Linguaggi di Programmazione

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Semantica Denotazionale HOFL-cap 9

Domini di interpretazioni

L'idea

Per ogni tipo introduciamo il corrispondente dominio $(V_\tau)_\perp$ definito induttivamente sulla struttura di τ in modo da poter assegnare un elemento del dominio $(V_\tau)_\perp$ ad ogni termine t chiuso e tipabile con tipo τ

Domini di interpretazioni

$$D_{int} \triangleq \mathbb{Z}_\perp$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

distinguere:
coppia di termini divergenti
da coppia divergente

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

distinguere:
prende arg e diverge
dalla divergenza senza prendere arg

Esempio

$$D_{int * int} \triangleq (\mathbb{Z}_\perp \times \mathbb{Z}_\perp)_\perp$$

$\mathbf{rec}\ p.\ p$ $\qquad (\mathbf{rec}\ x.\ x, \mathbf{rec}\ y.\ y)$

$\perp_{D_{int * int}}$ $\qquad (\perp_{D_{int}}, \perp_{D_{int}})$

Esempio

$$D_{int \to int} \triangleq [\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp$$

$\mathbf{rec}\ f.\ f$ $\qquad \lambda x.\ \mathbf{rec}\ y.\ y$

$\perp_{D_{int \to int}}$ $\qquad \lambda d.\ \perp_{D_{int}}$

Domini di interpretazioni

$$D_{int} \triangleq \mathbb{Z}_\perp \qquad D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp \qquad D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

Equivalentemente:

$$D_\tau \triangleq (V_\tau)_\perp$$

$$V_{int} \triangleq \mathbb{Z}$$

$$V_{\tau_1 * \tau_2} \triangleq D_{\tau_1} \times D_{\tau_2} = (V_{\tau_1})_\perp \times (V_{\tau_2})_\perp$$

$$V_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}] = [(V_{\tau_1})_\perp \to (V_{\tau_2})_\perp]$$

Domini di interpretazioni

$t : \tau \qquad \llbracket t \rrbracket \rho \in D_\tau$

ambiente $\qquad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

tipo coerente con assegnazione di valori alle variabili $\qquad x : \tau \Rightarrow \rho(x) \in D_\tau$

definiamo la funzione di interpretazione per ricorsione strutturale

Semantica Denotazionale

Constanti

$$t : \tau \qquad [\![t]\!]\rho \in D_\tau$$

$$\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\underbrace{[\![\underbrace{n}_{int}]\!]\rho}_{D_{int} = \mathbb{Z}_\perp} \triangleq \underbrace{\lfloor \underbrace{n}_{\mathbb{Z}} \rfloor}_{\mathbb{Z}_\perp}$$

Variabili

$$t : \tau \qquad [\![t]\!]\rho \in D_\tau$$

$$\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\underbrace{[\![\underbrace{x}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{\rho(x)}_{D_\tau}$$

Operazioni aritmetiche

da dimostrare: $\underline{\mathsf{op}}_\perp$ e' monotona e continua

$$\mathsf{op} \in \{+, - \times\}$$

$$\underbrace{[\![\underbrace{t_1}_{int}\ \mathsf{op}\ \underbrace{t_2}_{int}]\!]_{int}\rho}_{D_{int}=\mathbb{Z}_\perp} \triangleq \underbrace{\underbrace{[\![\underbrace{t_1}_{int}]\!]\rho}_{D_{int}=\mathbb{Z}_\perp}\ \underline{\mathsf{op}}_\perp\ \underbrace{[\![\underbrace{t_2}_{int}]\!]\rho}_{D_{int}=\mathbb{Z}_\perp}}_{D_{int}=\mathbb{Z}_\perp}$$

$$\underline{\mathsf{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1\ \underline{\mathsf{op}}_\perp\ v_2 \triangleq \begin{cases} \lfloor n_1\ \underline{\mathsf{op}}\ n_2 \rfloor & \text{se } v_1 = \lfloor n_1 \rfloor \text{ e } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{altrimenti } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ o } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

