Cutting Plane Algorithms

(Wolsey : Chapter 8)

Consider the Integer Linear Program (ILP):

$$(IP) \quad \begin{array}{l} \max\ cx \\ \left.\begin{array}{l} Ax \leq b \\ x \in \mathbb{Z}^n \end{array}\right\} x \in X \end{array}$$

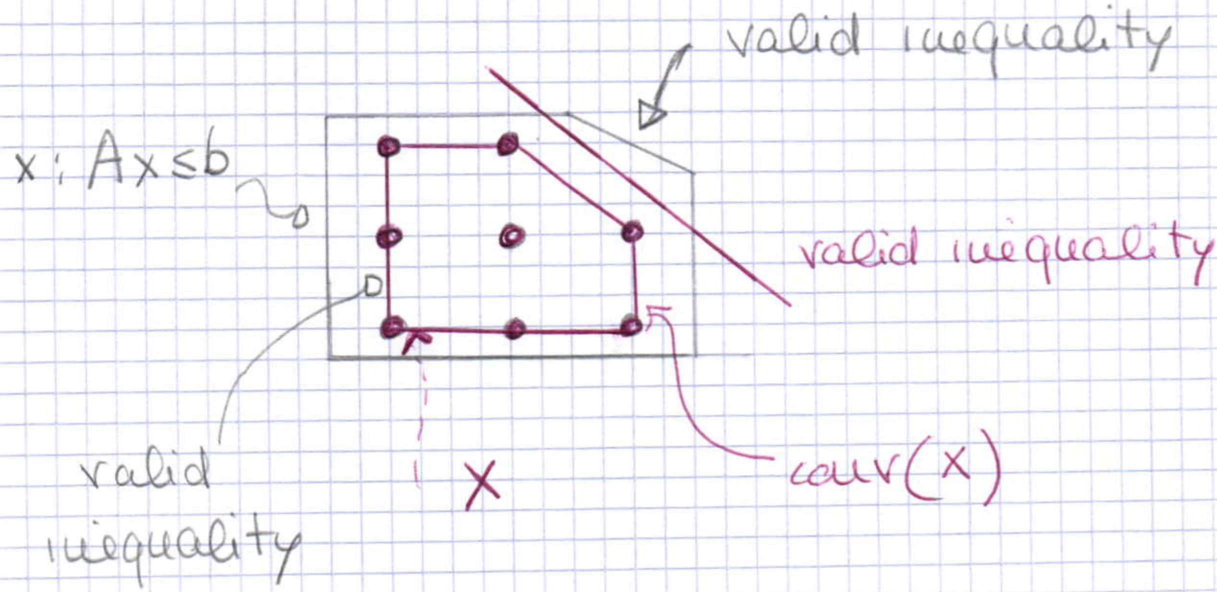
We know that the convex envelope of X, $conv(X)$ is a polyhedron. So, the optimal solution of (P) is an optimal solution of the LP:

$$\begin{array}{l} \max\ cx \\ x \in conv(X) \end{array}$$

Unfortunately, for NP-Hard problems we do not have an explicit, or a "good", description of $conv(X)$:

< how to find effective ways to approximate $conv(X)$? >

<u>Definition</u>: an inequality $\pi x \leq \pi_0$ is a <u>valid inequality</u> for X if $\pi x \leq \pi_0$ $\forall x \in X$



Simple valid inequalities

These are logical inequalities

1) <u>0-1 Knapsack set</u>

$$X = \{ x \in \{0,1\}^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2 \}$$

- if $x_2 = x_4 = 0$, then the l.h.s is $3x_1 + 2x_3 + x_5$, which is ≥ 0 but the r.h.s. is -2

impossible!

So $x_2 + x_4 \geq 1$ is valid for X

- similarly, if $x_1 = 1$ and $x_2 = 0$:

l.h.s, $3 + 2x_3 - 3x_4 + x_5 \geq 3 - 3 = 0$

but the r.h.s is -2

impossible!

