Optimization methods to multicommodity flows

(Ahuja - Magnanti - Orlin : 17.4 - 17.5 - 17.6 - 17.7)

1 Lagrangian relaxation

(Ahuja - Magnanti - Orlin : 17.4)

Recall the general LP model:

(MCF 1)
$$z = \mathrm{Min} \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij}^k x_{ij}^k$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^k - \sum_{(j,i)\in BS(i)} x_{ji}^k = b_i^k \qquad \forall i \in N,\ k = 1, \dots K$$

$$\boxed{\sum_{k=1}^{K} x_{ij}^k \leq u_{ij} \qquad \forall (i,j)\in A} \qquad \pi_{ij}$$

$$x_{ij}^k \geq 0 \qquad \forall (i,j)\in A,\ k = 1, \dots K$$

where:

K : number of commodities

x_{ij}^k : amount of flow pushed along (i,j) for commodity k

To apply Lagrangian relaxation, associate a nonnegative Lagrangian multiplier π_{ij} with each bundle constraint:

$$z(\pi) = \min \sum_{k=1}^{K} \sum_{(i,j)\in A} (c_{ij}^k + \pi_{ij}) x_{ij}^k - \sum_{(i,j)\in A} \pi_{ij} u_{ij}$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^k - \sum_{(j,i)\in BS(i)} x_{ji}^k = b_i^k \quad \forall i \in N, \; k = 1, \dots K$$

$$x_{ij}^k \geq 0 \quad \forall (i,j)\in A, \; k = 1, \dots K$$

For given $\{\pi_{ij}\}$ the term $\sum_{(i,j)\in A} \pi_{ij} u_{ij}$ is constant. Therefore, the resulting problem decomposes into separate minimum cost flow problems, one for each commodity, with costs $\{c_{ij}^k + \pi_{ij}\}$.

Note that, since (MCF 1) is a linear program, if we solve the corresponding Lagrangian Dual we get

$$Z = \max_{\pi : \pi_{ij} \geq 0} z(\pi)$$

For example, if we use the Subgradient algorithm to solve the Lagrangian Dual, then at each step we can update the Lagrangian multipliers according to:

$$\pi_{ij}^{t+1} = \max\left\{ \pi_{ij}^{t} + \underset{\substack{\uparrow \\ \text{step size}}}{\lambda_t} \left(\sum_{k=1}^{K} x_{ij}^{*k} - u_{ij} \right), 0 \right\},$$

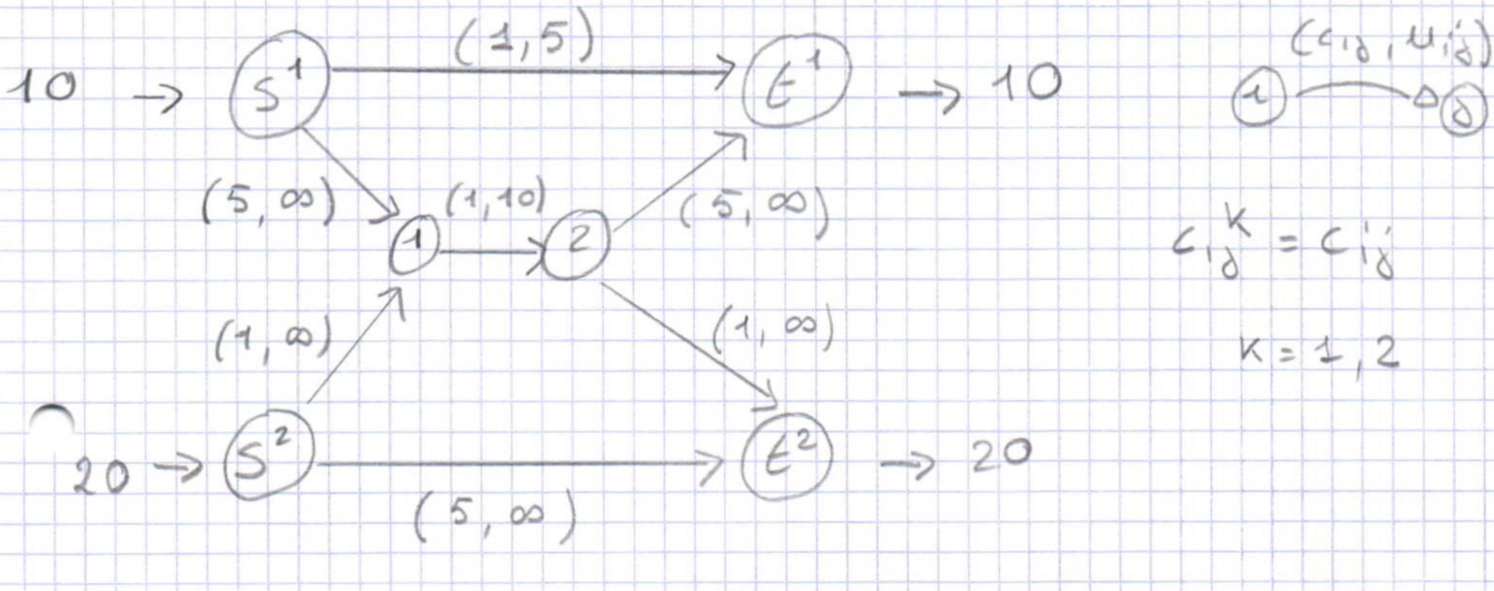
where:

- t current iteration
- $\{x_{ij}^{*k}\}$ optimal solution for commodity k

Recall, in fact, that $\left(\sum_{k=1}^{K} x_{ij}^{*k} - u_{ij} \right)$ is a subgradient of the Lagrangian function at $\{\pi_{ij}^{t}\}$.

659

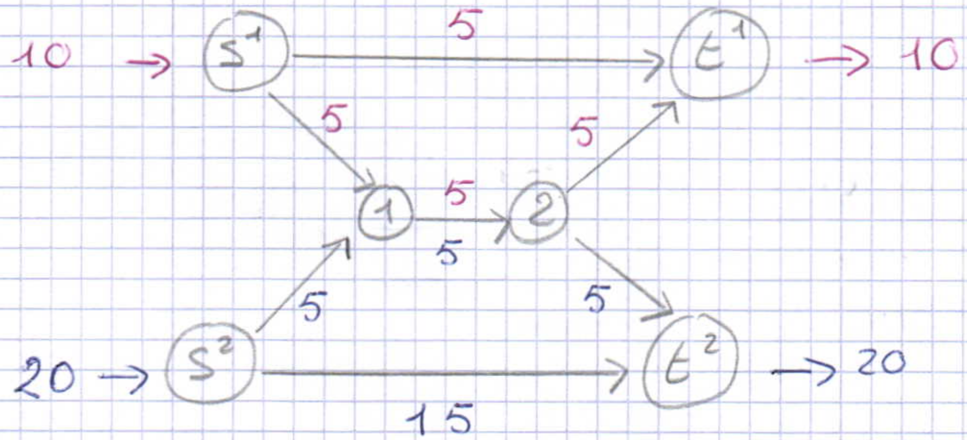
<u>example</u> (pages 661-662-663)



$c_{ij}^{k} = c_{ij}$

$k = 1, 2$

The optimal solution is



$z = 150$

- in this case, at each step we solve 2 shortest path problems
- $\lambda_t = \frac{1}{t}$ $\forall t$ (theory of subgradient optimization)

- From iteration 14 on, the values of the Lagrangian multipliers oscillate about, and converge to, their optimal values

- and $z(\pi)$ oscillates about its maximum value 150, i.e. z.

Advantages:

- shortest path (minimum cost flow) computations at each step
- easy to implement

Limitations:

- the subgradient method does not converge fast (small step sizes)
- the optimal solutions $\{x_{ij}^{*k}\}$ of the Lagrangian subproblems do not necessarily converge to an optimal solution of (MCF1): the bundle constraints may not be satisfied!

