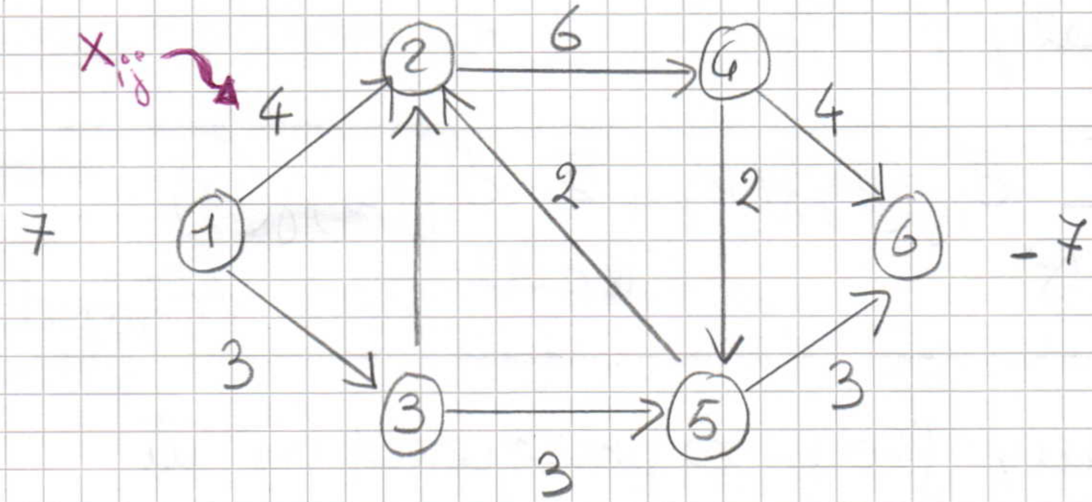
# Flow decomposition

(Ahuja - Magnanti - Orlin : Chapter 3 (3.5))

**Observation** : any flow x can be defined in terms of flows on arcs (as previously formulated) or in terms of flows on paths and cycles

**example**




< flows on arcs >

$x = (x_{ij})$

or

4 units along the path $(1, 2, 4, 6) = P_1$

3 units along the path $(1, 3, 5, 6) = P_2$

2 units along the cycle $(2, 4, 5) = W$

< flows on paths and cycle >

Let $\mathcal{P}$ : set of all paths

$f(P)$ : flow on $P \in \mathcal{P}$

$\mathcal{W}$ : set of all cycles

$f(w)$ : flow on $w \in \mathcal{W}$

$$\delta_{ij}(P)\ (w) = \begin{cases} 1 & \text{if } (i,j) \in P\ (w) \\ 0 & \text{otherwise} \end{cases}$$

Then : any flow representation in terms of path and cycle flows determines arc flows uniquely :

$$\boxed{x_{ij} = \sum_{P \in \mathcal{P}} \delta_{ij}(P) f(P) + \sum_{w \in \mathcal{W}} \delta_{ij}(w) f(w) \quad \forall (i,j) \in A}$$

Viceversa : any flow representation in terms of arc flows "decomposes" into path and cycle flow (not uniquely)

$\Downarrow$

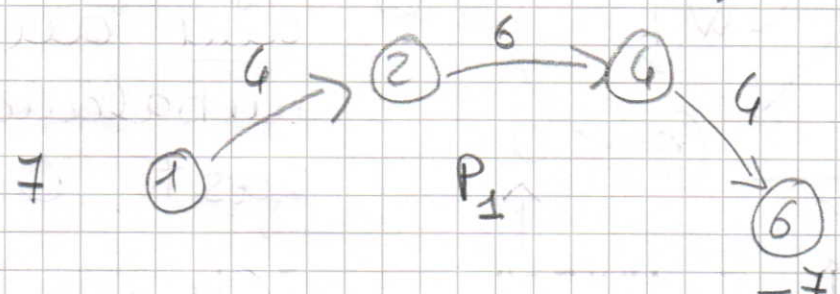
Flow decomposition Theorem : each path and cycle flow has a unique representation in terms of (nonnegative) arc flows. Conversely, each arc flow x can be represented as path and cycle flow (not uniquely) s.t.:

a) each directed path with a positive flow connects a source to a destination

b) at most n+m paths and cycles have a positive flow; out of these, at most m cycles have positive flow

<u>Proof</u> (intuition) < example cont. >

<u>select a source node</u> (1):

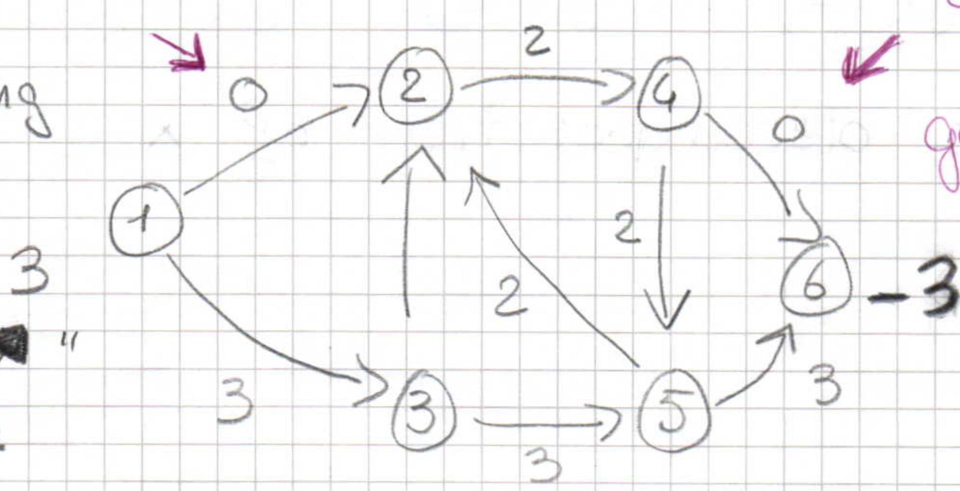



$\min\{r(P_1), b(1), -b(7)\} = 4$

subtract 4 units of flow along $P_1$:

$f(P_1) = 4$

"remaining flow"