So $x_1 \leq x_2$ is valid for X

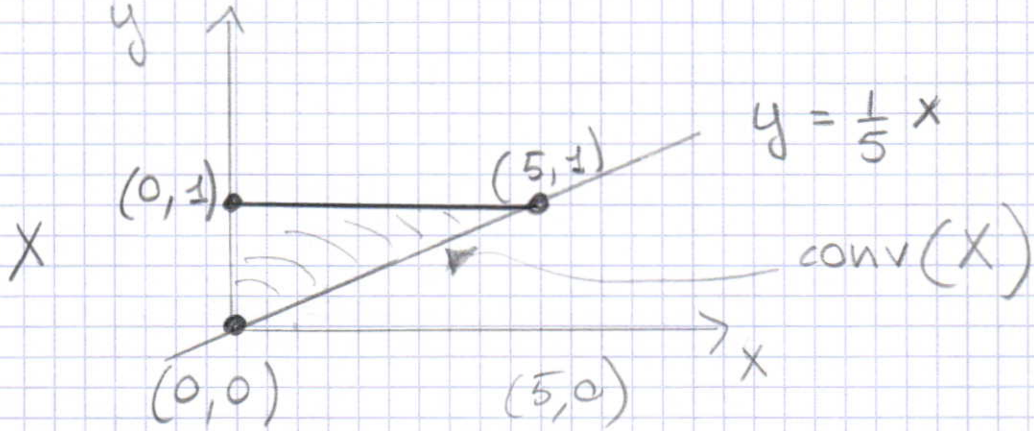
2) <u>Mixed 0-1 set</u>

$$X = \{(x,y) : x \leq 9999y, 0 \leq x \leq 5, \\ y \in \{0,1\}\}$$

Indeed:

$$X = \{(0,0), (x,1) \text{ with } 0 \leq x \leq 5\}$$

So $x \leq 5y$ is valid for X



Note that $\text{conv}(X) = \{(x,y) : 0 \leq x \leq 5,$
$0 \leq y \leq 1, x \leq 5y\}$

3) Integer rounding

$X = P \cap \mathbb{Z}^4$ where

$$P = \{ x \in \mathbb{R}^4_+ : 13 x_1 + 20 x_2 + 11 x_3 + 6 x_4 \geq 72 \}$$

$: 11 \qquad \frac{13}{11} x_1 + \frac{20}{11} x_2 + x_3 + \frac{6}{11} x_4 \geq \frac{72}{11}$

is valid for P

as $x \geq 0$, round up the coefficients on left to the nearest integer:

$$2 x_1 + 2 x_2 + x_3 + x_4 \geq \frac{72}{11}$$

(weaker) valid inequality for P

as $x \in \mathbb{Z}^4$ and the coefficients are integer, the l.h.s. must be integer:

$$2 x_1 + 2 x_2 + x_3 + x_4 \geq 7$$

is valid for X

< How can we generate valid inequalities for integer problems? >

Valid inequalities for LP

Let $P = \{x : Ax \leq b,\ x \geq 0\} \neq \emptyset$ polyhedron

<u>Proposition 1:</u> $\pi x \leq \pi_0$ is valid for P <u>if and only if</u> there exist $u \geq 0$, $v \geq 0$ such that $\pi = uA - v$ and $ub \leq \pi_0$, or equivalently if and only if there exists $u \geq 0$ s.t. $uA \geq \pi$ and $ub \leq \pi_0$.

<u>Proof</u>:

$\pi x \leq \pi_0$ is valid for P if and only if $\pi x \leq \pi_0 \ \forall x \in P$, i.e. if and only if

$$\max\{\pi x : x \in P\} \leq \pi_0 .$$

By strong (LP) duality:

$$\min\{ub : uA \geq \pi,\ u \geq 0\} \leq \pi_0 .$$

So, if and only if $\exists\ u \geq 0$ s.t.

- $uA \geq \pi$
- $ub \leq \pi_0$

$\square$

Valid inequalities for ILP

The basic, very simple, observation is:

Proposition 2: Let $X = \{ y \in \mathbb{Z} : y \leq b \}$.
Then $y \leq \lfloor b \rfloor$ is valid for X.

