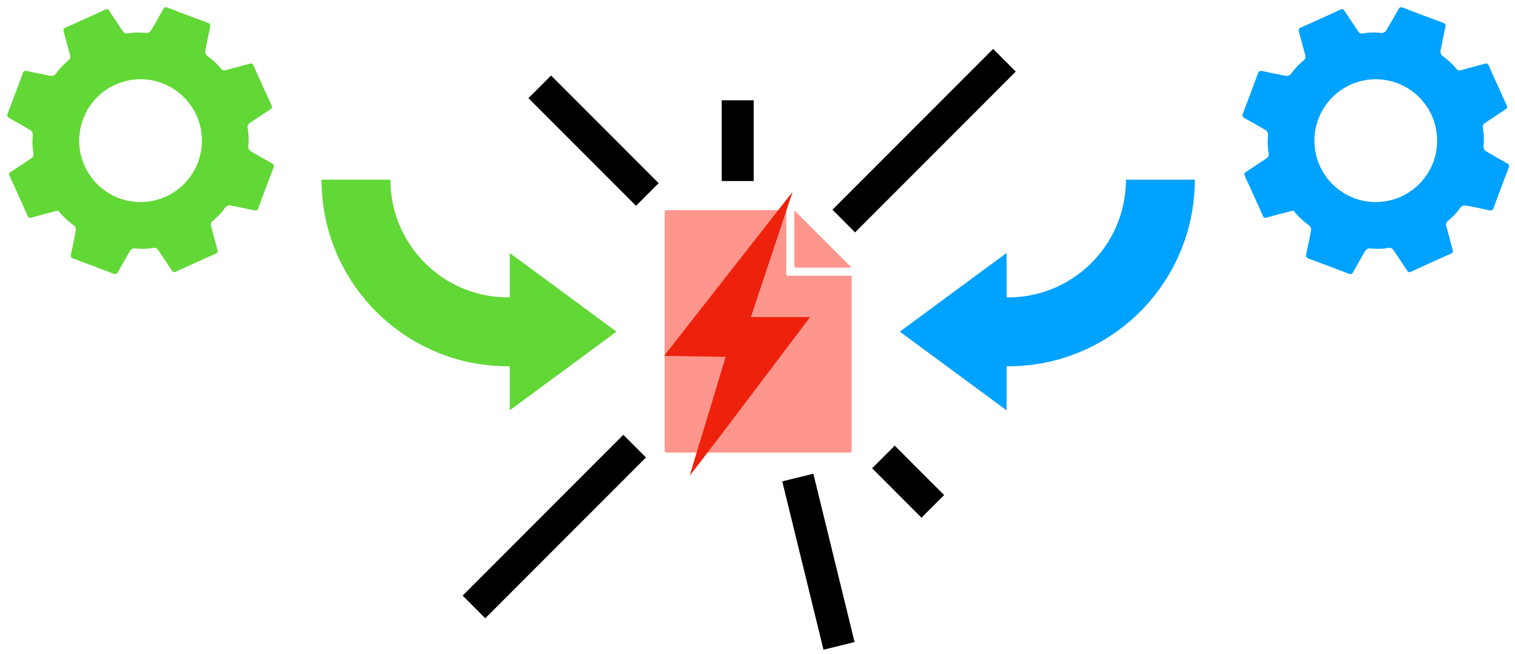
Linguaggi di Programmazione



Roberta Gori

HOFL Sintassi e Tipi - 7.1

HOFL pre-termini

(linguaggio funzionale di ordine superiore)

Sintassi HOFL

Attenzione : t=0 e' true
t!=0 e' false !!!

$n \in \mathbb{Z}$

op $\in \{+, -, \times\}$

$\textbf{if } t = 0 \textbf{ then } t_0 \textbf{ else } t_1$

$x \in Ide$

$$t \quad ::= \quad x \mid n \mid t_0 \ \mathsf{op} \ t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1$$ standard

$$\mid \quad (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t)$$ coppie proiezioni

$$\mid \quad \lambda x.\ t \mid t_0\ t_1$$ abstrazione applicazione

$$\mid \quad \textbf{rec } x.\ t$$ ricorsione

let $x = t$ in t

can contain x

Esercizio

$$\textbf{rec } f.\ \lambda x.\ \textbf{if } x \textbf{ then } 1 \textbf{ else } x \times (f\ (x-1))$$

quale e’ il significato del precedente pre-termine?