chiamata estensione strict

Condizionale

da dimostrare: Cond_τ is monotona e continua

$$\underbrace{[\![\mathbf{if}\ \underbrace{\overbrace{t}}_{int}\ \mathbf{then}\ \underbrace{t_1}_{\tau}\ \mathbf{else}\ \underbrace{t_2}_{\tau}]\!]}_{\tau}\rho \triangleq \mathsf{Cond}_\tau(\ \underbrace{[\![\underbrace{t}_{int}]\!]\rho}_{D_{int}=\mathbb{Z}_\perp}\ ,\ \underbrace{[\![\underbrace{t_1}_{\tau}]\!]\rho}_{D_\tau}\ ,\ \underbrace{[\![\underbrace{t_2}_{\tau}]\!]\rho}_{D_\tau}\ )$$

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Coppie

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

$$\underbrace{\llbracket \underbrace{(\underbrace{t_1}_{\tau_1} \; , \; \underbrace{t_2}_{\tau_2})}_{\tau_1 * \tau_2} \rrbracket \rho}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp} \triangleq \underbrace{\lfloor \; \underbrace{(\; \underbrace{\llbracket \underbrace{t_1}_{\tau_1} \rrbracket \rho}_{D_{\tau_1}} \; , \; \underbrace{\llbracket \underbrace{t_2}_{\tau_2} \rrbracket \rho}_{D_{\tau_2}} \;)}_{D_{\tau_1} \times D_{\tau_2}} \; \rfloor}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp}$$

Proiezioni

Equivalentemente: $[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \mathbf{let}\ d \Leftarrow [\![t]\!]\rho.\ \pi_1(d)$

$$\underbrace{[\![\underbrace{\mathbf{fst}(\ \underbrace{t}_{\tau_1 * \tau_2}\ )}_{\tau_1}]\!]\rho}_{D_{\tau_1}} \triangleq \underbrace{\underbrace{\underbrace{\pi_1}_{D_{\tau_1} \times D_{\tau_2} \to D_{\tau_1}}^*}_{(D_{\tau_1} \times D_{\tau_2})_\perp \to D_{\tau_1}}\ (\ \underbrace{[\![\underbrace{t}_{\tau_1 * \tau_2}]\!]\rho}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp}\ )}_{D_{\tau_1}}$$

$$[\![\mathbf{snd}(\ t\ )]\!]\rho \triangleq \pi_2^*\ (\ [\![t]\!]\rho\ )$$

Astrazione

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

$$\underbrace{[\![\lambda \underbrace{\underbrace{x}_{\tau_1}.\ \underbrace{t}_{\tau_2}}_{\tau_1 \to \tau_2}]\!]}_{D_{\tau_1 \to \tau_2} = [D_{\tau_1} \to D_{\tau_2}]_\perp}\rho \triangleq \underbrace{\lfloor\ \underbrace{\lambda \underbrace{d}_{D_{\tau_1}}.\ \underbrace{[\![\underbrace{t}_{\tau_2}]\!]\rho[^d/_x]}_{D_{\tau_2}}}_{[D_{\tau_1} \to D_{\tau_2}]}\ \rfloor}_{[D_{\tau_1} \to D_{\tau_2}]_\perp}$$

Applicazione (lazy)

Equivalentemente:

$$[\![\ t\ t_0\ ]\!]\rho \triangleq (\lambda\varphi.\ \varphi([\![t_0]\!]\rho))^*\ ([\![t]\!]\rho)$$

$$\underbrace{[\![\ \underbrace{\underbrace{t}_{\tau_0\to\tau}\ \underbrace{t_0}_{\tau_0}}_{\tau}\ ]\!]\rho}_{D_\tau} \triangleq \underbrace{\mathbf{let}\ \underbrace{\varphi}_{V_{\tau_0\to\tau}=[D_{\tau_0}\to D_\tau]} \Leftarrow \underbrace{[\![\underbrace{t}_{\tau_0\to\tau}]\!]\rho}_{D_{\tau_0\to\tau}=(V_{\tau_0\to\tau})_\bot}.\ \underbrace{\underbrace{\varphi}_{[D_{\tau_0}\to D_\tau]}(\underbrace{[\![\underbrace{t_0}_{\tau_0}]\!]\rho}_{D_{\tau_0}})}_{D_\tau}}_{D_\tau}$$