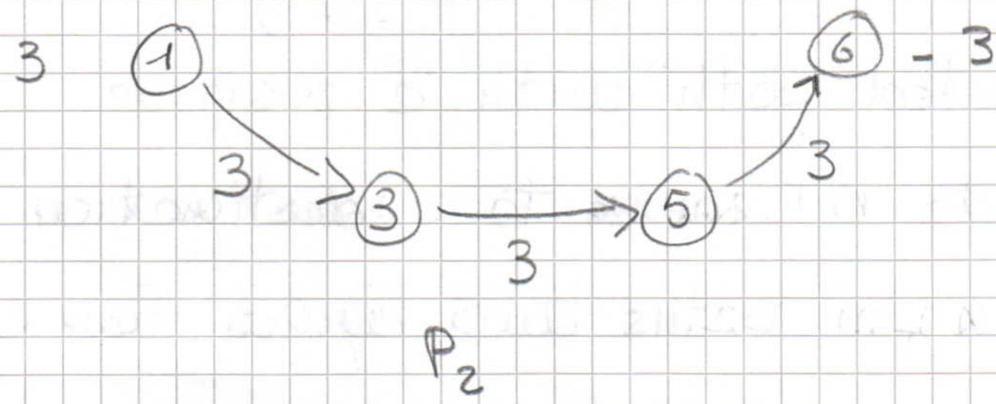



"updated balance"

at least one flow arc goes to 0

# select a source node (1):

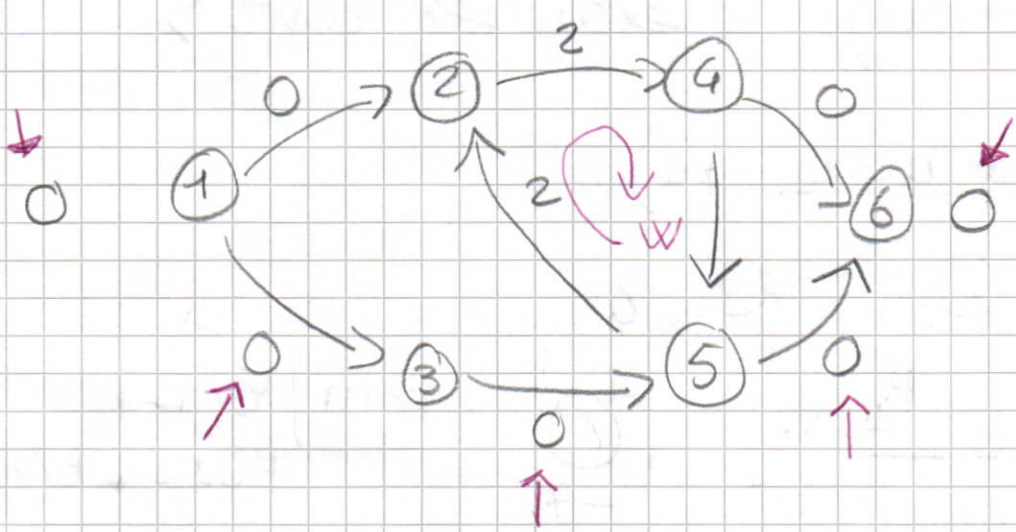





$\min\{r(P_2), b(1), -b(6)\} = 3$

subtract 3 units of flow along $P_2$:

$f(P_2) = 3$




at least one flow arc goes to 0 and one imbalance goes to 0

$(n + m)$

remaining cycle $w$ : $f(w) = 2$ at most $m$

$\Rightarrow f(P_1) \quad f(P_2) \quad f(w)$

is a decomposition of $x$

A <u>circulation</u> is a flow s.t.
$b(i) = 0 \quad \forall i \in N$

<u>Corollary</u>: a circulation $x$ can be represented in terms of cycle flow along $\leq m$ directed cycles.

## <u>Important consequences of the flow decomposition Theorem</u>

Let $x$ be a flow and $G(x)$ its residual network.

<u>Def</u>: an <u>augmenting cycle</u> $w$ w.r.t. $x$ is a directed cycle in $G(x)$; <u>its cost</u>

$$c(w) = \sum_{(i,j)\in w} c_{ij} = \sum_{(i,j)} c_{ij}\, \delta_{ij}(w)$$ is the

costs in $G(x)$

mistake in AMO

change of the cost of $x$ if we push 1 unit of flow along $w$.

Consider the minimum cost flow problem on $G=(N,A)$, and let $x$ and $x^{\circ}$ be any two feasible solutions: how can we compare $cx$ and $cx^{\circ}$?

Indeed, the flow decomposition theorem can be extended so as to estabish a relationship between $x$ and $x^{\circ}$:

<u>Flow decomposition theorem</u> (<u>ext 1</u>): given a feasible solution $x^{\circ}$ to the minimum cost flow problem, any other feasible solution $x$ can ben obtained from $x^{\circ}$ by sending flow along $\leq m$ augmenting cycles w.r.t. $x^{\circ}$:

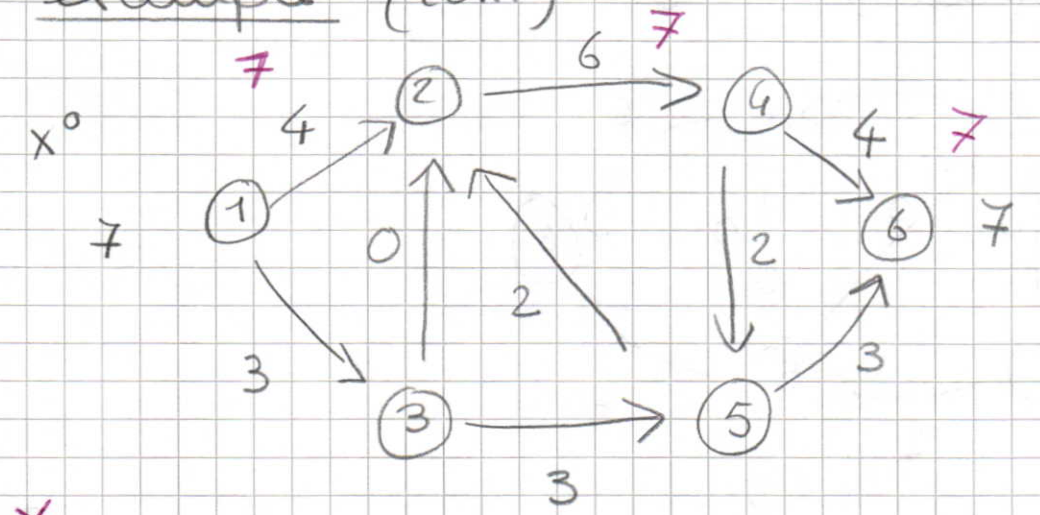
$$\boxed{x_{ij} = x^{\circ}_{ij} + \delta_{ij}(w_1)\, f(w_1) + \dots + \delta_{ij}(w_r)\, f(w_r) \qquad \forall (i,j) \in A}$$

$(r \leq m)$

<u>now</u>: $\delta_{ij}(w_k) = \begin{cases} +1 & \text{if } (i,j) \text{ is \underline{forward}} \\ -1 & \text{if } (i,j) \text{ is \underline{backward}} \\ 0 & \text{if } (i,j) \notin w_k \end{cases}$