Definisce la funzione fattoriale

Esercizio

$$\mathbf{rec}\ rep.\ \lambda n.\ \lambda f.\ \lambda x.\ \mathbf{if}\ n\ \mathbf{then}\ x$$
$$\mathbf{else}\ f\ (rep\ (n-1)\ f\ x)$$

quale e'significato del precedente pre-termine?

$$rep\ n\ f\ x = f^n\ x$$

Esercizio

$$\lambda x.\ \left(\left(\begin{array}{l} \textbf{rec}\ f.\ \lambda y.\ \textbf{if}\ (x - y)\ \textbf{then}\ 0 \\ \qquad\qquad\quad \textbf{else if}\ (x + y)\ \textbf{then}\ 1 \\ \qquad\qquad\quad \textbf{else}\ f\ (y + 1) \end{array} \right)\ 0 \right)$$

quale e' significato del precedente pre-termine?

maggiore o uguale a 0

Esercizio (da consegnare)

assumiamo $true = 0$

$false =$ qls $n \neq 0$

riempire al posto dei puntini (in HOFL)

$$
\begin{aligned}
or &\triangleq \lambda n.\, \lambda m.\, \cdots \\
and &\triangleq \lambda n.\, \lambda m.\, \cdots \\
not &\triangleq \lambda m.\, \cdots \\
implies &\triangleq \lambda n.\, \lambda m.\, \cdots \\
iff &\triangleq \lambda n.\, \lambda m.\, \cdots
\end{aligned}
$$

Pre-termini

$$
\begin{array}{rcl}
t & ::= & x \mid n \mid t_0 \text{ op } t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \\
& \mid & (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t) \\
& \mid & \lambda x.\ t \mid t_0\ t_1 \\
& \mid & \textbf{rec } x.\ t
\end{array}
$$

Perche' sono chiamati pre-termini?

- $x + 1$ ✔
- $\textbf{if } x \textbf{ then } x + 1 \textbf{ else } x - 1$ ✔
- $(0, \lambda x.\ x)$ ✔
- $\textbf{fst}(0, \lambda x.\ x)$ ✔
- $(\lambda x.\ x + 1)\ 3$ ✔
- $\textbf{rec } f.\ \lambda x.\ x + (f\ 0)$ ✔

abbiamo bisogno di un sistema di tipi

- ✘ $1 + (0, 5)$
- ✘ $2 \times \lambda x.\ x$
- ✘ $3\ \lambda x.\ x + 1$
- ✘ $\textbf{fst}(3)$
- ✘ $\textbf{if } x \textbf{ then } \lambda x.\ x \textbf{ else } (x, x)$
- ✘ $\textbf{rec } f.\ \lambda x.\ f + x$

tipi HOFL

Tipi

$$
\begin{array}{rcl}
t & ::= & x \mid n \mid t_0 \ \mathsf{op}\ t_1 \mid \mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1 \\
& \mid & (t_0, t_1) \mid \mathbf{fst}(t) \mid \mathbf{snd}(t) \\
& \mid & \lambda x.\ t \mid t_0\ t_1 \\
& \mid & \mathbf{rec}\ x.\ t
\end{array}
$$

quali tipi?

infinite combinazioni!

coppie

funzioni

int

$int * int$

$int \to int$

$int * (int \to int)$

$(int * int) \to int$

$int * (int * int)$

$(int \to int) \to int$

$(int \to int) * (int \to int)$

$(int \to int) \to (int \to (int * int))$

Sintassi dei tipi

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \rightarrow \tau_1$$

$\mathcal{T}$

insieme di tutti i tipi

assumiamo variabili tipate

$$Ide = \{Ide_\tau\}_{\tau \in \mathcal{T}}$$

$$\widehat{\cdot} : Ide \rightarrow \mathcal{T}$$

$\widehat{x}$ denota il tipo di x

Type judgements

si legge “ha tipo”

formula: $t : \tau$

sono assegnati ai pre-termini
usando un insieme di regole di inferenza
(induzione strutturale sulla sintassi HOFL)