<u>example</u> (cont) optimal $\{x_{ij}^{*k}\}$

$$s^1 \xrightarrow[u_{s^1t^1} = 5]{10} t^1$$

$$s^2 \xrightarrow[20]{} t^2$$

2 Column generation

Recall the path flow formulation:

$$(MCF\,2)\quad \text{Min} \sum_{k=1}^{K} \sum_{p \in P^k} c^k(p) \cdot f(p)$$

$$(\sigma^k) \quad \sum_{p \in P^k} f(p) = d^k \quad , \; k = 1, .. K$$

$$(w_{ij}) \quad \sum_{k=1}^{K} \sum_{p \in P^k} \delta_{ij}(p) \cdot f(p) \leq u_{ij} \quad \forall (i,j) \in A$$

$$f(p) \geq 0 \qquad \forall p \in P^k, \; k = 1, .. K$$

where :

- P^k set of directed paths from s^k to t^k, for each commodity k
- d^k amount of flow to push from s^k to t^k, for each commodity k
- $\delta_{ij}(p) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases} \quad \forall (i,j) \in A, \; p \in P^k.$

< link-path formulation >

- (MCF2) is a LP with an exponential number of variables: how to deal with them?

- Consider the dual of (MCF2):

(DMFC2) $$\text{Max} \sum_{k=1}^{K} d^k \cdot \sigma^k - \sum_{(i,j)\in A} u_{ij} \cdot w_{ij}$$

$$\boxed{\sigma^k - \sum_{(i,j)\in A} \delta_{ij}(p)\, w_{ij} \leq c^k(p)} \quad \forall p \in P^k,\ k=1,\dots K$$

$$w_{ij} \geq 0 \qquad \forall (i,j) \in A$$

rewrite as

$$\boxed{c^k(p) + \sum_{(i,j)\in p} w_{ij} - \sigma^k \geq 0} \quad \forall p \in P^k,\ k=1,\dots K$$

reduced cost of p

Since $c^k(p) = \sum_{(i,j)\in p} c_{ij}^k$, we can write the reduced cost of p as:

$$\boxed{c_p^{\sigma,w} = \sum_{(i,j)\in p} (c_{ij}^k + w_{ij}) - \sigma^k}$$

That is: it is the cost of p w.r.t. modified costs $(c_{ij}^k + w_{ij})$, minus a value, σ^k, which depends on the commodity.

Given a feasible solution to (MCF2), $\{f(p)\}$, this is optimal if and only if there are "arc prices" $w_{ij} \geq 0$, $\forall (i,j) \in A$, such that:

- $c_p^{G,w} \geq 0 \quad \forall p \in P^k, \forall k$ <u>dual feasibility</u>
- $w_{ij}\left(u_{ij} - \sum_{k=1}^{K} \sum_{p \in P^k} \delta_{ij}(p) f(p)\right) = 0 \quad \forall (i,j)$
- $f(p) \cdot c_p^{G,w} = 0 \quad \forall p \in P^k, \forall k$

(c.s.c.) complementary slackness conditions (LP theory)

<u>Key idea in column generation</u>: instead of listing explicitly all columns ($\equiv f(p)$) in (MCF2), use a subset, say corresponding to the paths $\tilde{P}^1, \tilde{P}^2, \ldots, \tilde{P}^K$, and generate the remaining columns only "as needed".

- So, consider (MCF2) restricted to $\tilde{P}^1, \ldots, \tilde{P}^K$, and find an optimal solution $f^*(p)$ via a Simplex algorithm: note that by setting $f(p)=0 \ \forall p \notin (\tilde{P}^1 \cup \tilde{P}^2 \ldots \cup \tilde{P}^K)$, then we have a feasible solution to (MCF2)

- Let $\{w^*_{ij}\}, \{\sigma^{*k}\}$ the optimal solution of the dual of the restricted (MCF2):

$$\mathrm{Max} \sum_{k=1}^{K} d^k \sigma^k - \sum_{(i,j)\in A} w_{ij} u_{ij}$$

$c_p^{\sigma^*, w^*} \geq 0$ $\quad c_p^{\sigma, w} \geq 0 \quad \forall p \in \underline{\tilde{P}^k}, \ k=1, \ldots K$