Example

$$X = P \cap \mathbb{Z}^2, \text{ where}$$

$$P: \quad \begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- $\frac{2}{7}$
- $\frac{37}{63}$
- 0

1) $u = \left(\frac{2}{7}, \frac{37}{63}, 0\right)$:

$$2x_1 + x_2\left(\frac{37}{63} - \frac{4}{7}\right) \leq 4 + \cancel{3}\cdot\frac{37}{\underset{21}{\cancel{63}}}$$

$$2x_1 + \frac{1}{63}x_2 \leq \frac{121}{21}$$ valid for P (Prop. 1)

2) reduce the coefficients on the left to the nearest integer:

$$2x_1 + 0 \cdot x_2 \leq \frac{121}{21}$$ valid for P (Prop. 1)

3) now reduce the r.h.s to the nearest integer:

$$2x_1 \leq \left\lfloor \frac{121}{21} \right\rfloor = 5 \qquad \text{valid for } X \text{ (Prop. 2)}$$

If we repeat the procedure with weight of $\frac{1}{2}$ for this new constraint:

$$x_1 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2 \qquad \underline{\text{Tighter valid inequality}}$$

<u>General procedure we have used:</u>

<u>Chvátal-Gomory procedure</u> (to generate a valid inequality for the set $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$:

- A : $m \times n$ matrix
- $\{a^1, a^2, .. a^n\}$ denote the columns of A
- $u \in \mathbb{R}^m_+$ is a vector of nonnegative multipliers (one per row)

(i) $\sum_{j=1}^{n} u a^j x_j \leq u b$

is valid for P (and so for X)

(ii) $\sum_{j=1}^{n} \lfloor u a^j \rfloor x_j \leq u b$

is valid for P (and so for X), since $x_j \geq 0$, $j = 1, .. n$

(iii) $\sum_{j=1}^{n} \lfloor u a^j \rfloor x_j \leq \lfloor u b \rfloor$

is valid for X since x_j are integer, and so $\sum_{j=1}^{n} \lfloor u a^j \rfloor x_j$ is integer $\blacksquare$

This simple procedure is able to generate <u>all</u> valid inequalities for ILP:

<u>Theorem</u>: Each valid inequality for X can be obtained by applying the Chvátal-Gomory procedure a finite number of times.

<u>No proof</u>

- This is mainly a theoretical result
- There is a variety of "ad hoc" and also general ways to generate valid inequalities

< How to use them ? >

1 A priori addition

Idea: If $P = \{ x : Ax \leq b,\ x \geq 0 \}$, where $X = P \cap \mathbb{Z}^n$, is the given formulation of the ILP problem to be solved, find a set $\Phi x \leq q$ of valid inequalities for X, enhance P, obtaining $P' = \{ x : Ax \leq b,\ \Phi x \leq q,\ x \geq 0 \}$, and then apply the favorite algorithm (e.g. Branch and Bound) to P'

Advantages: if P' is significantly smaller than P, then the bounds should be improved and hence Branch and Bound should be more effective.

Drawbacks:

- LPs very big (long time for their solution)
- standard Branch and Bound tools not usable (too many constraints)

2 Cutting Plane algorithms

Let:

- (IP): $\max cx$
  $x \in X$

  where $X = P \cap \mathbb{Z}^n$

- F: known family of valid inequalities for X

In many cases F contains too many inequalities to be added a priori:

< how can we generate "useful" valid inequalities of F? >

Cutting Plane Algorithm

Initialization : $t=0$, $P^0 = P$;

Iteration t : solve the LP :

$$\bar{z}^t = \max_{x \in P^t} cx$$

Let x^t be an optimal solution.
If $x^t \in \mathbb{Z}^n$, STOP : x^t solves (IP).
If $x^t \notin \mathbb{Z}^n$, then solve the separation problem for x^t and the family F :

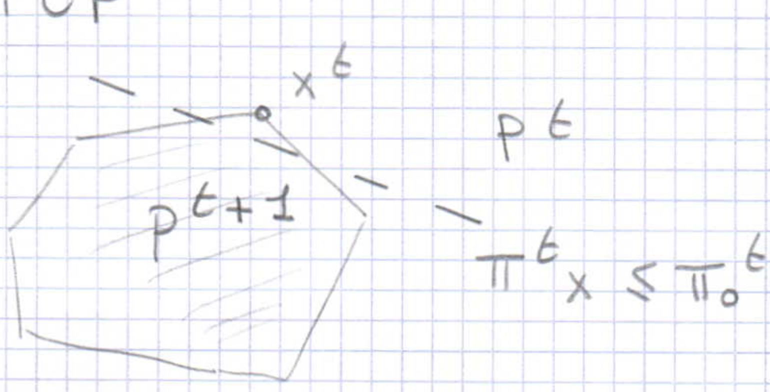
- if $\pi^t x \leq \pi_0^t \in F$ is found such that

  $$\boxed{\pi^t x^t > \pi_0^t}$$ (it cuts off x^t)

  then $P^{t+1} = P^t \cap \{x : \pi^t x \leq \pi_0^t\}$;