Sistema di tipi

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathsf{op} \ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if} \ t \ \mathbf{then} \ t_0 \ \mathbf{else} \ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

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$$\frac{x : \tau \quad t : \tau}{\mathbf{rec} \ x.\ t : \tau}$$

Termini ben formati

$$\begin{array}{rcl} t & ::= & x \mid n \mid t_0 \text{ op } t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \\ & \mid & (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t) \\ & \mid & \lambda x.\ t \mid t_0\ t_1 \\ & \mid & \textbf{rec}\ x.\ t \end{array}$$

$$\tau \ ::= \ int \mid \tau_0 * \tau_1 \mid \tau_0 \to \tau_1$$

$\mathcal{T}$ insieme di tutti i tipi

un pre-termine t e' ben formato se $\exists \tau \in \mathcal{T}.\ t : \tau$

cioe' se possiamo assegnargli un tipo
anche detto ben tipato *o tipabile*

T_τ insieme di tutti i termini ben formati di tipo τ

Controllo dei tipi
Church Type Theory

Church Type Theory

- Le variabili sono etichettate con tipi (dichiarati)

- Deduciamo il tipo dei termini per induzione strutturale cioe' usando le regole di inferenza in modo bottom-up

Esempio

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$$\frac{}{fact : int \to int}$$

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$$\frac{f : int \to int \qquad \lambda x.\ \textbf{if } x \textbf{ then } 1 \textbf{ else } (x \times (f(x-1))) : int \to int}{fact : int \to int}$$

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scritto in maniera più semplice

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$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \mathbf{rec}\ \underbrace{f}_{int \to int} .\lambda\ \underbrace{x}_{int} .\mathbf{if}\ \underbrace{x}_{int}\ \mathbf{then}\ \underbrace{1}_{int}\ \mathbf{else}\ \underbrace{\underbrace{x}_{int} \times (\underbrace{\underbrace{f}_{int \to int}\ (\underbrace{\underbrace{x}_{int} - \underbrace{1}_{int}}_{int}}_{int}))}_{int}$$

Esempio

$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

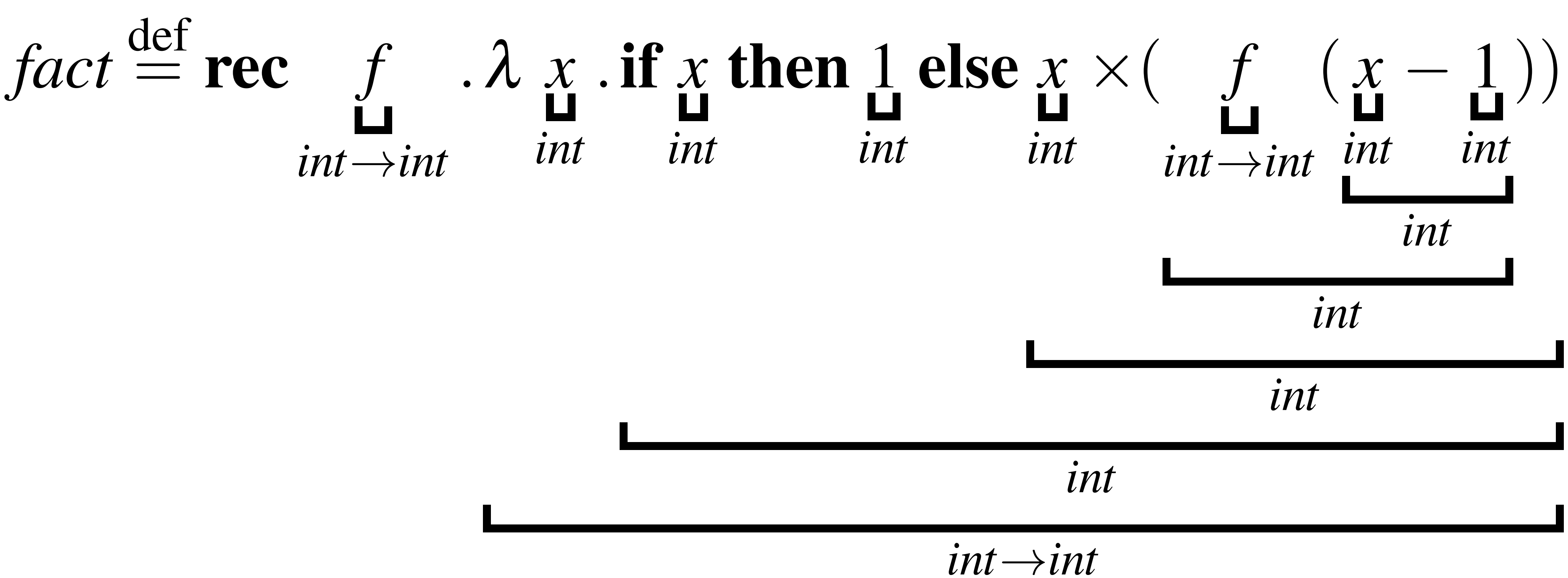
scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \mathbf{rec}\ \underbrace{f}_{int \to int} . \lambda \underbrace{x}_{int} . \underbrace{\mathbf{if}\ \underbrace{x}_{int}\ \mathbf{then}\ \underbrace{1}_{int}\ \mathbf{else}\ \underbrace{\underbrace{x}_{int} \times (\underbrace{\underbrace{f}_{int \to int}\ (\underbrace{\underbrace{x}_{int} - \underbrace{1}_{int}}_{int})}_{int})}_{int}}_{int}$$

