Decision Theory

Example: buying a used car

We have the option of buying an used car, then repairing and reselling it.

The car can be a *peach* or a *lemon*.

Let us esteem probabilities and utilities to decide if buying it or not.

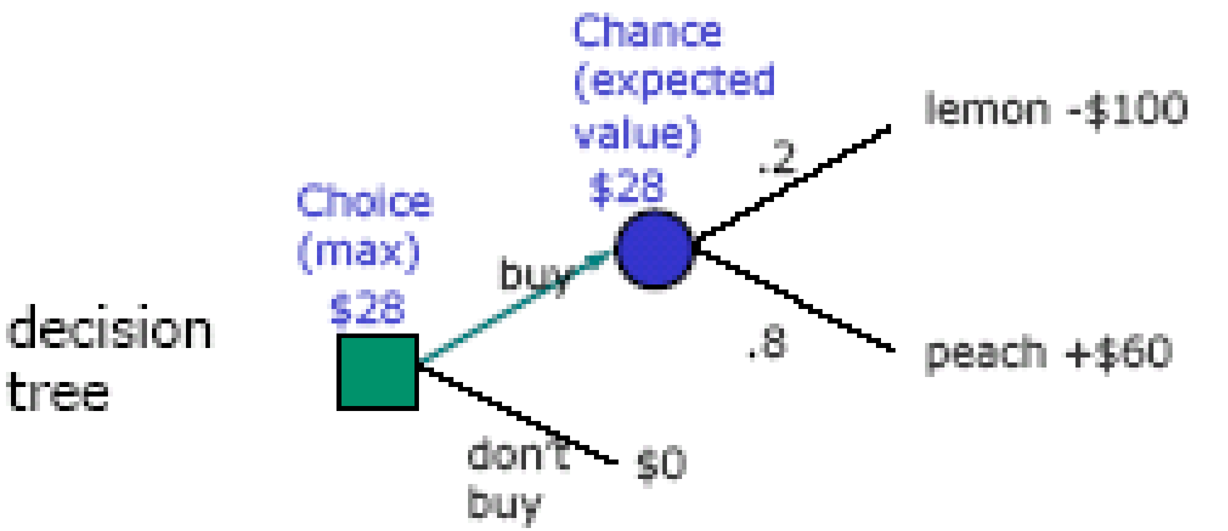
We are risk neutral: utility is proportional to money amount, 1,000 $ are worth exactly the double of 500 $.

Procedure (recursive: backward, from leaves to root)

1. Draw the tree of possible scenarios
2. Label each leaf with the scenario utility (if it happens)
3. Label each event node with the sum of children nodes utilities weighted with probabilities (i.e. expected utilities)
4. Label each decision node with the max of children nodes utilities

Buying a Used Car

- Costs $1000
- Can sell it for $1100, $100 profit
- Every car is a lemon or a peach
- 20% are lemons
- Costs $40 to repair a peach, $200 to repair a lemon
- Risk neutral



The *buy* decision dominates the *don't buy* decision.

The decision tree says we have to buy, if we want to be rational.

Here *rational* means *maximizing utility*.

Not only: the tree tells us how much utility we'll have choosing the best decision, i.e. 28 $.

The event node (blue circle) is worth 28 because after having bought and before observing the peach/lemon event we are virtually gaining 28 $.

Virtually means *on the average in a game repeated many times*.

The decision node (green square) is worth 28 because before making the decision we are virtually gaining 28 $, under the condition we are rational.

An unhappy employee of the car-shop offer us consulting: he/she knows if the car is a peach or a lemon and can reveal us this piece of info at a price to be negotiated.

What is the max amount it is rational to pay?

In other words, what is the *indifference price*, below which we must accept, above which we must refuse, at which we can indifferently accept or refuse?

Again: we must accept/refuse if we want to be rational, i.e. utility maximizers.

The answer is: the indifference price is the difference between the expected utility of the game played paying for information and the no-information game already seen. This is the value added by buying info.

We call it the *Expected Value of Perfect Information*.

(Perfect because we can know the truth, without uncertainty)

Let us build a second decision tree, having the event node as root, because now the first step is observing, the second deciding.

