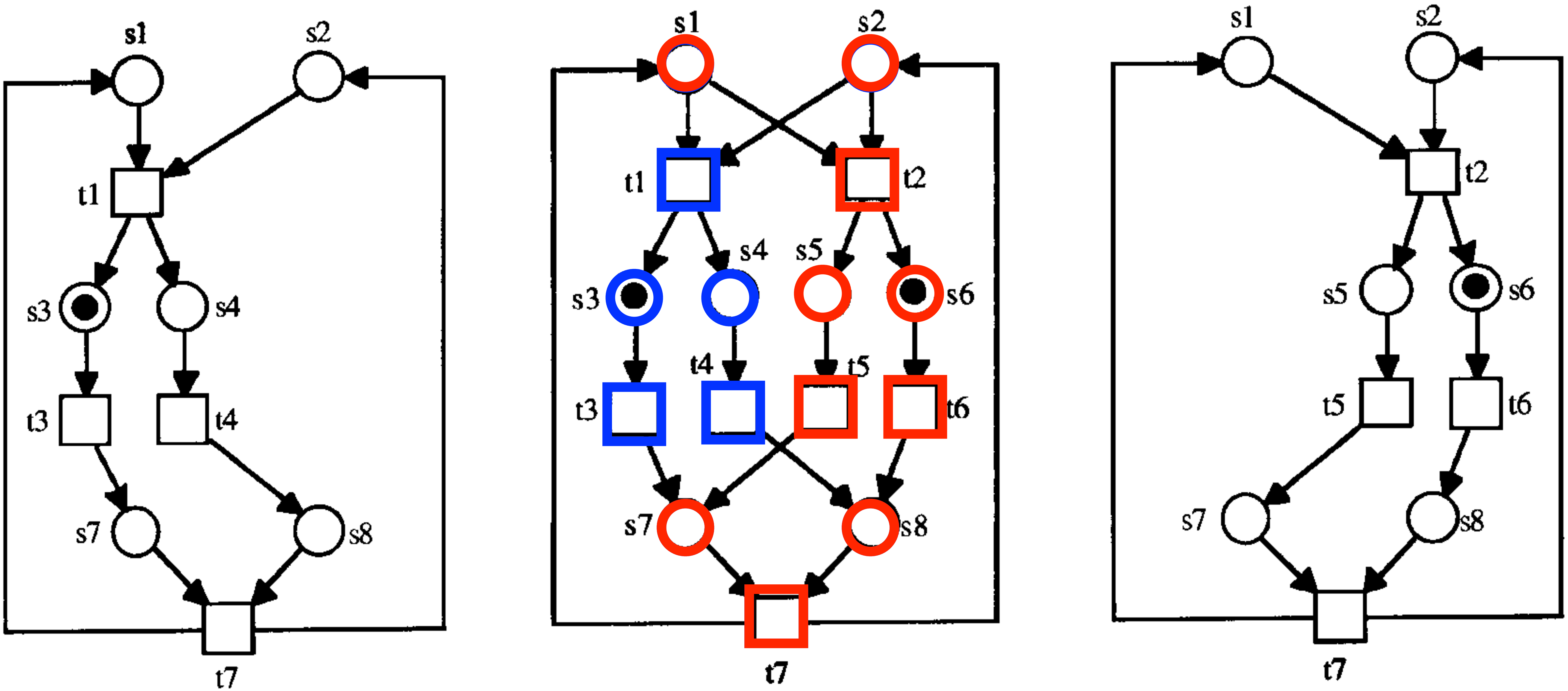
T-cover

Definition: Let **C** be a set of T-components of a net N

C is a **T-cover** if every transition t of N belongs to one or more T-components in **C**