Ricorsione

$$\underbrace{[\![\mathbf{rec}\ \underbrace{\underbrace{x}_{\tau}.\ \underbrace{t}_{\tau}}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{[\![\underbrace{t}_{\tau}]\!]\rho[\underbrace{{}^{[\![\mathbf{rec}\ x.\ t]\!]\rho}}_{D_\tau}/\underbrace{x}_{\tau}]}_{D_\tau}$$

Ricorsione

$$\underbrace{[\![\underbrace{\mathbf{rec}\ \underbrace{x}_{\tau}.\ \underbrace{t}_{\tau}}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{\underbrace{\mathit{fix}}_{[[D_\tau \to D_\tau] \to D_\tau]}\ \underbrace{\lambda \underbrace{d}_{D_\tau}.\ \underbrace{[\![\underbrace{t}_{\tau}]\!]\rho[{}^{\underbrace{d}_{D_\tau}}/_{\underbrace{x}_{\tau}}]}_{D_\tau}}_{[D_\tau \to D_\tau]}}_{D_\tau}$$

Recap

$$[\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![x]\!]\rho \triangleq \rho(x)$$

$$[\![t_1 \ \mathsf{op} \ t_2]\!]\rho \triangleq [\![t_1]\!]\rho \ \underline{\mathsf{op}}_\perp \ [\![t_2]\!]\rho$$

$$[\![\mathbf{if} \ t \ \mathbf{then} \ t_1 \ \mathbf{else} \ t_2]\!]\rho \triangleq \mathsf{Cond}_\tau(\ [\![t]\!]\rho \ , \ [\![t_1]\!]\rho \ , \ [\![t_2]\!]\rho \ )$$

$$[\![(t_1 \ , \ t_2)]\!]\rho \triangleq \lfloor \ (\ [\![t_1]\!]\rho \ , \ [\![t_2]\!]\rho \ ) \ \rfloor$$

$$[\![\mathbf{fst}(\ t \ )]\!]\rho \triangleq \pi_1^* \ (\ [\![t]\!]\rho \ )$$

$$[\![\mathbf{snd}(\ t \ )]\!]\rho \triangleq \pi_2^* \ (\ [\![t]\!]\rho \ )$$

$$[\![\lambda x. \ t]\!]\rho \triangleq \lfloor \ \lambda d. \ [\![t]\!]\rho[^d/_x] \ \rfloor$$

$$[\![\ t \ t_0 \ ]\!]\rho \triangleq \mathbf{let} \ \varphi \Leftarrow [\![t]\!]\rho. \ \varphi([\![t_0]\!]\rho)$$

$$[\![\mathbf{rec} \ x. \ t]\!]\rho \triangleq \mathit{fix} \ \lambda d. \ [\![t]\!]\rho[^d/_x]$$

Esempi

I seguenti termini hanno la stessa semantica?

$$f \stackrel{\text{def}}{=} \lambda x : int.\ 3$$
$$g \stackrel{\text{def}}{=} \lambda x : int.\ \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3$$
$$h \stackrel{\text{def}}{=} \textbf{rec}\, y : int \to int.\ \lambda x : int.\ 3$$

Esempio

$$f \stackrel{\text{def}}{=} \lambda x : int\,.\, 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor \qquad [\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![f]\!]\rho = [\![\lambda x.\, 3]\!]\rho = \lfloor \lambda d.\, [\![3]\!]\rho[^d/_x] \rfloor = \lfloor \lambda d.\, \lfloor 3 \rfloor \rfloor$$