$k = 1, \dots r$

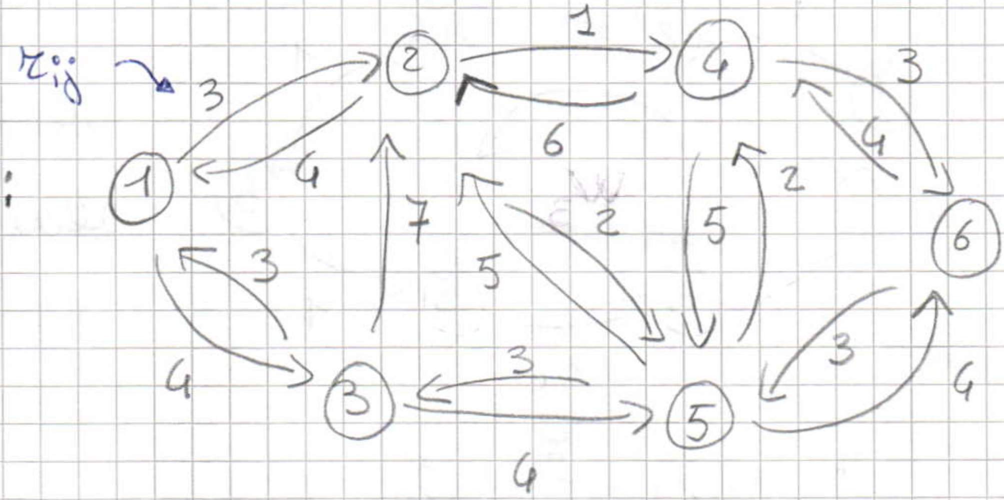
## example (cont)



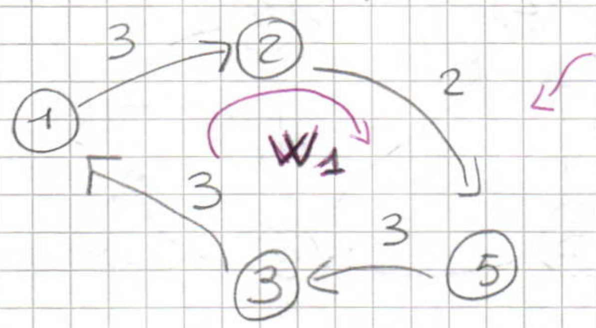

$u_{ij} = 7$

$\forall (i,j) \in A$

$G(x^0)$:



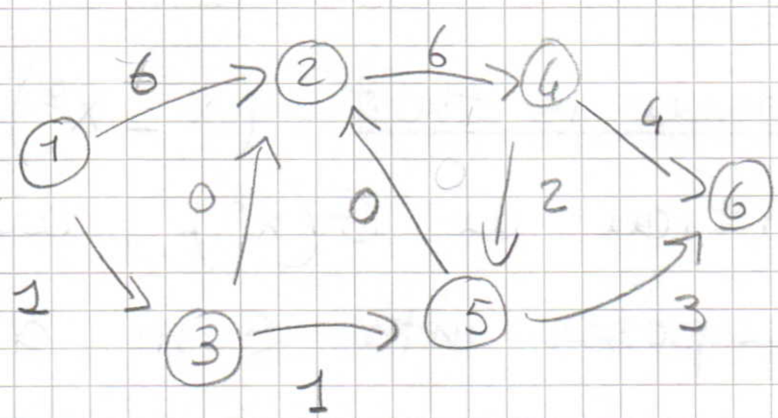

**In fact:** start from $x^0$



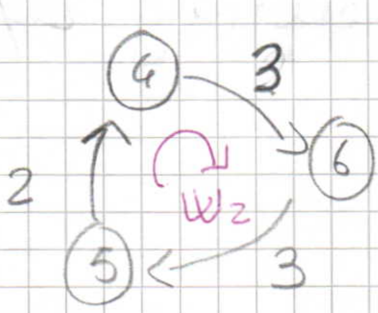

$\delta_{25} = -1$

send 2 along $w_1$

< intermediate flow >

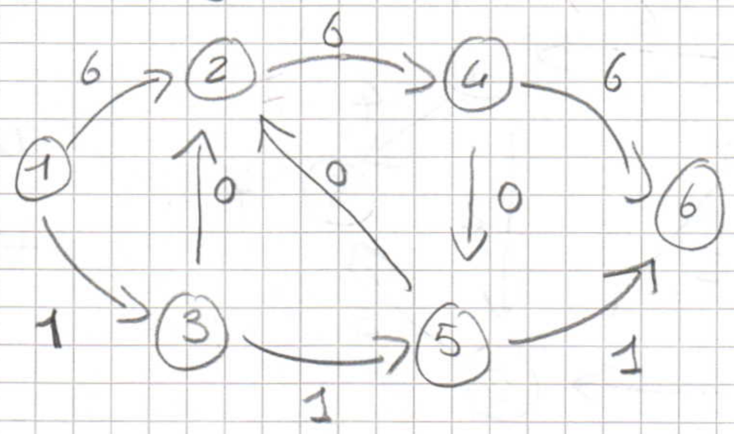


**then**

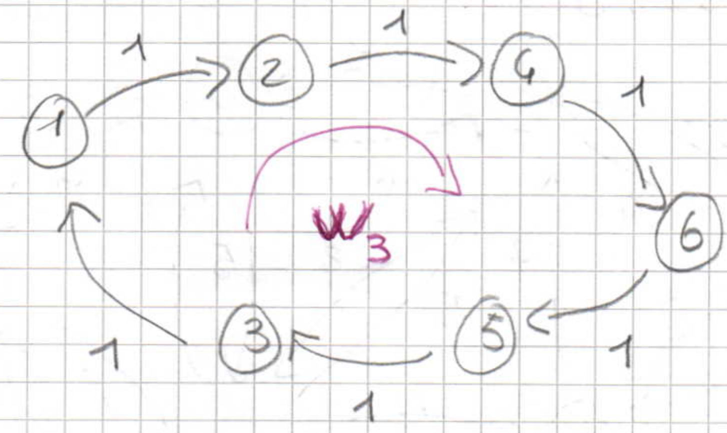



send 2 along $w_2$

<intermediate flow>

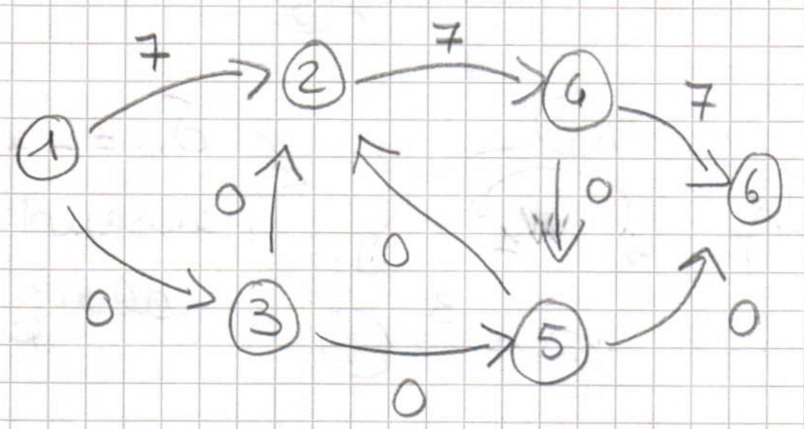


and finally




send 1 along $w_3$

$x$ !!