Esempio

$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

scritto in maniera più semplice

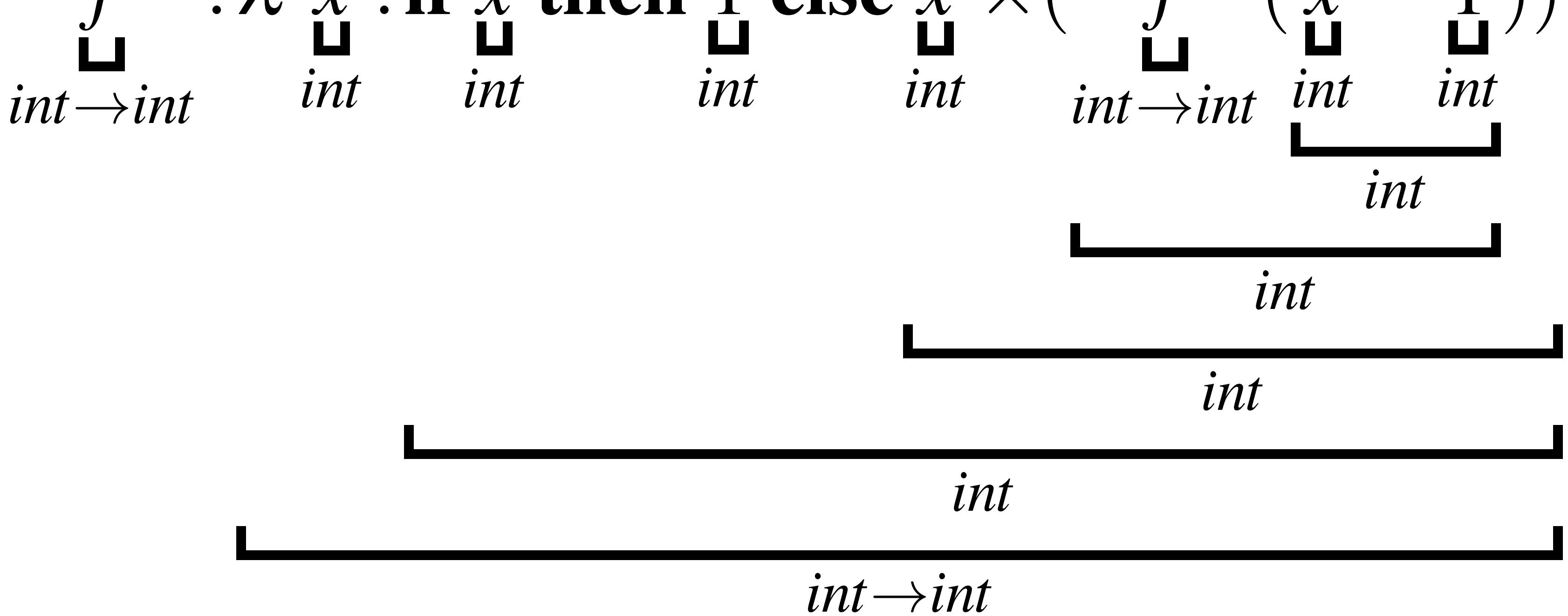


Esempio

$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \mathbf{rec}\ f\ .\lambda\ x\ .\mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (\ f\ \ (x - 1)) \qquad : int \to int$$