Esempio

$$g \stackrel{\text{def}}{=} \lambda x : int.\ \textbf{if } x \textbf{ then } 3 \textbf{ else } 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

$$\begin{aligned}
[\![g]\!]\rho &= [\![\lambda x.\ \textbf{if } x \textbf{ then } 3 \textbf{ else } 3]\!]\rho \\
&= \lfloor \lambda d.\ [\![\textbf{if } x \textbf{ then } 3 \textbf{ else } 3]\!]\rho[^d/_x] \rfloor
\end{aligned}$$

$$[\![\textbf{if } t \textbf{ then } t_1 \textbf{ else } t_2]\!]\rho \triangleq \mathsf{Cond}_\tau(\ [\![t]\!]\rho\ ,\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )$$

$$= \lfloor \lambda d.\ Cond(d, \lfloor 3 \rfloor, \lfloor 3 \rfloor) \rfloor$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \bot_{D_\tau} & \text{se } v = \bot_{\mathbb{Z}_\bot} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$$= \lfloor \lambda d.\ \textbf{let } x \Leftarrow d.\ \lfloor 3 \rfloor \rfloor$$

$$[\![f]\!]\rho \neq [\![g]\!]\rho$$

$$\lfloor \lambda d.\ \lfloor 3 \rfloor \rfloor \qquad f \stackrel{\text{def}}{=} \lambda x : int.\ 3$$

Esempio

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int\, .\, \lambda x : int\, .\, 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[^{d_y}/_y] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[^{d_y}/_y, ^{d_x}/_x] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$

$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$\begin{aligned}
d_0 &= \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp} \\
d_1 &= \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\perp = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor \\
d_2 &= \Gamma_h(d_1) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = d_1
\end{aligned}$$

Esempio

$$h \stackrel{\text{def}}{=} \textbf{rec}\, y : int \to int\,.\, \lambda x : int\,.\, 3$$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$d_0 = \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp}$$
$$d_1 = \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\perp = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

Elemento massimale in $[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp$
potremmo gia' fermarci

Esempio

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int\,.\ \lambda x : int\,.\ 3$$

$$[\![h]\!]\rho = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = [\![f]\!]\rho$$

$$[\![f]\!]\rho = [\![\lambda x.\ 3]\!]\rho = \lfloor \lambda d.\ [\![3]\!]\rho[^d/_x] \rfloor = \lfloor \lambda d.\ \lfloor 3 \rfloor \rfloor$$

$$f \stackrel{\text{def}}{=} \lambda x : int\,.\ 3$$

Esempio

$x : \tau$

$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$

$$[\![\mathbf{rec}\ x.\ x]\!]\rho = \mathit{fix}\ \lambda d_x.\ [\![x]\!]\rho[^{d_x}/_x]$$

$$= \mathit{fix}\ \lambda d_x.\ d_x$$

$$d_0 = \bot_{D_\tau}$$

$$d_1 = (\lambda d_x . d_x)\ d_0 = d_0 = \bot_{D_\tau}$$

$$[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{D_\tau}$$

$x : int \to int$ $\qquad$ $[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$

$x : int * int$ $\qquad$ $[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{(\mathbb{Z}_\bot \times \mathbb{Z}_\bot)_\bot}$

Esempi

I seguenti termini hanno la stessa semantica?

$$\mathbf{rec}\ x.\ x$$

$$\lambda y.\ \mathbf{rec}\ z.\ z$$

Esempio

$y : \tau_1 \qquad z : \tau_2$

$\llbracket \lambda x.\ t \rrbracket \rho \triangleq \lfloor\ \lambda d.\ \llbracket t \rrbracket \rho[^d/_x]\ \rfloor$

$$\llbracket \lambda y.\ \mathbf{rec}\ z.\ z \rrbracket \rho = \lfloor \lambda d_y.\ \llbracket \mathbf{rec}\ z.\ z \rrbracket \rho[^{d_y}/_y] \rfloor$$

$$= \lfloor \lambda d_y.\ \bot_{D_{\tau_2}} \rfloor$$

$$= \lfloor \bot_{[D_{\tau_1} \to D_{\tau_2}]} \rfloor$$

$$= \lfloor \bot_{V_{\tau_1 \to \tau_2}} \rfloor$$

$$\neq \bot_{D_{\tau_1 \to \tau_2}} \quad = \bot_{(V_{\tau_1 \to \tau_2})_\bot}$$