<u>The reason is that</u> $(x - x^\circ)$ is a circulation in $G(x^\circ)$, and therefore it decomposes into $\leq m$ directed cycles ($\equiv$ augmenting cycles) in $G(x^\circ)$

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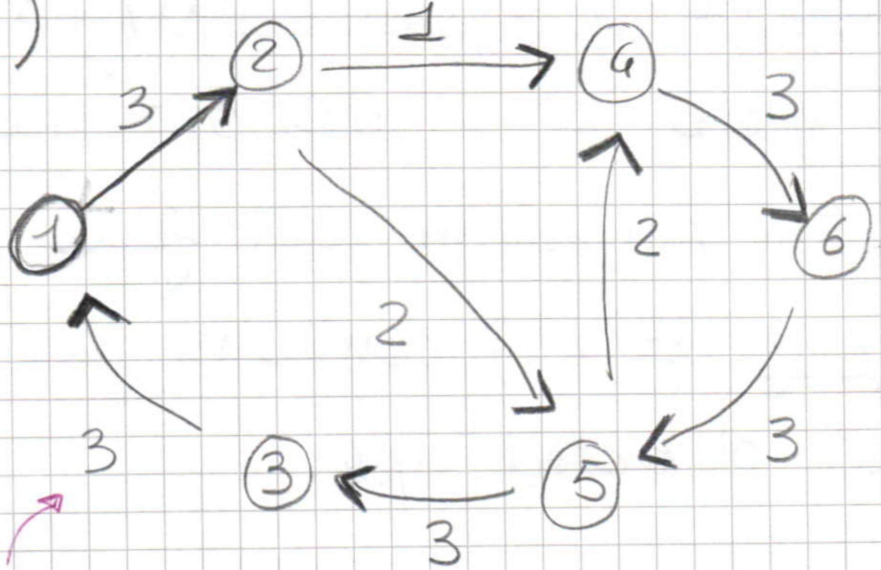
$(x - x^0)$




circulation in $G(x^0)$

$x_{13} - x^0_{13} = -3$

(reverse arc in $G(x^0)$)

<u>Therefore</u>:

$$\boxed{\sum_{(i,j)\in A} c_{ij} x_{ij}} = \sum_{(i,j)\in A} c_{ij} x^0_{ij} + \sum_{(i,j)\in A} c_{ij}\delta_{ij}(w_1) f(w_1) +$$

$$+ \dots + \sum_{(i,j)\in A} c_{ij}\delta_{ij}(w_r) f(w_r) =$$

$$= \underline{\sum_{(i,j)\in A} c_{ij} x^0_{ij} + c(w_1) f(w_1) + \dots + c(w_r) f(w_r)}$$

We can thus derive the following <u>optimality conditions</u> for the minimum cost flow problem:

Theorem (Negative Cycle Optimiality):
a feasible flow $x^*$ is an optimal solution for the minimum cost flow problem if and only if $G(x^*)$ contains no negative cost directed cycle.

## Algorithms for minimum cost flow

(Ahuja - Magnanti - Orlin : Chapter 9
(9.1, 9.3 ("Negative Cycle Optimality Conditions" and "Reduced Cost Optimiality Conditions"), 9.6, 9.7)

Notation: $z(x) = \sum_{(i,j)\in A} c_{ij} x_{ij}$ cost of flow $x$

Assumptions:
- integral data
- $\sum_{i\in N} b(i) = 0$ and feasibility
- $c_{ij} \geq 0 \quad \forall (i,j) \in A$

# Optimality Conditions

## 1 Negative Cycle Optimality Conditions

< already introduced >

Proof : to be studied

## 2 Reduced Cost Optimality Conditions

Let us associate $\pi(i) \in \mathbb{R}$ with each node $i$

potential of $i$

Given a potential vector $\pi$, let:

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$$ reduced cost of $(i,j)$

N.B.: reduced costs are defined also for the arcs in a ~~residual~~ network (using the residual cost instead of $c_{ij}$)

mistake in AMO

We already proved:

**Property:**

(a) for any directed path $P$ from $k$ to $\ell$, $\displaystyle\sum_{(i,j)\in P} c_{ij}^{\pi} = \sum_{(i,j)\in P} c_{ij} - \pi(k) + \pi(\ell)$

(b) for any directed cycle $W$,

$$\sum_{(i,j)\in W} c_{ij}^{\pi} = \sum_{(i,j)\in W} c_{ij}.$$

**Theorem** (Reduced Cost Optimality Conditions):
a feasible flow $x^*$ is an optimal solution to the minimum cost flow problem if and only if there exists a set of node potentials $\pi$ s.t.

$$c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x^*).$$

**Proof**

$\Leftarrow$: assume $x^*$ s.t. $c_{ij}^{\pi} \geq 0 \quad \forall (i,j)\in G(x^*)$.

Therefore $\displaystyle\sum_{(i,j)\in W} c_{ij}^{\pi} \geq 0 \quad \forall W$ directed cycle in $G(x^*)$

costs in $G(x^*)$ ("augmenting cycle")

From Property (b), $\displaystyle\sum_{(i,j)\in W} c_{ij} \geq 0 \quad \forall W$ cycle in $G(x^*)$

Therefore, $G(x^*)$ contains no negative cost directed cycle. From the "Negative Cycle Optimality Conditions" $x^*$ is an optimal flow!

$\Rightarrow$: Let $x^*$ optimal. Then $G(x^*)$ contains no negative cost directed cycle.

Compute a shortest path tree of root 1 in $G(x^*)$ (well-defined due to ) ), and $d(i)$ denote the shortest path label of node $i$.

From Bellman's optimality conditions

$$d(j) \leq d(i) + \underset{\text{cost in } G(x^*)}{c_{ij}} \quad \forall (i,j) \in G(x^*)$$

$$\Updownarrow$$

$$c_{ij} - (\underbrace{-d(i)}_{\pi(i)}) + (\underbrace{-d(j)}_{\pi(j)}) \geq 0$$

So $c^{\pi}_{ij} \geq 0 \quad \forall (i,j) \in G(x^*)$ if we set $\pi(i) = -d(i)$

In other words: $x^*$ satisfies the reduced cost optimality conditions. $\square$

Economic interpretation of reduced cost:

if . $c_{ij}$ : cost of transporting 1 unit of commodity from $i$ to $j$

. $\mu(i) = -\pi(i)$ cost of obtaining 1 unit of commodity at $i$

$$c_{ij} - \pi(i) + \pi(j) \geq 0$$

$$\Updownarrow$$

$$c_{ij} + \mu(i) - \mu(j) \geq 0$$

$$\Updownarrow$$

$$\mu(j) \leq c_{ij} + \mu(i)$$

The cost of obtaining 1 unit of commodity at node $j$ must be no more than the cost of obtaining the unit at $i$ plus the cost of sending it from $i$ to $j$.