Inferenza di tipi
Curry Type theory

Curry Type Theory

- I tipi delle variabili non devono necessariamente essere (dichiarati)

- Inferiamo il tipo dei termini e' inferito usando le regole per derivare vincoli di tipo (equazioni di tipo) le cui soluzioni (tramite unificazione) definiscono il tipo principale

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\, \lambda x.\, (x, (p\ (x+2)))$$

intuivamente

$$t\ 0 \equiv (0, (t\ 2)) \equiv (0, (2, (t\ 4))) \equiv \cdots \equiv (0, (2, (4, \ldots)))$$

sequenza di tutti i numeri pari

possiamo digitare sequenze di interi di lunghezza fissa
non abbiamo un tipo per sequenze di lunghezza qualsiasi/infinita

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

Haskell

```
Prelude> let p x = (x, p (x+2))

<interactive>:…:5: error:
    • Occurs check: cannot construct the infinite type: b ~ (t, b)
      Expected type: t -> b
        Actual type: t -> (t, b)
    • Relevant bindings include
        p :: t -> b (bound at <interactive>:…:5)
```

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

$$t = \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2))) : \tau$$

$\nwarrow$ $\widehat{p}=\tau$ $\lambda x.\ (x, (p\ (x+2))) : \tau$

$\nwarrow$ $\tau=\tau_1\to\tau_2$, $\widehat{x}=\tau_1$ $(x, (p\ (x+2))) : \tau_2$

$\nwarrow$ $\tau_2=\tau_3*\tau_4$ $x : \tau_3,\ (p\ (x+2)) : \tau_4$

$\nwarrow$ $\widehat{x}=\tau_3$ $(p\ (x+2)) : \tau_4$

$\nwarrow$ $p : \tau_5 \to \tau_4,\ (x+2) : \tau_5$

$\nwarrow$ $\widehat{p}=\tau_5\to\tau_4$ $(x+2) : \tau_5$

$\nwarrow$ $\tau_5=int$ $x : int,\ 2 : int$

$\nwarrow^*$ $\widehat{x}=int$ $\square$

$$\left.\begin{array}{l}\widehat{x} = \tau_1\\ \widehat{x} = \tau_3\\ \widehat{x} = int\end{array}\right\}\ \tau_1 = \tau_3 = int$$

$$\left.\begin{array}{l}\widehat{p} = \tau = \tau_1 \to \tau_2\\ \widehat{p} = \tau_5 \to \tau_4\end{array}\right\}\ \begin{array}{l}\tau_1 = \tau_5 = int\\ \tau_2 = \tau_4\end{array}$$

$$\left.\begin{array}{l}\tau_2 = \tau_4\\ \tau_2 = \tau_3 * \tau_4\end{array}\right\}\ \text{fail! (occur check)}$$

$$\frac{}{x : \widehat{x}} \quad \frac{}{n : int} \quad \frac{t_0 : int \quad t_1 : int}{t_0\ \textbf{op}\ t_1 : int} \quad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \quad \frac{t : \tau_0 * \tau_1}{\textbf{fst}(t) : \tau_0} \quad \frac{t : \tau_0 * \tau_1}{\textbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \quad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\textbf{rec}\ x.\ t : \tau}$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda\ x\ .\ (\ x\ , (\quad p \quad (\ x + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \textbf{rec}\ p.\quad \lambda\ x\ .\ (\ x\ , (\quad p \quad (\underbrace{x}_{int} + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda \underbrace{x}_{int}.\ (\underbrace{x}_{int}, (\quad p \quad (\underbrace{x}_{int} + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda \underbrace{x}_{int}.\ (\underbrace{x}_{int}, (\quad p \quad \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda \underbrace{x}_{int}.\ (\underbrace{x}_{int}, (\underbrace{p}_{int \to \tau_4}\ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ \underbrace{p}_{int \to \tau_4}.\ \lambda \underbrace{x}_{int}.\ (\underbrace{x}_{int}, (\underbrace{p}_{int \to \tau_4}\ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