$w^*_{ij} \geq 0$ $\quad w_{ij} \geq 0 \quad \forall (i,j) \in A$

Since they are optimal, the <u>C.S.C.</u> are satisfied:

- $w^*_{ij}\left(u_{ij} - \sum_{k=1}^{K} \sum_{p\in\tilde{P}^k} \delta_{ij}(p) f^*(p)\right) = 0$

- $f^*(p)\, c_p^{\sigma^*, w^*} = 0 \quad \forall p \in \tilde{P}^k, \forall k$

Since $f(p)=0 \ \forall p \notin (\tilde{P}^1 \cup \tilde{P}^2 \cup \ldots \cup \tilde{P}^K)$, the

C.S.C are always satisfied.

<u>Therefore</u> : $\{f^*(p)\}$ extended with 0 $\forall p \notin (\hat{P}^1 \cup \ldots \cup \hat{P}^K)$ is optimal to (MCF2) if $\boxed{c_p^{\sigma^*,w^*} \geq 0 \quad \forall p}$.

* How can we check this condition without listing explicitly all p? *

$$c_p^{\sigma^*,w^*} \geq 0 \qquad \forall p \in P^k, \; k = 1, \ldots K$$

is equivalent to :

$$\sum_{(i,j)\in p} (c_{ij}^k + w_{ij}^*) - \sigma^{*k} \geq 0 \quad \forall p \in P^k, \; k = 1, \ldots K,$$

which in turn is equivalent to :

$$\min_{p \in P^k} \sum_{(i,j)\in p} (c_{ij}^k + w_{ij}^*) \geq \sigma^{*k} \qquad k = 1, \ldots K$$

$\forall k$: shortest path from s^k to t^k w.r.t. the modified costs $(c_{ij}^k + w_{ij}^*)$

Therefore:

- compute the K shortest paths (price-out)
- if the shortest path cost is $\geq \sigma^{*k}$ for each commodity k, then STOP: $\{f^*(p)\}$ extended with 0 for each p not considered is optimal to (MCF2)
- otherwise, i.e. for some k the shortest path q has a cost $< \sigma^{*k}$ (i.e. $\sigma_q^{\sigma^*, w^*} < 0$), then add q to $\tilde{P}^k$ and iterate:

column generation

Note that we are adding a column to (MCF2) and therefore a row to (DMCF2):

< cutting plane algorithm applied to (DMCF2): the separation problem consists in shortest path computations >

Interpretation in terms of Dantzig-Wolfe decomposition

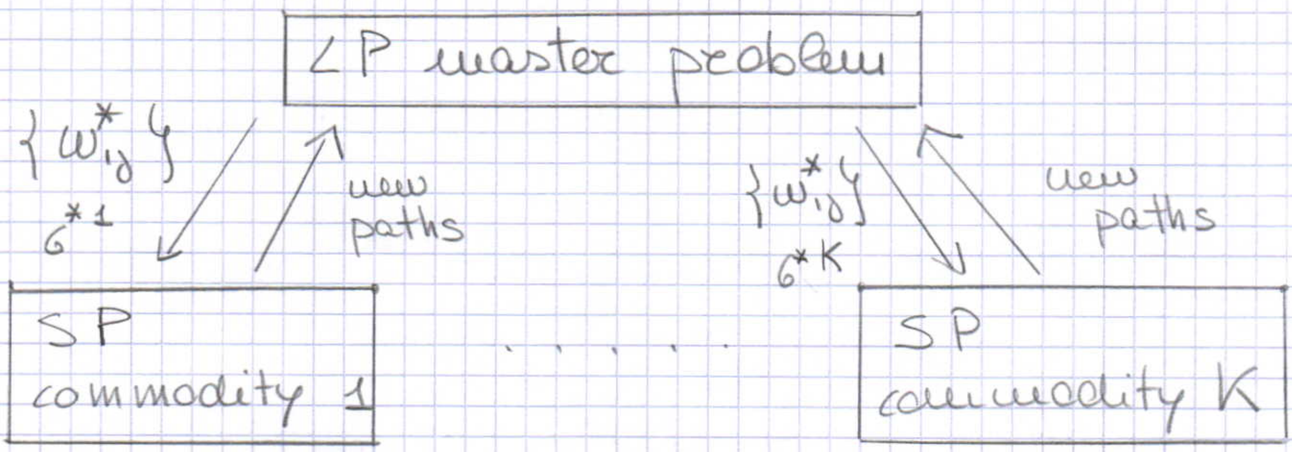
- coordinator or restricted master problem:

  < smaller LP ≡ (MCF2) with a subset of columns >

- K decision makers:

  < K shortest path problems >

The K decision makers generate new columns guided by the coordinator:



Advantages :

- save old columns (generating new columns is time consuming), and/or add a packet of columns (flexibility)

- since the shortest path problems are independent of each other, they can be solved in parallel (parallel computation)

Observation : the K subproblems correspond to a Lagrangian problem with Lagrangian multipliers $\pi_{ij} = w_{ij}^* \ \forall (i,j) \in A$: in fact, we are solving the Lagrangian Dual (whose optimal value is the optimal value of (MCF2), since they are LPs). lower bound to z

Advantages : at each step of the column generation we have both a lower bound as well as an upper bound, and therefore an optimality gap : we can stop the algorithm when the optimality gap is $\leq \varepsilon$, for a given $\varepsilon > 0$.

3 Resource-directive decomposition

- Previous approaches are price-directive: they decompose the multicommodity flow problem into singlecommodity flows (e.g. shortest paths), placing "prices" (π_{ij} or w_{ij}^*) on the bundle constraints

- Alternative approach: allocate the joint bundle capacity of each arc (u_{ij}) to the single commodities, i.e. resource-directive decomposition

- Consider (MCF 1) with the individual capacities $\{u_{ij}^k\}$ in addition to the global ones $\{u_{ij}\}$

- Let $r_{ij}^k \leq u_{ij}^k$ be the units of u_{ij} allocated to commodity k, $\forall (i,j) \in A$, $k = 1, \ldots K$

Then (MCF 1) can be restated as:

$$z = \mathrm{Min} \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k}$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^{k} - \sum_{(j,i)\in BS(i)} x_{ji}^{k} = b_i^{k} \quad \forall i \in N, \; k = 1, .. K$$

$$0 \leq x_{ij}^{k} \leq r_{ij}^{k} \quad \forall (i,j)\in A, \; k = 1, .. K$$

$$\sum_{k=1}^{K} r_{ij}^{k} \leq u_{ij} \quad \forall (i,j) \in A$$

$$0 \leq r_{ij}^{k} \leq u_{ij}^{k} \quad \forall (i,j)\in A, \; k = 1, .. K$$

<u>resource-directive problem</u>

In order to solve it, first fix $\{r_{ij}^{k}\} = r$ and then determine $\{x_{ij}^{k}\}$ according to:

$$z = \mathrm{Min}\; z(r)$$

$$\sum_{k=1}^{K} r_{ij}^{k} \leq u_{ij} \quad \forall (i,j)\in A$$

$$0 \leq r_{ij}^{k} \leq u_{ij}^{k} \quad \forall (i,j) \in A, \; k = 1, .. K$$

<u>resource-allocation problem</u>

where $z(r) = \sum_{k=1}^{K} z^{k}(r^{k})$, and

$$z^k(r^k) = \min \sum_{(i,j)\in A} c_{ij}^k x_{ij}^k$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^k - \sum_{(j,i)\in BS(i)} x_{ji}^k = b_i^k \quad \forall i \in N$$

$$0 \leq x_{ij}^k \leq r_{ij}^k \quad \forall (i,j)\in A$$

K minimum cost flow (e.g. shortest path) subproblems for each given $\{r_{ij}^k\}$!

Therefore : resource - directive decomposition allows one to reformulate (MCF 1) as a "Knapsack - like" problem, i.e. the resource - allocation problem, with a complex objective function $z(r)$.