$x : int \to int$ $\qquad \llbracket \mathbf{rec}\ x.\ x \rrbracket \rho = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$ $\qquad$ diverge

$y : int, z : int$ $\qquad \llbracket \lambda y.\ \mathbf{rec}\ z.\ z \rrbracket \rho = \lfloor \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]} \rfloor$ $\qquad$ aspetta arg e diverge

Esercizio

$$x : int * int \;,\; y : int \;,\; z : int$$

$$[\![\ \mathbf{rec}\ x.\ x\ ]\!]\rho \quad \stackrel{?}{=} \quad [\![\ (\ \mathbf{rec}\ y.\ y\ ,\ \mathbf{rec}\ z.\ z\ )\ ]\!]\rho$$

diverge

una coppia
di computazioni divergenti

$$\perp_{D_{int*int}}$$

$$\lfloor(\perp_{D_{int}}, \perp_{D_{int}})\rfloor$$

Lazy vs Eager

Applicazioni Eager

returns $\perp$ quando $[\![t]\!]\rho = \perp$

lazy $[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$

eager $[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \mathbf{let}\ d \Leftarrow [\![t_0]\!]\rho.\ \varphi(\lfloor d \rfloor)$

returns $\perp$ quando $[\![t]\!]\rho = \perp$ o $[\![t_0]\!]\rho = \perp$

Definizioni ben costruite

Ben definite

Dobbiamo garantire che tutte le funzioni che abbiamo usato siano monotone e continue,
in modo che la teoria dei punti fissi di Kleene sia applicabile

$\pi_1 \;\; \pi_2 \;\; (\cdot)^* \;\; apply \;\; \mathit{fix} \quad \mathbf{let}$ gia' analizzate

$\underline{\mathsf{op}}_\perp \quad \mathsf{Cond}_\tau \quad \lambda$ da controllare

TH. $\underline{\text{op}}_\perp$ e' monotono e continuo

$$\underline{\text{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1 \ \underline{\text{op}}_\perp \ v_2 \triangleq \begin{cases} \lfloor n_1 \ \underline{\text{op}} \ n_2 \rfloor & \text{se } v_1 = \lfloor n_1 \rfloor \text{ e } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{altrimenti } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ o } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

Omettiamo il controllo di monotonicita'
Dal momento che il dominio ha solo catene finite, e' anche continuo

TH. Cond_τ e' monotona e continua

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Omettiamo il controllo di monotonicità

Dimostriamo la continuità su ogni parametro separatamente

Il primo parametro e' in $\mathbb{Z}_\perp$

sono possibili solo catene finite, quindi la continuità è garantita

Dimostriamo la continuità sul secondo parametro

Per il terzo parametro la prova è analoga e viene omessa

(continua)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Continuita' sul secondo parametro

prendiamo $v \in \mathbb{Z}_\perp, d \in D_\tau, \{d_i\}_{i\in\mathbb{N}} \subseteq D_\tau$

vogliamo provare $\mathsf{Cond}_\tau\left(v, \bigsqcup_{i\in\mathbb{N}} d_i, d\right) = \bigsqcup_{i\in\mathbb{N}} \mathsf{Cond}_\tau(v, d_i, d)$

procediamo per analisi dei casi $v \begin{cases} \perp_{\mathbb{Z}_\perp} \\ \lfloor 0 \rfloor \\ \lfloor n \rfloor\ ,\ n \neq 0 \end{cases}$

(continua)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$v = \perp_{\mathbb{Z}_\perp}$

$$\mathsf{Cond}_\tau\left(\perp_{\mathbb{Z}_\perp}, \bigsqcup_{i \in \mathbb{N}} d_i, d\right) = \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\perp_{\mathbb{Z}_\perp}, d_i, d)$$