  $t := t+1$

- otherwise STOP



- If the algorithm terminates without an integer solution, then the "last" P^t is an improved formulation (better than P): it can be given in input to a Branch and Bound
- It is often preferable to add several violated cuts at each iteration (rather than one at a time)

A specific implementation is:

Gomory's fractional cutting plane algorithm

Idea: at each iteration t, if x^t is not integer, then choose a basic variable which is not integer, and generate a Chvátal-Gomory inequality so as to cut off x^t.

Assume that (IP) is written in the form:

$$\max\ cx$$
$$Ax = b$$
$$x \geq 0,\ \text{integer}$$

So, the LP to be solved at each iteration of the Cutting Plane Algorithm is:

$$\max\ cx$$
$$Ax = b$$
$$x \geq 0$$

[Obs: by rewriting $\max cx = -\min -cx$, this is the Dual of the Asymmetric Pair]

Solve it by the Dual Simplex Algorithm and let B an optimal basis, and (x_B^*, x_N^*) an optimal solution, where N corresponds to the nonbasic variables:

$$\underset{\text{basis matrix}}{A_B}\, x_B + A_N x_N = b$$

$$x_B + \underbrace{A_B^{-1} A_N}_{[\tilde{a}_{kj}]} \underset{0}{\overset{}{x_N}} = \underbrace{A_B^{-1} b}_{[\tilde{b}_k]}$$

So: $x_B^* = A_B^{-1} b \geq 0$

$x_N^* = 0$

If x^* (i.e. x_B^*) is non integer, there is $k \in B$ s.t. $x_k^* = (A_B^{-1} b)_k = \tilde{b}_k \notin \mathbb{Z}$.

Choose row k; then

$$x_k + \sum_{j \in N} \tilde{a}_{kj} x_j = \tilde{b}_k \qquad (\circ)$$

So:

$$x_k + \sum_{j \in N} \lfloor \tilde{a}_{kj} \rfloor x_j \leq \lfloor \tilde{b}_k \rfloor$$

is a Chvátal-Gomory cut.

Since $x_k = \tilde{b}_k - \sum_{j \in N} \tilde{a}_{kj} x_j$ from $(\circ)$, rewrite the Chvátal-Gomory cut by eliminating x_k:

$$\sum_{j \in N} \underbrace{(\tilde{a}_{kj} - \lfloor \tilde{a}_{kj} \rfloor)}_{f_{kj}} x_j \geq \underbrace{\tilde{b}_k - \lfloor \tilde{b}_k \rfloor}_{\alpha_k}$$

or

$$\sum_{j \in N} f_{kj} x_j \geq \alpha_k$$

Since :

- $0 \leq f_{kj} < 1 \quad \forall j$
- $0 < \alpha_k < 1 \quad$ (in fact $x_k^* = \tilde{b}_k \notin \mathbb{Z}$)

then $\displaystyle\sum_{j \in N} f_{kj} x_j \geq \alpha_k$

is a valid inequality which cuts off $(x_B^*, \underset{0}{\underset{\parallel}{x_N^*}})$:

$0 \geq \alpha_k$ <u>no</u> since $0 < \alpha_k < 1$.