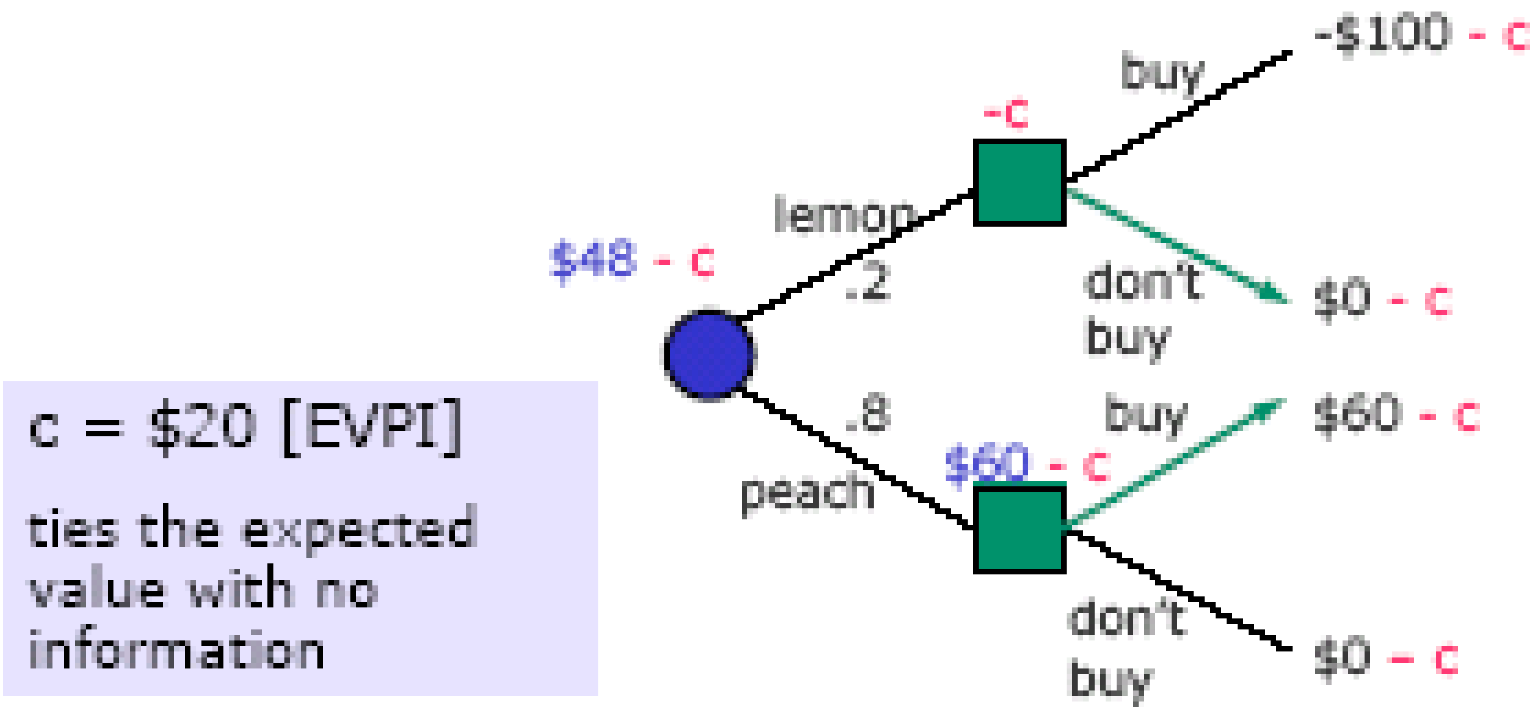
Being c the cost of information, the value added is

$$(48 - c) - 28$$

the difference between the new game value and the old one.

We are indifferent when *c = 20*.

The indifference price is 20.

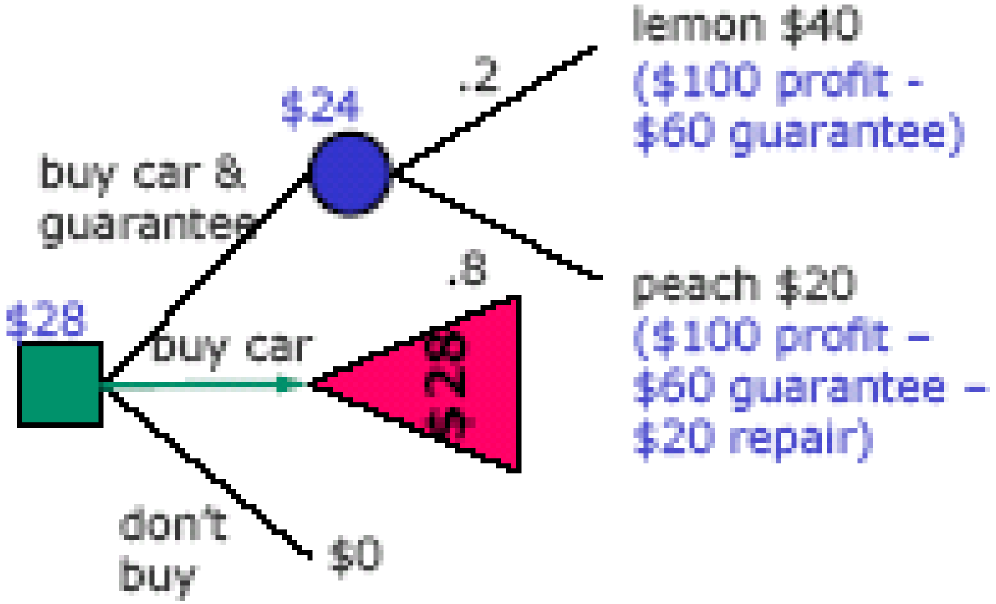


We have the option of buying a guarantee.
Its price is 60 $.
It covers the full price of repairing if greater or equal to 100 $, the half part if lesser.

Now we have 3 alternatives.
The red sub-tree is already evaluated to 28 $.

If we are both rational and risk neutral, we have to buy the car without the guarantee.

- Costs $60
- Covers 50% of repair costs
- If repairs > $100, covers them all



Most people would buy the guarantee in this situation, due to risk aversion.

