Sorting atomic items

Chapter 5

Distribution based sorting paradigms

The distribution-based sorting

QuickSort is an in place algorithm, but…
Consider the stack for the recursive calls.
For balanced partitions: $O(\log n)$ space
Worst case of unbalanced partitions: $\Omega(n)$ calls, $\Theta(n)$ space!!
QuickSort is modified.

We can bound the recursive depth. Algorithm Bounded
Based on the fact that: Quicksort does not depend on the order in which recursive calls are executed!

Small arrays can be better sorted with InsertionSort (when n is typically of the order of tens).

The modified version mixes one recursive call with an iterative while loop.

Algorithm Bounded(S, I, j)

Algorithm 5.6 The binary quick-sort with bounded recursive-depth:

```
while (j − i > n_0) do
    r = pick the position of a "good pivot";
    swap S[r] with S[i];
    p = PARTITION(S, i, j);
    if (p ≤ (i+j)/2) then                n > no
        BOUNDEDQS(S, i, p − 1);
        i = p + 1;
    else
        BOUNDEDQS(S, p + 1, j);
        j = p − 1;
    end if
end while
INSERTIONSORT(S, i, j);                  n ≤ no
```

Algorithm Bounded(S, I, j)

- The recursive call is executed on the smaller part of the partition
- It drops the recursive call on the larger part of the partition in favor of another execution of the while-loop.
- Ex:

10	3	7	15	21	18	2	11

pivot =S[2]=3

Partition

2	3	10 7 15 21 18 11

Only one recursive call on S(i,p-1); For the larger part S(i+1,j) we iterate to the while loop

- Technique: Elimination of Tail Recursion
- Bounded takes O(nlogn) time in average and O(logn) space.

Multi-way QuickSort

- 2-level model: M =internal memory size ; B = block size ;
- Split the sequence S into $k = \Theta(M/B)$ sub-sequences using k-1 pivots.
- We would like balanced partitions, that is of $\Theta(n/k)$ items each.
- Select k-1 $s_1, s_2, \ldots s_{k-1}$ "good pivots" is not a trivial task! (later)

Multi-way QuickSort

Let bucket Bi the portion of S between pivot s_{i-1} and s_i.

We want $|Bi| = \Theta(n/k)$ for all buckets!

So:

at the first step the size of portions n/k

at the second step the size of portions n/k^2

at the third step the size of portions n/k^3

....

Stop when $n/k^i \le M$: $n/M \le k^i$, i is the number of recursion steps

$$i \ge \log_k n/M = \log_{M/B} n/M$$

This number of steps is enough to have portions shorter than M, and sorted in internal memory!

Partition takes O(n/B) I/O's (dual to multiway Merge) : 1 input block, k output blocks (used to write into the k-partition under formation).

Multi-way QuickSort

Find k good pivots efficiently.

Randomized strategy called oversampling.

$\Theta(ak)$ items are sampled, $a \geq 0$ parameter of the oversampling.

Algorithm 5.7 Selection of $k-1$ good pivots via oversampling

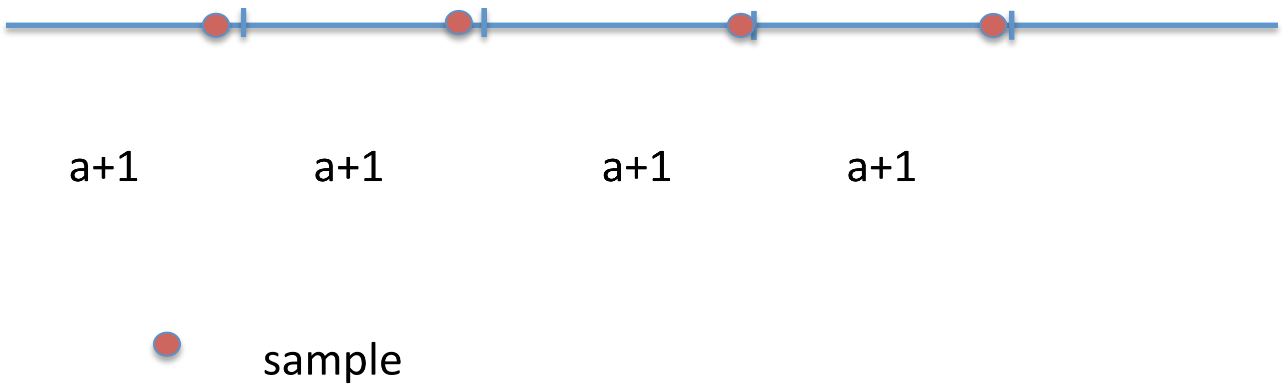
Take $(a+1)k-1$ samples at random from the input sequence;
Sort them into an ordered sequence A;
For $i = 1, \ldots, k-1$, pick the pivot $s_i = A[(a+1)i]$;
return the pivots s_i;