<u>Example</u>

(IP) $z = \max 4x_1 - x_2$

$$
\begin{aligned}
7x_1 - 2x_2 + x_3 &= 14 \\
x_2 + x_4 &= 3 \\
2x_1 - 2x_2 + x_5 &= 3 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0,
\end{aligned}
$$

integer

Solving the LP we get the optimum basis $B=\{1,2,5\}$, so:

$$x_1 \qquad + \frac{1}{7}x_3 + \frac{2}{7}x_4 \qquad = \frac{20}{7}$$

$$x_2 \qquad + x_4 \qquad = 3$$

$$-\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7}$$

i.e. $x^* = \left(\frac{20}{7}, 3, 0, 0, \frac{23}{7}\right)$ is an optimal LP solution; since x^*_1 is fractional, use the first row to generate the cut:

$$\left(\frac{1}{7} - \left\lfloor \frac{1}{7} \right\rfloor\right) x_3 + \left(\frac{2}{7} - \left\lfloor \frac{2}{7} \right\rfloor\right) x_4 \geq \frac{20}{7} - \left\lfloor \frac{20}{7} \right\rfloor$$

i.e.

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

Add this cut (which cuts off x^*:

$$0 + 0 \ngeq \frac{6}{7} \,)$$

and go to the next iteration of the Cutting Plane Algorithm.

Ad-hoc valid inequalities

1 0-1 Knapsack

$$X = \{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \le b \}$$

where:

- $a_j > 0 \qquad j = 1, \dots n$
- $b > 0$

Let $N = \{1, \dots n\}$

Definition : $C \subseteq N$ is a cover if $\sum_{j \in C} a_j > b$.

A cover is minimal if $C \setminus \{j\}$ is not a cover $\forall j \in C$.

Proposition : If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \le |C| - 1$$

is valid for X.

Example

$$X = \{ x \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + \\ + 5x_5 + 4x_6 + x_7 \leq 19$$

minimal cover inequalities for X are:

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &\leq 2 \\ x_1 + x_2 \qquad\qquad + x_6 &\leq 2 \\ x_1 \qquad\qquad + x_5 + x_6 &\leq 2 \\ x_3 + x_4 + x_5 + x_6 &\leq 3 \end{aligned}$$ ∎

There is a simple way to strengthen the cover inequalities:

Proposition: If C is a cover for X, then the extended cover inequality:

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid for X, where $E(C) = C \cup \{ j : a_j \geq a_i \ \forall i \in C \}$

Example (cont.)

if $C = \{3, 4, 5, 6\}$, then the (unique) extended cover inequality for C is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

Note that the cover inequality

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

is dominated by such an extended inequality.

Separation for cover inequalities

Let F the family of cover inequalities for X.

Separation problem for F: given $x^* \notin \mathbb{Z}^n$, with $0 \leq x_j^* \leq 1 \ \forall j \in N$, does x^* satisfy all cover inequalities?

Formally:

since each cover inequality can be rewritten as

$$\sum_{j \in C} (1 - x_j) \geq 1,$$

then the separation problem is:

$*$ Does there exist $C \subseteq N$ with $\sum_{j \in C} a_j > b$ such that $\sum_{j \in C} (1 - x_j^*) < 1$? $*$

Equivalently:

- compute

$$(P_s) \qquad \alpha = \min_{C \subseteq N} \left\{ \sum_{j \in C} (1 - x_j^*) : \sum_{j \in C} a_j > b \right\}$$

- is $\alpha < 1$?

Since C is unknown, we can formulate (P_s) as a 0-1 program:

$$z_j = \begin{cases} 1 & \text{if } j \in C \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in N$$

so we can restate as:

$$\alpha = \min \sum_{j \in N} (1 - x_j^*) z_j$$

$$\sum_{j \in N} a_j z_j > b$$

$$z_j \in \{0, 1\} \qquad j = 1, \dots n$$

<u>Proposition</u>:

- if $\alpha \geqslant 1$ then x^* satisfies all cover inequ.
- if $\alpha < 1$, then if z^* is an optimal solution, and $C^* = \{ j \in N : z_j^* = 1 \}$, $\sum_{j \in C^*} x_j \leq |C^*| - 1$ cuts off x^*.