(continua)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$v = \lfloor 0 \rfloor$

$$\mathsf{Cond}_\tau \left(\lfloor 0 \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\lfloor 0 \rfloor, d_i, d)$$

(continua)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$v = \lfloor n \rfloor \ , \ n \neq 0$

$$\mathsf{Cond}_\tau\left(\lfloor n \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d\right) = d = \bigsqcup_{i \in \mathbb{N}} d = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\lfloor n \rfloor, d_i, d)$$

TH. la lambda astrazione e' monotona e continua

$t : \tau$ $\lambda d.\ [\![t]\!]\rho[^d/_x]$ e' continua

ci concentriamo su una proprietà più forte

$\tilde{x}$ ennupla $x_1, ..., x_n$ $\tilde{d}$ ennupla $d_1, ..., d_n$

$\lambda\widetilde{d}.\ [\![t]\!]\rho[^{\widetilde{d}}/_{\widetilde{x}}]$ e' continua

la prova e' per induzione strutturale su t

(provateci)

Corollary $t : \tau_0 \rightarrow \tau$ $\mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$ e' continua

(il limite di funzioni continue è continuo)

Proprieta’ principali

Lemma di sostituzione

$$\begin{array}{r} x, t_0 : \tau_0 \\ t : \tau \end{array} \qquad [\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$$

update dell'ambiente

sostituzione sintattica

la prova e' per induzione strutturale su t

(provateci)

Composizionalita'

Il lemma di sostituzione $[\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$ è importante perché garantisce la composizionalità della semantica denotazionale

TH. $[\![t_1]\!]\rho = [\![t_2]\!]\rho \quad \Rightarrow \quad [\![t[^{t_1}/_x]]\!]\rho = [\![t[^{t_2}/_x]]\!]\rho$

proof. assumiamo $[\![t_1]\!]\rho = [\![t_2]\!]\rho$

$$[\![t[^{t_1}/_x]]\!]\rho \underbrace{=}_{\text{lemma di sostituzione}} [\![t]\!]\rho[^{[\![t_1]\!]\rho}/_x] \underbrace{=}_{[\![t_1]\!]\rho = [\![t_2]\!]\rho} [\![t]\!]\rho[^{[\![t_2]\!]\rho}/_x] \underbrace{=}_{\text{lemma di sostituzione}} [\![t[^{t_2}/_x]]\!]\rho$$

Solo le variabili free hanno importanza

TH. $t : \tau$

$\forall x \in \mathsf{fv}(t).\ \rho(x) = \rho'(x) \quad \Rightarrow \quad [\![t]\!]\rho = [\![t]\!]\rho'$

la prova e' per induzione strutturale su t

(provateci)

Corollary t chiuso $\quad \Rightarrow \quad \forall \rho, \rho'.\ [\![t]\!]\rho = [\![t]\!]\rho'$

TH. I termini canonici non sono bottom

$$c \in C_\tau \;\Rightarrow\; \forall \rho.\ [\![c]\!]\rho \neq \bot_{D_\tau}$$

prova. per induzione sulle regole dei termini canonici

$$P(c \in C_\tau) \triangleq \forall \rho.\ [\![c]\!]\rho \neq \bot_{D_\tau}$$

$$\frac{}{n \in C_{int}} \qquad [\![n]\!]\rho = \lfloor n \rfloor \neq \bot_{D_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}} \qquad [\![(t_0, t_1)]\!]\rho = \lfloor ([\![t_0]\!]\rho, [\![t_1]\!]\rho) \rfloor \neq \bot_{D_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x.\, t \text{ closed}}{\lambda x.\, t \in C_{\tau_0 \to \tau_1}} \qquad [\![\lambda x.\ t]\!]\rho = \lfloor \lambda d.\ [\![t]\!]\rho[^d/_x] \rfloor \neq \bot_{D_{\tau_0 \to \tau_1}}$$