$z(r)$, however, is easy to evaluate :

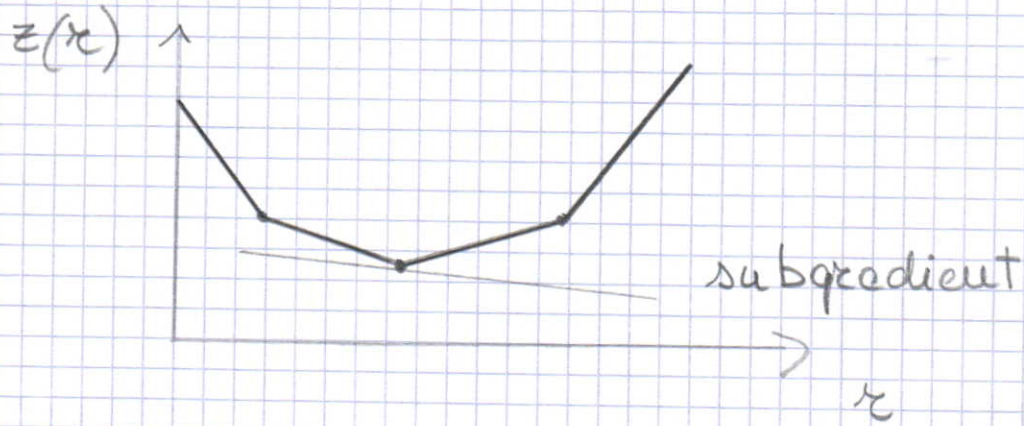
via K minimum cost flow problems ...

< How to exploit this from an algorithmic point of view ? >

Property : $z(r)$ is a piecewise linear convex function of r.

No proof

example when r is a singleton :



Therefore : "resource-directive" decomposition is equivalent to minimize the piecewise linear convex function $z(r)$ on $R = \{ r : \sum_{k=1}^{K} r_{ij}^k \le u_{ij}, \forall (i,j) \in A, \ 0 \le r_{ij}^k \le u_{ij}^k, \forall (i,j) \in A, k = 1, \dots K \}$

- the evaluation of function $z(r)$ for given r involves K minimum cost flow computations.

Approaches :

1) Heuristic (local improvement) : given the current $\bar{r}$, choose two commodities, say k' and k'', and an arc (i,j), and:

- $r_{ij}^{k'} := \bar{r}_{ij}^{k'} + \theta$
- $r_{ij}^{k''} := \bar{r}_{ij}^{k''} - \theta$

($r_{ij}^{\tilde{k}}$ unchanged for $\tilde{k} \neq k', k''$)

- perform the move which gives the greatest decrease of $z(r)$

advantage : very easy to implement

limitation : it is a heuristic, so no guarantee to converge to an optimal solution.

2) Borrowing ideas from subgradient optimization, minimize $z(r)$ by choosing as a movement direction a subgradient γ of $z(r)$ at the current $r = \bar{r}$, i.e.:

$$z(r) \geq z(\bar{r}) + \gamma(r - \bar{r}) \quad \forall r \in R$$

- If $\bar{r} = (\bar{r}^1, \ldots \bar{r}^K)$, it is possible to prove that, if $\bar{\gamma}^1, \ldots \bar{\gamma}^K$ are subgradients of $z^k(r^k)$ at $\bar{r}^k$, $k = 1, \ldots K$, then $\bar{\gamma} = (\bar{\gamma}^1, \ldots \bar{\gamma}^K)$ is a subgradient of $z(r)$ at $\bar{r} = (\bar{r}^1, \ldots \bar{r}^K)$.

- So, decomposition also at the level of subgradient computation!

<u>Property</u>: $\bar{\gamma}^1, \bar{\gamma}^2, \ldots \bar{\gamma}^K$ can be found as a by-product of the K minimum cost flow computations (to evaluate $z^k(\bar{r}^k)$, $k = 1, \ldots K$).

<u>No proof</u>