Example

$$X = \{ x \in \{0,1\}^6 : 45 x_1 + 46 x_2 + 79 x_3 + 54 x_4 + 53 x_5 + 125 x_6 \leq 178 \}$$

consider $x^* = (0, 0, \frac{3}{4}, \frac{1}{2}, 1, 0)$

The corresponding separation problem is:

$$\alpha = \min \; z_1 + z_2 + \frac{1}{4} z_3 + \frac{1}{2} z_4 + 0 z_5 + z_6$$

$$45 z_1 + 46 z_2 + 79 z_3 + 54 z_4 + 53 z_5 + 125 z_6 >$$
$$> 178 \quad (\equiv \geq 179)$$
$$z_i \in \{0,1\} \quad i = 1, \ldots 6$$

An optimal solution is $z^* = (0, 0, 1, 1, 1, 0)$ with $\alpha = \frac{3}{4}$. Thus the cover inequality

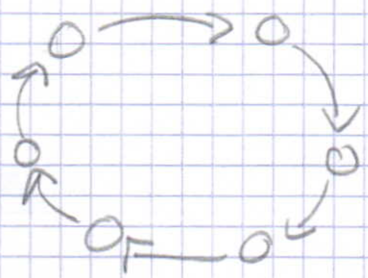
$$x_3 + x_4 + x_5 \leq 2$$

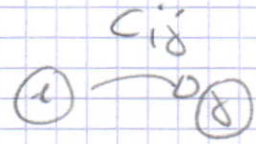
cuts off x^*. Note that it is violated by the amount $(1 - \alpha) = \frac{1}{4}$.

Observation: the separation problem is a knapsack problem wich can be solved exactly or via heuristics

(2) Asymmetric TSP

Let $G=(N,A)$ be a directed graph





Find a minimum cost directed Hamiltonian tour.

ILP formulation:

$$(ATSP) \quad \min \sum_{(i,j)\in A} c_{ij}\, x_{ij}$$

$$\sum_{(j,i)\in BS(i)} x_{ji} = 1 \qquad \forall i \in N$$

$$\sum_{(i,j)\in FS(i)} x_{ij} = 1 \qquad \forall i \in N$$

$$\sum_{i\in S} \sum_{j\in N\setminus S} x_{ij} \geq 1 \qquad \forall S \subset N,\ S \neq \emptyset$$

cut constraints

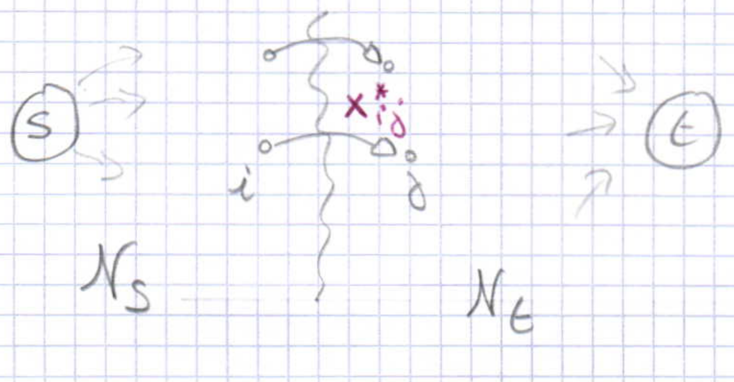
$$x_{ij} \in \{0,1\} \qquad \forall (i,j)\in A$$

In some cases, the point is not to separate "additional" valid inequalities, but to manage an exponential number of known inequalities, such as the cut constraints. This also to solve the linear relaxation!

Consider the linear relaxation of the previous (ATSP) formulation: we can not manage the exponential number of cut constraints:

Separation problem for cut constraints

- select two nodes s and t in G
- if $\{x^*_{ij}\}$ is the optimal solution of the "current" LP, then define the capacity of (i,j) equal to x^*_{ij}, $\forall (i,j) \in A$
- solve a maximum flow problem from s to t in G equipped with the capacities $\{x^*_{ij}\}$:



From the max flow - minimum cut theorem, the maximum flow value is equal to the minimum capacity of the cuts <u>separating s from t</u> : let (N_s, N_t) be the found minimum cut

- if the maximum flow value is < 1, then it means

$$\sum_{i \in N_s} \sum_{j \in N_t} x^*_{ij} < 1$$

so we have found a violated cut constraint that we can add to the LP (cutting off x^*); solve the new LP

- otherwise choose another pair (s, t) and iterate

At the termination, if $\sum_{i \in N_s} \sum_{j \in N_t} x^*_{ij} \geq 1$ $\forall$ pair (s, t), then the cut constraints are satisfied by $\{x^*_{ij}\}$, which is an optimal solution of the linear relaxation of ATSP !

< <u>Cutting Plane for LP</u> >