# Cycle-canceling algorithm

(Ahuja - Magnanti - Orlin : 9.6 (until page 319, "Augmenting flow..." excluded))

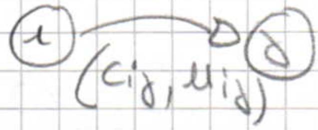
Given a feasible flow $x$ (e.g. via a maximum flow algorithm), at each iteration find a negative augmenting cycle $W$ in $G(x)$, and push $\delta = \min\{r_{ij} : (i,j) \in W\}$ along $W$, until no negative augmenting cycle exists (minimum cost flow)

- See Figure 9.7, page 317

How to find a negative cycle in $G(x)$?
via Bellman-Ford's shortest path algorithm ($\leftarrow$ course RO) : $O(m \cdot n)$

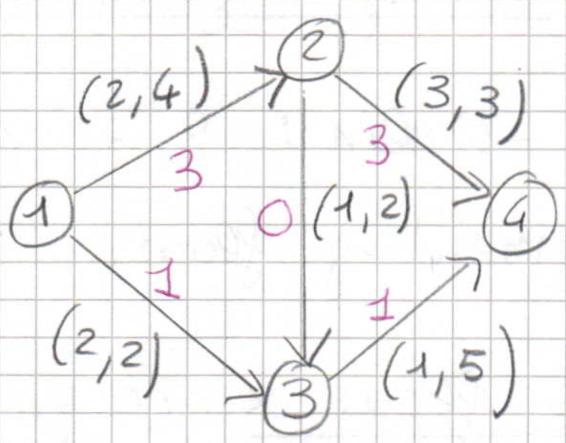
Obs : a feasible flow at each iteration

# Example



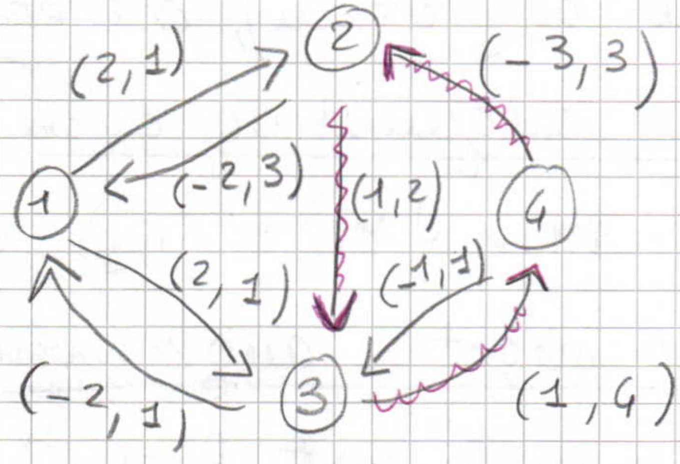

1)

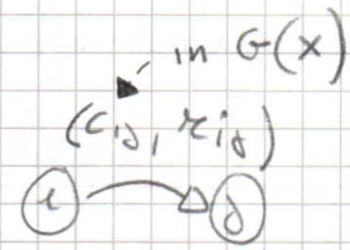
$\{x_{ij}\}$




$z(x) = cx = 18$

$G(x)$




A negative augmenting cycle:

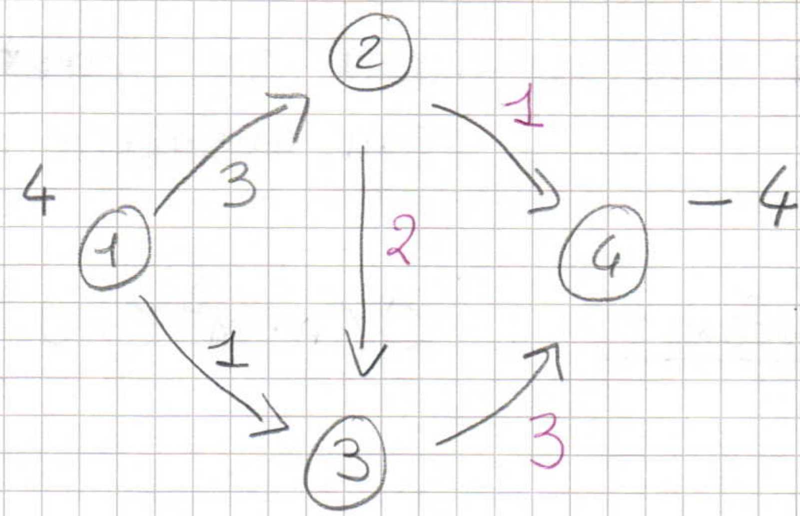
$$W = (2, 3, 4)$$

$$c(W) = -3 + 1 + 1 = -1$$

$$\delta = \min\{3, 2, 4\} = 2$$

< Augment $\delta$ along $W$ >

2) $\{x_{ij}\}$




$$z(x) = cx = 18 + \underset{c(W)}{\underset{\shortparallel}{(-1)}} \cdot \underset{\delta}{\underset{\shortparallel}{(2)}} = 16$$

⋮

As a by-product:

<u>Theorem</u> (<u>integrality</u>): if arc capacities and supplies/demands are integer, then there exists an integer minimum cost flow.

<u>Proof</u>

- maximum flow algorithm finds an integer initial flow $x$;
- at each iteration $r_{ij} \in \mathbb{Z}^+$, so $\delta$ is integer. $\square$

Let $C$ be the maximum arc cost.
(in absolute value)

Then :

Time complexity : $O(n \cdot m^2 C \cdot U)$

Proof

1) • $mCU$ : upper bound on the initial flow cost
$(c_{ij} \leq C \ , \ x_{ij} \leq U \quad \forall (i,j))$

• $-mCU$ : lower bound on the optimal flow cost
$(c_{ij} \geq -C \ , \ x_{ij} \leq U \quad \forall (i,j))$

due to integrality, at each iteration
$z(x)$ decreases by an integer $\geq 1$
$\Rightarrow \quad O(mCU)$ iterations

2) cost per iteration : $O(m \cdot n)$

1) & 2) $\Rightarrow O(n \cdot m^2 \cdot C \cdot U)$

$\square$

# Successive shortest path algorithm

(Ahuja - Magnanti - Orlin : 9.7 (until page 323, "The successive shortest path ..." excluded))

<In contrast to cycle-canceling algorithm, the successive shortest path algorithm maintains the reduced cost optimality, and strives to attain feasibility >

In fact, it maintains "pseudoflows":

Def : a pseudoflow $x : A \to \mathbb{R}^+ \cup \{0\}$ satisfies the capacity constraints; it may not satisfy the flow conservation constraints.