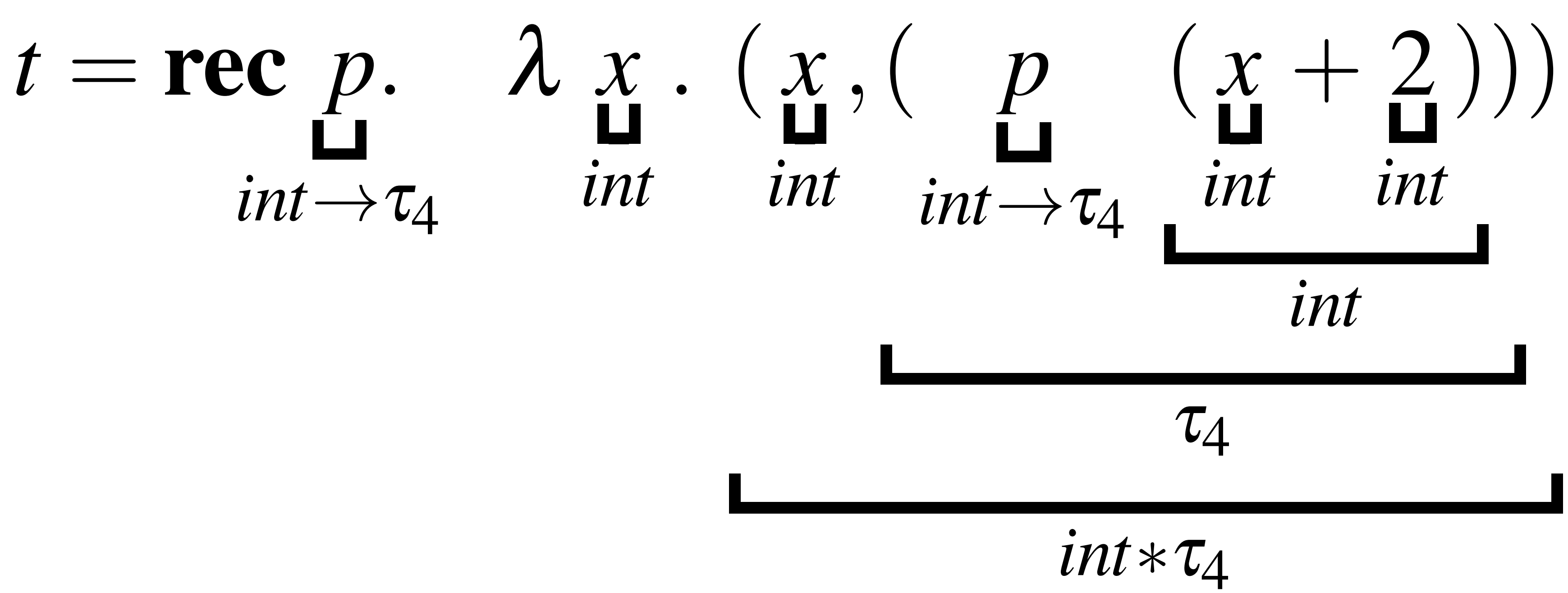
scritto in maniera più semplice

$$t = \textbf{rec}\ \underbrace{p}_{int \to \tau_4}.\ \lambda\ \underbrace{x}_{int}\ .\ (\underbrace{x}_{int}, (\underbrace{\underbrace{p}_{int \to \tau_4}\ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int}}_{\tau_4}))$$

