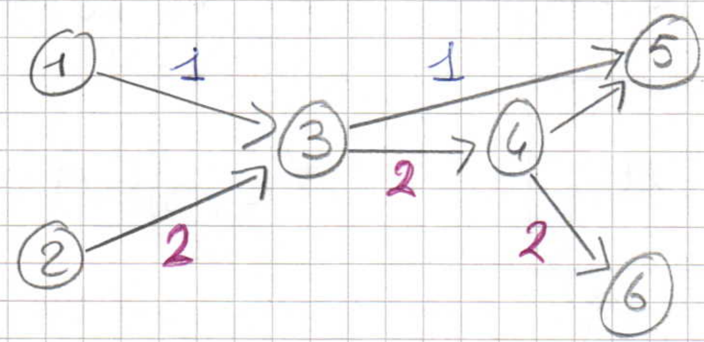
Multicommodity flows

(Ahuja - Magnanti - Orlin : 17.1)

In many application contexts, several commodities share network resources (common arc capacities)

example



$u_{ij} = 2$
$\forall (i,j) \in A$

commodity 1 : send one message from 1 to 5

commodity 2 : send two messages from 2 to 6

Obs: commodity 1 can not use the path (1 3 4 5), due to commodity 2 (which must use arc (3,4), of capacity 2)

General LP model:

x_{ij}^{k} : amount of flow pushed along (i,j) for commodity k

(MCF1) $$\text{Min} \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k}$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^{k} - \sum_{(j,i)\in BS(i)} x_{ji}^{k} = b_i^{k} \qquad \forall i \in N,\ k = 1, \dots K$$

$$0 \leq x_{ij}^{k} \leq u_{ij}^{k} \qquad \forall (i,j)\in A,\ k = 1, \dots K$$

$$\boxed{\sum_{k=1}^{K} x_{ij}^{k} \leq u_{ij}} \qquad \forall (i,j)\in A$$

Bundle constraints

common arc capacities

where:

K : number of commodities

u_{ij}^{k} : individual flow bounds (may be $+\infty$)

Assumptions:

1) Linear costs

This is not a proper assumption e.g. in traffic networks, where often the cost function is nonlinear, e.g. to model "congestion":

$$\text{e.g.} \quad \min \sum_{(i,j)\in A} \frac{x_{ij}}{(u_{ij} - x_{ij})}$$

where $x_{ij} = \sum_{k=1}^{K} x_{ij}^k$

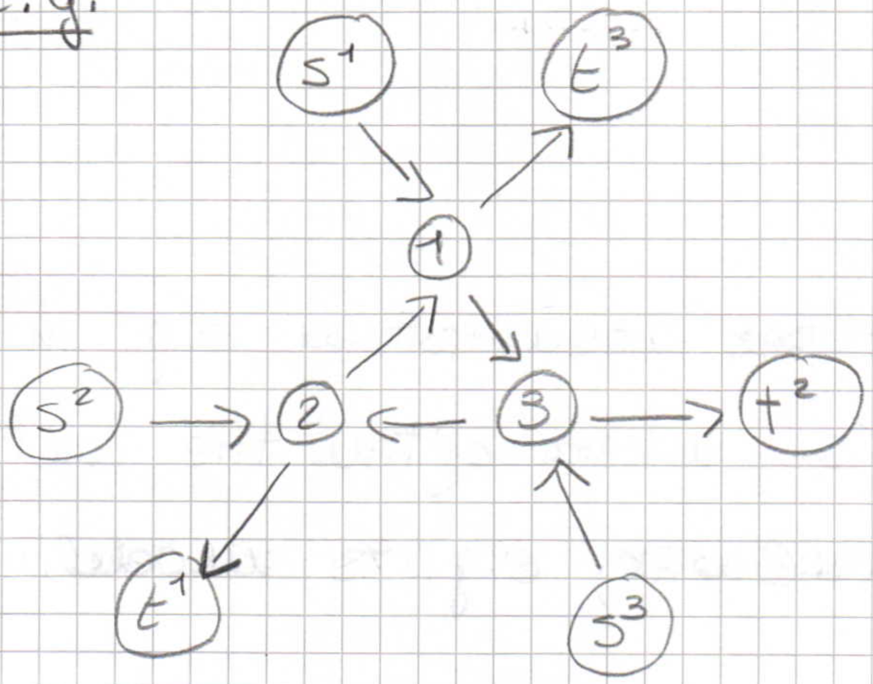
* when the flow approaches the arc capacity, then the "delay" goes to $+\infty$ .... "