NB: a flow is a special pseudoflow

Then

$$\boxed{e(i) = b(i) + \sum_{(j,i) \in BS(i)} x_{ji} - \sum_{(i,j) \in FS(i)} x_{ij} \quad \forall i \in N}$$

imbalance of $i$

If:

$e(i) > 0$ __excess node__ (must send flow)

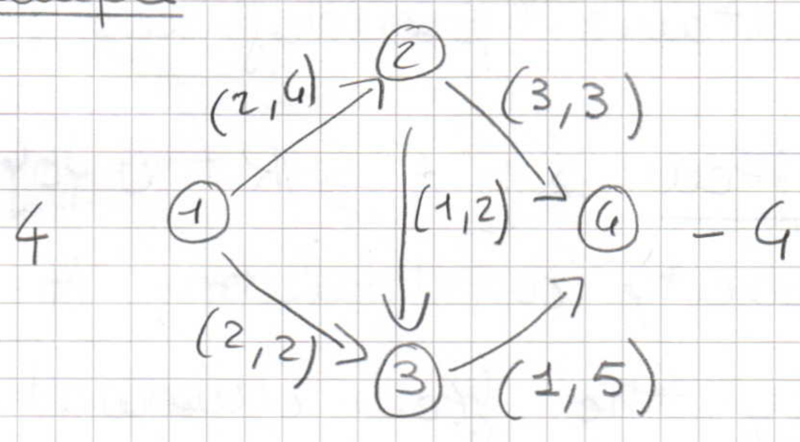
$e(i) < 0$ __deficit node__ (must receive flow)

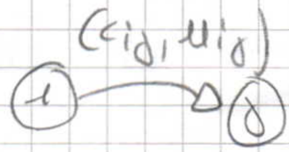
$e(i) = 0$ __balanced__

If $E$ set of excess nodes and $D$ set of deficit nodes, then:

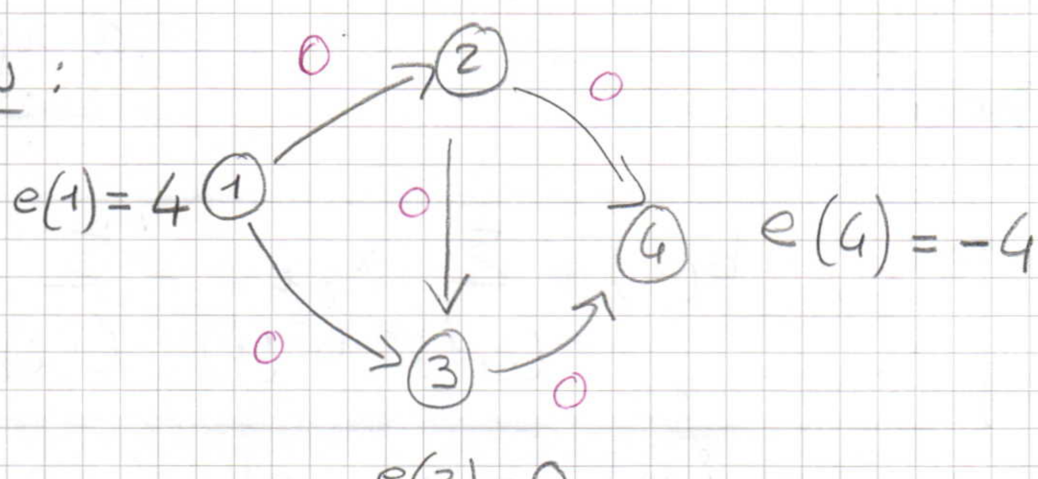
$$\sum_{i \in N} e(i) = \sum_{i \in N} b(i) = 0 \Rightarrow \sum_{i \in E} e(i) = -\sum_{i \in D} e(i)$$

__example__




__a pseudoflow__:




$E = \{1\}$

$D = \{4\}$

Define the residual network $G(x)$ as before. 

<u>Lemma</u> : Suppose a pseudoflow $x$ satisfies the reduced cost optimality w.r.t. some potentials $\{\pi(i)\}$. Let $d(i)$ the shortest path distance from some node $s$ to $i$ in $G(x)$ w.r.t. $c_{ij}^{\pi}$. Then:

(a) $x$ also satisfies the reduced cost optimality w.r.t. $\{\pi'(i)\}$, with (i.e. $c_{ij}^{\pi'} \geq 0$ $\forall (i,j)$)

$$\pi'(i) = \pi(i) - d(i) \quad \forall i \in N$$

(b) $c_{ij}^{\pi'} = 0$ if $(i,j)$ belongs to a shortest path (w.r.t. $c_{ij}^{\pi}$) from $s$ to some other node.

<u>Proof</u> :

(a) • $c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x)$ (by hyp.)

$\Rightarrow$ • $d(j) \leq d(i) + c_{ij}^{\pi} \quad \forall (i,j) \in G(x)$

$(d(s) = 0)$ shortest path optimality conditions

Remember that:

$$c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j)$$ ; by substituting in $\Rightarrow$ we get:

$$d(j) \leq d(i) + c_{ij} - \pi(i) + \pi(j) \qquad \forall (i,j) \in G(x)$$

$$c_{ij} - \underbrace{(\pi(i) - d(i))}_{\pi'(i)} + \underbrace{(\pi(j) - d(j))}_{\pi'(j)} \geq 0 \qquad \text{"} \quad \text{"}$$

i.e. $c^{\pi'}_{ij} \geq 0 \quad \forall (i,j) \in G(x)$.

(b) Consider a shortest path from s to some node K:

$$P \qquad s \rightsquigarrow i \xrightarrow{c^{\pi}_{ij}} j \rightsquigarrow K$$

it is $d(j) = d(i) + c^{\pi}_{ij} \qquad \forall (i,j) \in P$

Since $c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j)$, we get

$$c_{ij} - \underbrace{(\pi(i) - d(i))}_{\pi'(i)} + \underbrace{(\pi(j) - d(j))}_{\pi'(j)} = 0$$

$$\Updownarrow$$

$$c^{\pi'}_{ij} = 0$$

$\square$

<u>Lemma</u>: Suppose a pseudoflow $x$ satisfies the reduced cost optimality conditions. If $x'$ is obtained from $x$ by sending flow along a shortest path (w.r.t. $c^{\pi}_{ij}$) from some node $s$ to some node $k$, then $x'$ also satisfies the reduced cost optimality conditions.