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

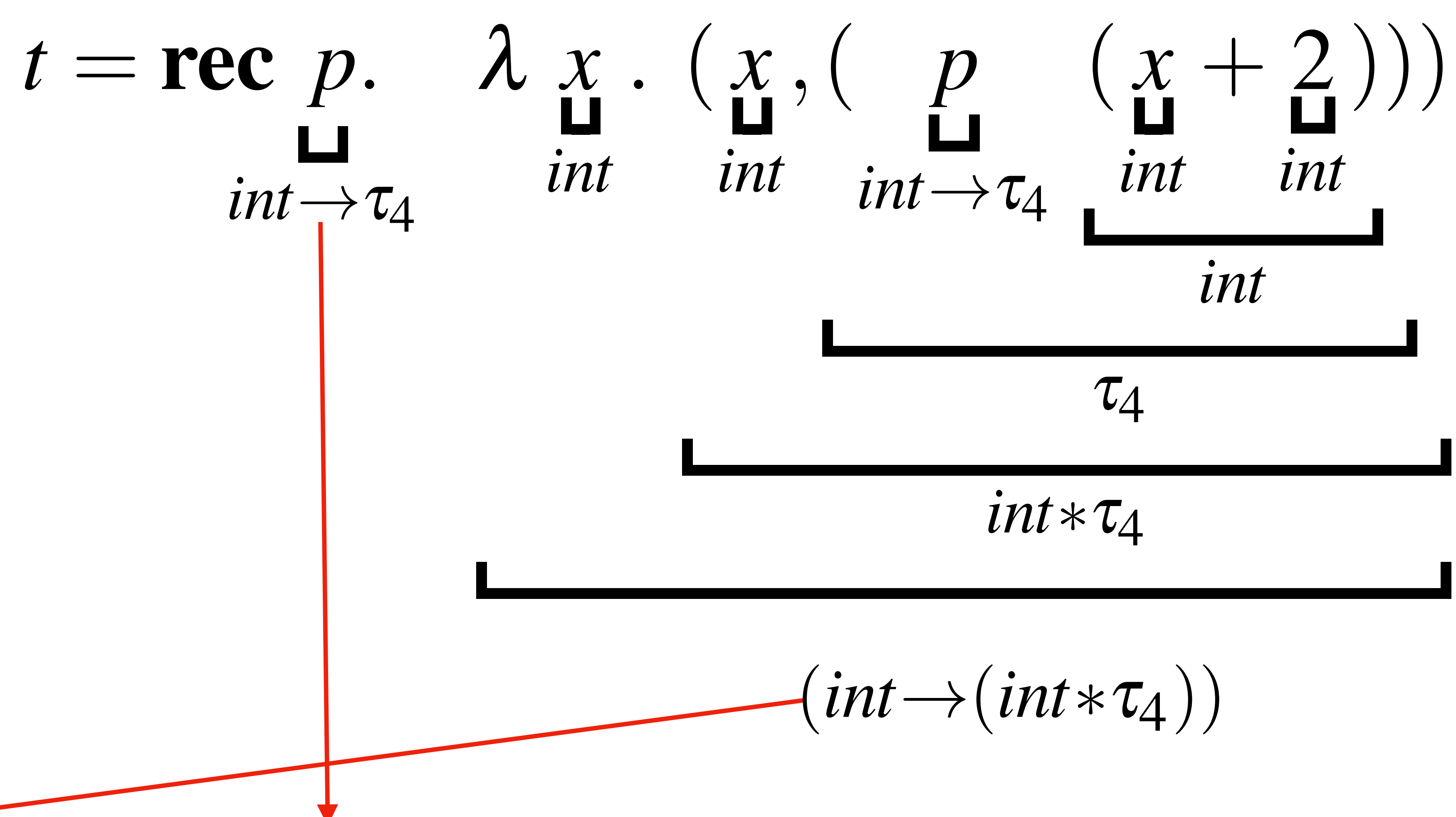
scritto in maniera più semplice



Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice



$$(int \to (int * \tau_4)) = (int \to \tau_4) \Rightarrow \tau_4 = (int * \tau_4)$$

fallisce (occur check)