Now, imagine a worker offer us to carry out a test.
The test is not perfect: a peach car can fail (false negative outcome) as well as a lemon car can succeed (false positive outcome).
The price of the test is 9 $.
In order to make an informed decision we have to know probabilities.
Then, we'll be able to use the Bayes rule.

The symbol | stays for conditioned probability
H is an hypothesis (peach or lemon)
D is evidence (pass or fail)
The Bayes rule is

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

We can have the car inspected for $9

P("pass" | peach) = 0.9 P("fail" | peach) = 0.1

P("pass" | lemon) = 0.4 P("fail" | lemon) = 0.6

P("pass") =

P("pass" | lemon)P(lemon) + P("pass" | peach)P(peach)

P("pass") = 0.4*0.2 + 0.9*0.8 = 0.8

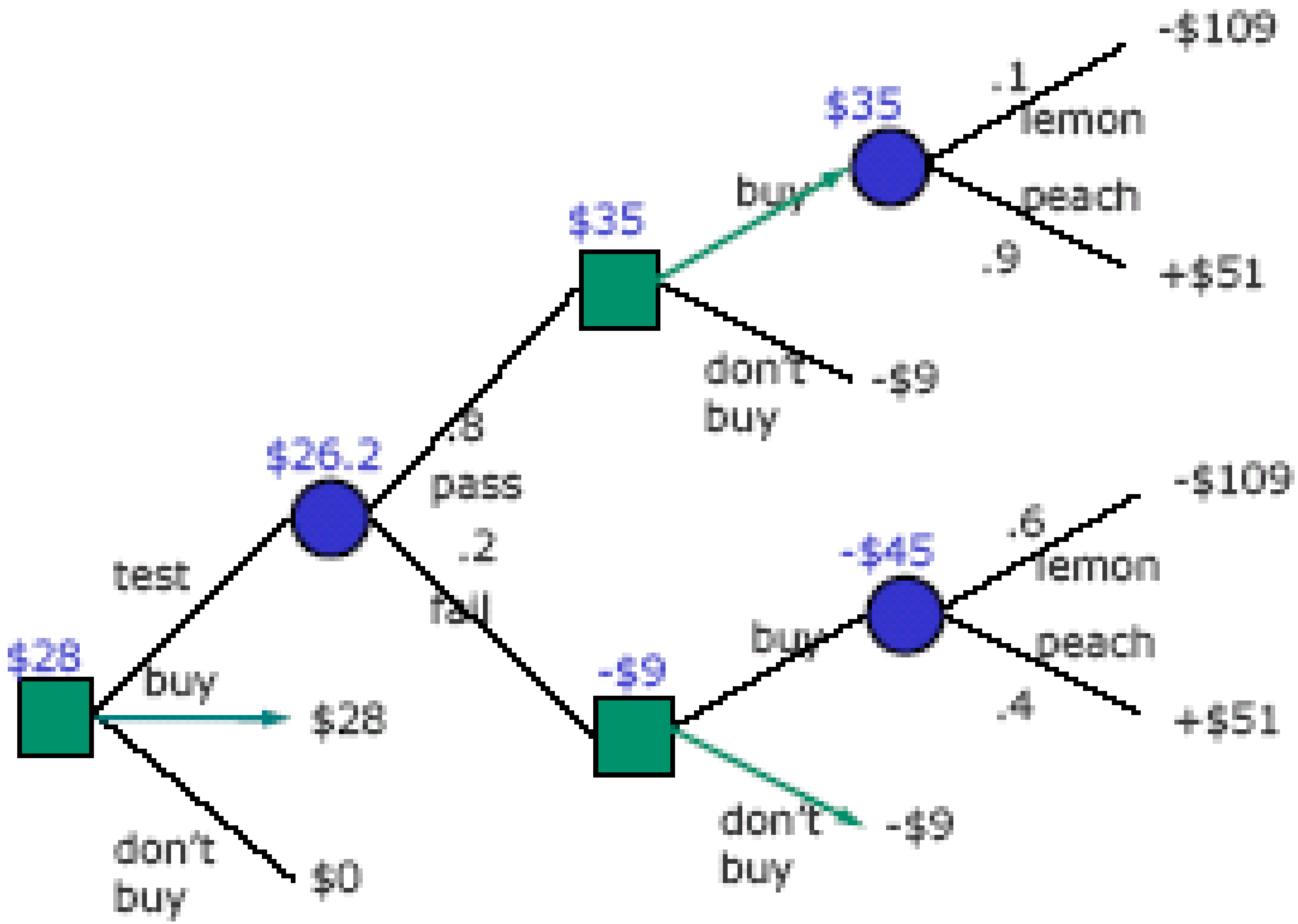
P("fail") = 0.2

P(lemon | "pass") = P("pass" | lemon) P(lemon)/P("pass")

P(lemon | "pass") = 0.4 * (0.2 / 0.8) = 0.1

P(lemon | "fail") = P("fail" | lemon) P(lemon)/P("fail")

P(lemon | "fail") = 0.6 * (0.2 / 0.2) = 0.6



The best decision is buying without testing.