<u>Proof</u>: Let $\{\pi(i)\}$ and $\{\pi'(i)\}$ as in the previous lemma.

$$x: \quad \left.\begin{array}{ll} c^{\pi'}_{ij} \geq 0 & \forall (i,j) \in G(x) \\ c^{\pi'}_{ij} = 0 & \forall (i,j) \text{ in the shortest path tree rooted at } s \end{array}\right\} \text{previous lemma}$$

send $\delta$ to obtain $x'$:

$$s \rightsquigarrow \circ \rightsquigarrow i \longrightarrow \underset{c^{\pi'}_{ij}=0}{} j \rightsquigarrow \circ \; k$$

The augmentation may add $(j,i)$ to $G(x')$ : $c^{\pi'}_{ji} = 0$, so also this arc satisfies the reduced cost optimality! $\square$
(w.r.t. $\{\pi'(i)\}$)

# Successive shortest path

$x := 0$, $\pi := 0$; (NB: $c^{\pi}_{ij} \geq 0$ $\forall (i,j)$)

$e(i) := b(i)$ $\forall i \in N$;

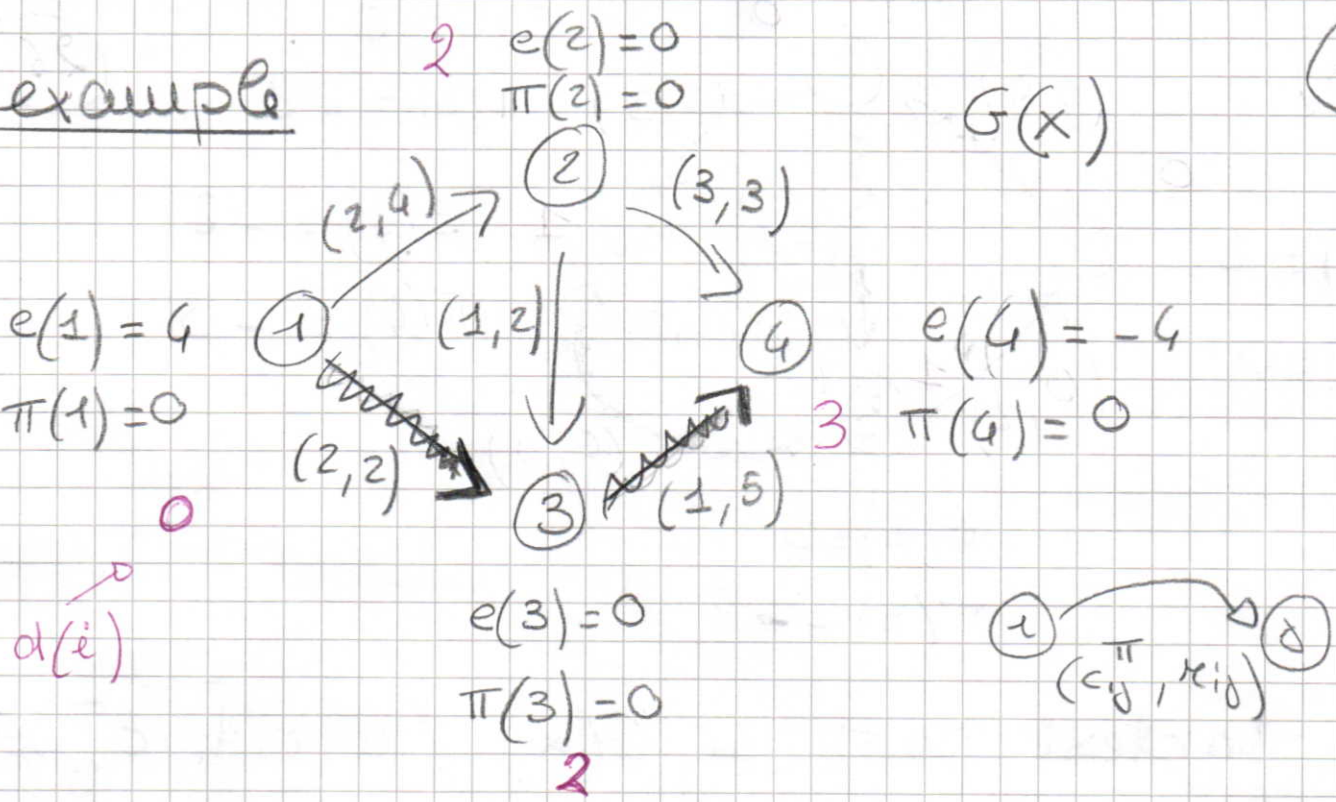
$E = \{ i : e(i) > 0 \}$; $D = \{ i : e(i) < 0 \}$;

__while__ $E \neq \emptyset$ __do__

- select $s \in E$ and $k \in D$;
- find the shortest path tree of root $s$ in $G(x)$ w.r.t. $\{ c^{\pi}_{ij} \}$; ← NB: $c^{\pi}_{ij} \geq 0$
- let $d(i)$ be the shortest distance of $i$, $\forall \in N$, and $P$ the shortest path from $s$ to $k$;
- $\pi(i) := \pi(i) - d(i)$, $\forall i \in N$; ← $\pi'(i)$ in the lemmas
- $\delta = \min \{ e(s), - e(k), \min \{ r_{ij} : (i,j) \in P \} \}$;
- send $\delta$ units of flow along $P$;
- update $x$, $G(x)$, $E$, $D$ and the reduced costs (now $\{ c^{\pi'}_{ij} \}$)

__end__

## example

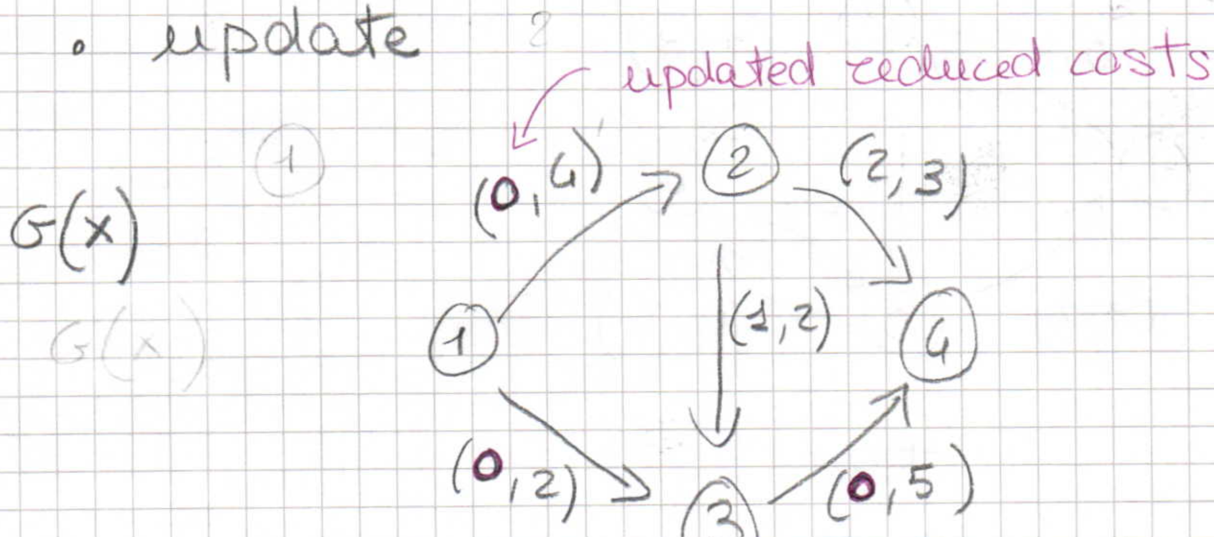



1) • shortest path in $G(x)$ (w.r.t. $c^{\pi}_{ij} \geq 0$) from $1 \in E$ to $4 \in D$: $(1, 3, 4)$