Esercizio

inferire il tipo del termine sotto

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathbf{op} \ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if} \ t \ \mathbf{then} \ t_0 \ \mathbf{else} \ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1 \ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\mathbf{rec} \ x.\ t : \tau}$$

$$\mathbf{rec}\ rep.\ \lambda n.\ \lambda f.\ \lambda x.\ \mathbf{if}\ n\ \mathbf{then}\ x$$
$$\mathbf{else}\ f\ (rep\ (n-1)\ f\ x)$$

$$\tau_0 \to \tau_0$$

$$(\tau_1 \to \tau_0) \to \tau_0 \to \tau_0$$

$$int \to (\tau_1 \to \tau_0) \to \tau_0 \to \tau_0$$

$n : int$

$x : \tau_0$

$f : \tau_1 \to \tau_0$

$rep : int \to (\tau_1 \to \tau_0) \to \tau_0 \to \tau_1$ ➡ $\tau_0 = \tau_1$

Esercizio

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathsf{op}\ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

inferire il tipo del termine sotto

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\mathbf{rec}\ x.\ t : \tau}$$

$$\lambda x. \left(\left(\begin{array}{l} \mathbf{rec}\ f.\ \lambda y.\ \mathbf{if}\ (x - y)\ \mathbf{then}\ 0 \\ \qquad\qquad\quad \mathbf{else\ if}\ (x + y)\ \mathbf{then}\ 1 \\ \qquad\qquad\quad \mathbf{else}\ f\ (y+1) \end{array} \right)\ 0 \right)$$

$$\frac{\frac{\frac{}{}}{int \to int}}{int}$$

$$\overline{int \to int}$$

int

$x : int$

$y : int$

$f : int \to int$

Substituzioni capture-avoiding
(di nuovo)

Variabili libere

$$\mathrm{fv}(n) \stackrel{\mathrm{def}}{=} \varnothing$$

$$\mathrm{fv}(x) \stackrel{\mathrm{def}}{=} \{x\}$$

$$\mathrm{fv}(t_0 \text{ op } t_1) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t) \cup \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}((t_0, t_1)) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\textbf{fst}(t)) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t)$$

$$\mathrm{fv}(\textbf{snd}(t)) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t)$$

$$\mathrm{fv}(\lambda x.\ t) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t) \setminus \{x\}$$

$$\mathrm{fv}((t_0\ t_1)) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\textbf{rec } x.\ t) \stackrel{\mathrm{def}}{=} \mathrm{fv}(t) \setminus \{x\}$$

Substituzioni

$$n[^t/_x] = n$$

$$y[^t/_x] \stackrel{\text{def}}{=} \begin{cases} t & \text{if } y = x \\ y & \text{if } y \neq x \end{cases}$$

$$(t_0 \text{ op } t_1)[^t/_x] \stackrel{\text{def}}{=} t_0[^t/_x] \text{ op } t_1[^t/_x] \quad \text{with op} \in \{+, -, \times\}$$

$$(\mathbf{if}\ t'\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1)[^t/_x] \stackrel{\text{def}}{=} \mathbf{if}\ t'[^t/_x]\ \mathbf{then}\ t_0[^t/_x]\ \mathbf{else}\ t_1[^t/_x]$$

$$(t_0, t_1)[^t/_x] \stackrel{\text{def}}{=} (t_0[^t/_x], t_1[^t/_x])$$

$$\mathbf{fst}(t')[^t/_x] \stackrel{\text{def}}{=} \mathbf{fst}(t'[^t/_x])$$

$$\mathbf{snd}(t')[^t/_x] \stackrel{\text{def}}{=} \mathbf{snd}(t'[^t/_x])$$

$$(t_0\ t_1)[^t/_x] \stackrel{\text{def}}{=} (t_0[^t/_x]\ t_1[^t/_x])$$

$$(\lambda y.\ t')[^t/_x] \stackrel{\text{def}}{=} \lambda z.\ (\ t'[^z/_y][^t/_x]\ ) \quad \text{for } z \notin \text{fv}(\lambda y.\ t') \cup \text{fv}(t) \cup \{x\}$$

$$(\mathbf{rec}\ y.\ t')[^t/_x] \stackrel{\text{def}}{=} \mathbf{rec}\ z.\ (t'[^z/_y][^t/_x]) \quad \text{for } z \notin \text{fv}(\mathbf{rec}\ y.\ t') \cup \text{fv}(t) \cup \{x\}$$

I tipi sono rispettati

TH. $x_0 : \tau_0$ $\quad$ $t : \tau \quad \Rightarrow \quad t[{}^{t_0}/_{x_0}] : \tau$

$t_0 : \tau_0$

prova omessa
(per induzione strutturale)