$\Theta(ak)$ candidates
$\Theta(ak)\log(ak)$ time
Select k-1 pivots evenly distributed

Balanced selection of $s_i = A[(a+1)i]$ should provide good pivots!!

Multi-way QuickSort

- K=5



Multi-way QuickSort

- The larger is a the closer to $\Theta(n/k)$

- If $a=n/k$ the elements of set A cannot be sorted in M !

- If $a = 0$ the selection is fast, but unbalanced partitions are is more probable.

- Good choice for a: $\Theta(\log k)$. Pivot-selection costs $\Theta(k\log^2 k)$

  Pivot selection costs $\Theta(ak \log ak)$

  $a=\log k$ $\quad k\log k \log(k\log k) = k\log k (\log k + \log\log k) = \Theta(k \log^2 k)$

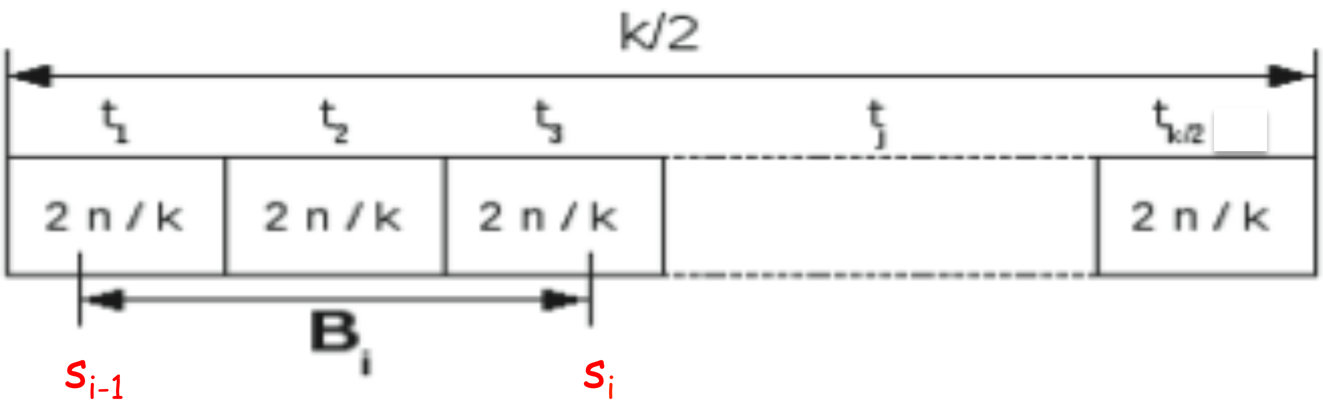
Multi-way QuickSort

Lemma. Let $k \geq 2$, $a + 1 = 12 \ln k$. A sample of $(a+1)k-1$ suffices to ensure that all buckets receive less than $4n/k$ elements, with probability at least ½.

Proof. We find un upper bound to complement event: there exists 1 bucket containing more than $4n/k$ elements with probability at most ½. Failure sampling.

Consider the sorted version of S, S'. Logically split S' in $k/2$ segments $(t_1, t_2, \ldots, t_{k/2})$ of $2n/k$ elements each.

Multi-way QuickSort



- The event is that there exists a bucket Bi with more that 4n/k items. It spans more than one segment: pivots s_{i-1} and s_i fall outside t_2.
- In t_2 fall less than (a+1) samples (see selection algorithm: between 2 pivots there are a+1 samples, hence in t_2 there are less).