• $\pi(i) := \pi(i) - d(i) \quad \forall i \quad (0, -2, -2, -3)$

• $\delta = \min\{4, 4, \min\{2, 5\}\} = 2$

• update




+

sending of 2 units along $(1, 3, 4)$

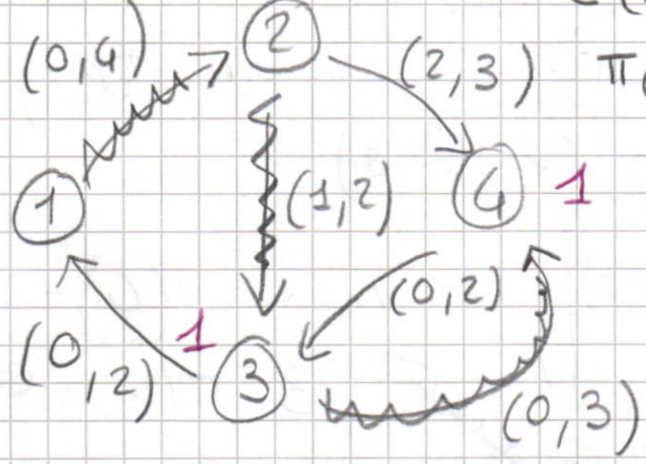
$G(x)$

$e(2)=0$

$\pi(2) = -2$

$e(1) = 2$

$\pi(1) = 0$




$e(4) = -2$

$\pi(4) = -3$

$e(3)=0$

$\pi(3) = -2$

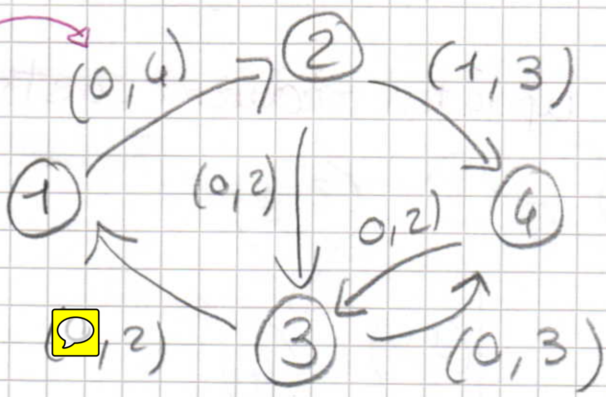
2) shortest path in $G(x)$ (w.r.t. $c_{ij}^{\pi} \geq 0$) from $1 \in E$ to $4 \in D$: (1 2 3 4)

- $\pi(i) := \pi(i) - d(i) \quad \forall i$

  $(0, -2, -3, -4)$

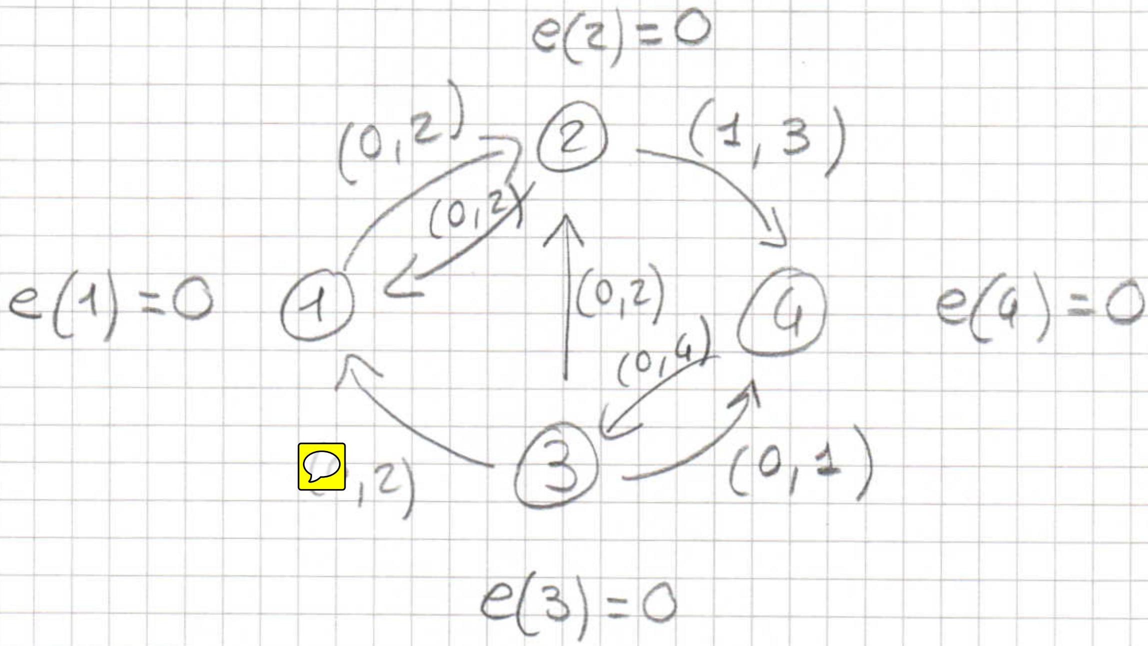
- $\delta = \min \{2, 2, \min\{4, 2, 3\}\} = 2$

updated reduced costs

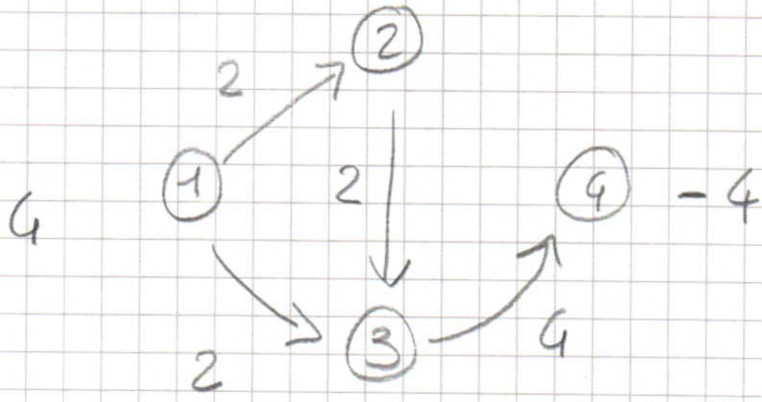



+

sending of 2 units along (1 2 3 4)




STOP : The pseudoflow is a feasible (and optimal) flow




Time Complexity : $O(nU \cdot S(n,m))$

Proof

- $\leq nU$ iterations
  (the excess of some nodes strictly decreases at each iteration)
- $S(n,m)$ time to compute a shortest path tree
  (e.g. $O(n^2) \leftarrow$ Dijkstra)