2) the model assumes that $\{x_{ij}^k\}$ can be fractional

This is not appropriate in several application contexts; however:

* Multicommodity flows do not satisfy the integrality property *

e.g.



$u_{ij} = 1$
$\forall (i,j)$

Problem: maximize the total flow from s^1 to t^1, from s^2 to t^2, from s^3 to t^3

Optimal (fractional) solution: each source sends 0,5 to its destination

1.5 units

Optimal (integer) solution: just one of the three commodities sends 1 unit of flow

1 unit !!

Solution approaches

1. Price - directive decomposition
2. Resource - directive decomposition

⋮

< in the second part of the course >

... although the problem is solvable via LP solver (if its size allows this...)

Formulation with path flows

(Ahuja - Magnanti - Orlin ; pages 665 - 666)

Assumptions (for simplicity)

- single source s^k
- single destination t^k
- no individual bounds (i.e. u_{ij}^k $+\infty$)
- d^k amount of flow from s^k to t^k

for each commodity k

Further assumption: $c_{ij}^{k} \geq 0 \quad \forall (i,j) \in A$
$k = 1, \ldots K$

Consequence 1: the cost of each directed cycle is ≥ 0; therefore, there exist minimum cost multicommodity flows where the flow on every cycle is 0.

Consequence 2: from the flow decomposition theorem, we can assume that the flow of each commodity k decomposes into flows on paths from s^k to t^k (no cycle flows!)

- Let P^k set of directed paths from s^k to t^k, $\forall k$
- $\delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \text{ is in } P \\ 0 & \text{otherwise} \end{cases}$

$\forall (i,j) \in A, \; \forall P \in P^k$

Introduce:

$f(P)$: flow along P (variable)

Then

$$\boxed{x_{ij}^k = \sum_{P \in P^k} \delta_{ij}(P)\, f(P)} \qquad \forall (i,j) \in A,\ k = 1, \dots K$$

If we replace $\{x_{ij}^k\}$ in formulation (MCF1), we get the equivalent path flow formulation:

$$\text{(MCF2)} \quad \text{Min} \sum_{k=1}^{K} \underbrace{\sum_{(i,j)\in A} c_{ij}^k \Big[\overbrace{\sum_{P\in P^k} \delta_{ij}(P)\, f(P)}^{x_{ij}^k} \Big]}_{= \sum_{P \in P^k} c^k(P) \cdot f(P)}$$

$c^k(P)$ ← cost of P w.r.t commodity k

Link-path formulation v.s. Node-link formulation

flow decomposition Theorem

$$\sum_{P \in P^k} f(P) = d^k, \qquad k = 1, \dots K$$

$$\sum_{k=1}^{K} \overbrace{\sum_{P \in P^k} \delta_{ij}(P) \cdot f(P)}^{x_{ij}^k} \le u_{ij} \qquad \forall (i,j) \in A$$

$$f(P) \geq 0 \qquad \forall P \in P^k, \ k = 1, \dots K$$

Obs :

- (MCF 2) has $m + K$ constraints (plus nonnegativity constraints)
- (MCF 1) has $m + nK$ constraints (plus nonnegativity)

However : (MCF2) has one variable per path (maybe exponential...)

How can we manage this ?

< Column generation approach >
(second part of the course)

* (MCF 2) is very used for modelling real network flow optimization problems ... see some examples in the following ... *