- Pr (exists B_i : $|B_i| \geq 4$ n/k) $\leq$ Pr (exists t_j : contains < (a+1) samples)

  $\leq$ k/2 Pr (a specific segment contain < (a+1) samples)

Since k/2 is the number of segments.

Multi-way QuickSort

- Pr (1 sample goes in a given segment) = $(2n/k)/n = 2/k$
  If drawn uniformly at random from S (and S').
- Let X the number of samples going in a given segment, we want to compute:

$$\Pr(X < a+1)$$

- Observe : $E(X) = ((a+1)k-1) \times 2/k \geq 2(a+1)-2/k$, per $k \geq 2$

$$E(X) \geq 2(a+1) - 1 \geq 3/2(a+1) \text{ for all } a \geq 1$$

$$a+1 \leq 2/3\, E(X) = (1-1/3)\, E(X)$$

By Chernoff bound

$$\Pr(X < (1 - \delta) E(X) \leq e\hat{}\{\delta^2 /2)E(X)\})$$

Setting $\delta=1/3$ and assume $a+1 = 12\ln k$

$$\Pr(X < a+1) \leq P(X \leq (1-1/3)E(X)) \leq$$

$$e^{-E(X)/18} \leq e^{-(a+1)/12} = e^{-\ln k} = 1/k$$

Multi-way QuickSort

- $\Pr(X < a+1) \leq 1/k$

We have already derived:

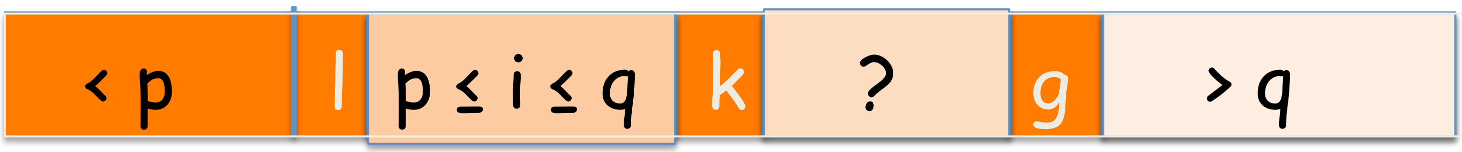
- $\Pr(\text{exists } B_i : |B_i| \geq 4\, n/k) \leq k/2 \times \Pr(\text{a segment contain< } (a+1) \text{ samples})$

- $\Pr(\text{exists } B_i : |B_i| \geq 4\, n/k) \leq 1/2$ complement event of the lemma

- All buckets receives less than 4n/k elements with probability > 1/2

Dual Pivot QuickSort

- Good strategy in practice no theoretical result.
- Empirical good results in average.



- p, q pivots l, k, g indices ➔ 4 pieces

1. items smaller than p
2. items larger or equal to p and smaller or equal to q.
3. items not jet considered
4. items greater than q

Dual Pivot QuickSort

Similar to the 3-ways Partition: maintains the invariants.

- Items equal to the pivot are not treated separately.
- 2 indices move rightward , l and k, while g moves leftward.
- Termination: $k \geq g$.

- For item k, compare S[k] : p,
  if S[k] < p exchange S[k] and S[l] and increment pointers

  else if S[k] > q decrease g while S[g] > q and $g \neq k$
  the last value of g : $S[g] \leq q$
  exchange S[k] and S[g]
  ……

The comparison with S[k] drives the phases possibly including a long shift to the left.   The nesting of comparison is the key for the efficiency of the algortihm.

Dual Pivot Partition

l k g g

5 12 9 12 13 15 17 19 12 26 18 22 20

p=12 q= 17 g--

exchange 19 and 12, k++, g--

g

l k

5 12 9 12 13 15 17 12 19 26 19 18 22 20

no exchange , k++, g--

Dual Pivot QuickSort

You can find the complete code description and the visualization of the algorithm on youtube by searching for Dual pivot QuickSort.

Conclusions:

Even a very old, classic algorithm such as QuickSort can be speed up and innovated!