We say that N is **covered by T-components** if it has a T-cover

T-cover: example



T-coverability theorem

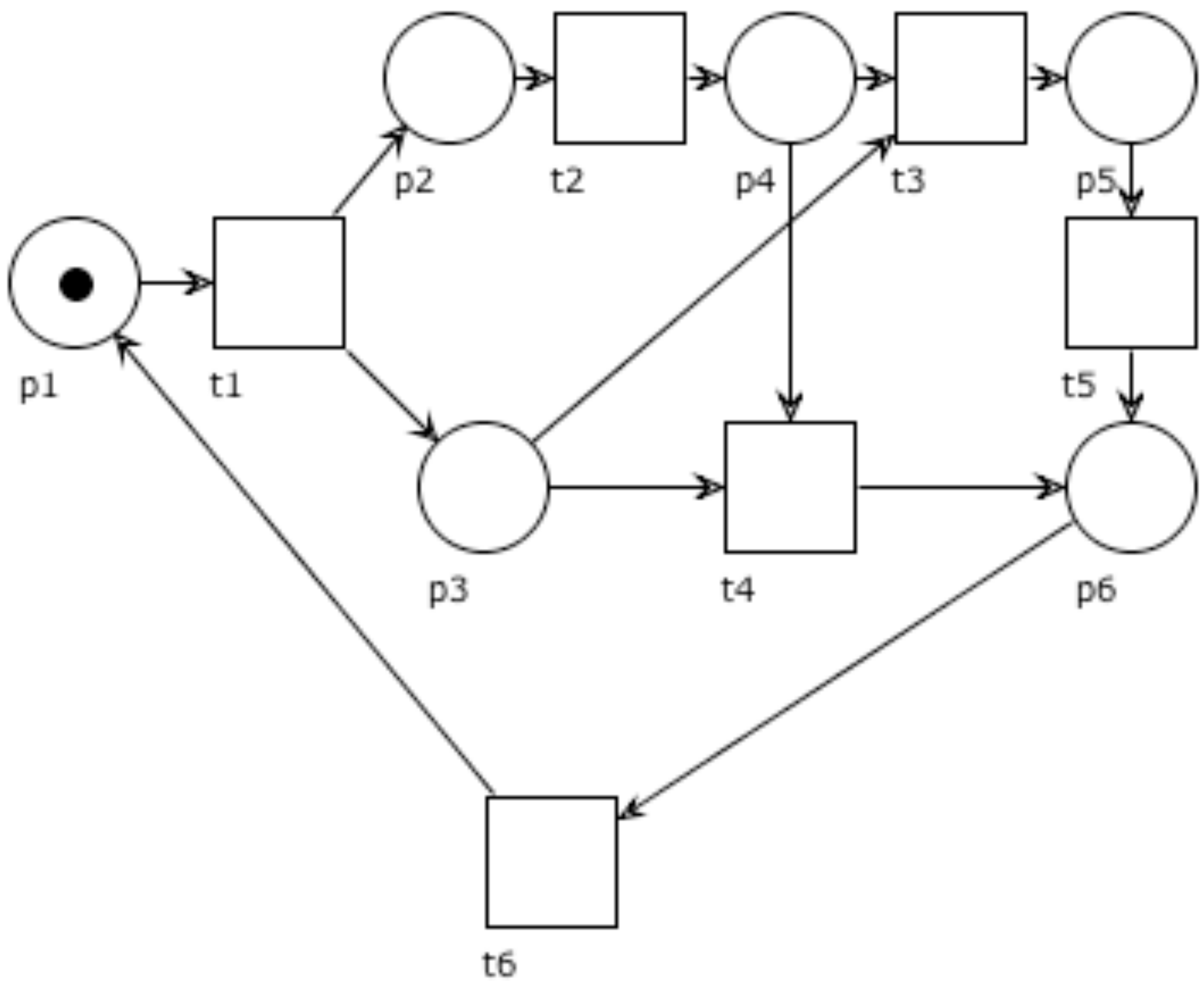
Theorem: If a free-choice system is live and bounded then it is T-coverable

(proof omitted)

Consequence:
free-choice + not T-coverable => not (live and bounded)

Exercise

Find an S-cover and a T-cover for the net below and derive suitable S- and T-invariants



Compositionality

Compositionality of sound free-choice nets

Lemma:
If a free-choice workflow net N is sound then it is safe

(because N* is S-coverable and M_0=i has just one token)

Proposition:
If N and N' are sound free-choice workflow nets then N[N'/t] is a sound free-choice workflow net

(N, N' are safe; we just need to show that N[N'/t] is free-choice)