Master Program in *Data Science and Business Informatics*

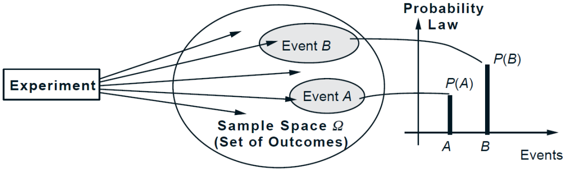
Statistics for Data Science

Lesson 04 - Discrete random variables

Salvatore Ruggieri

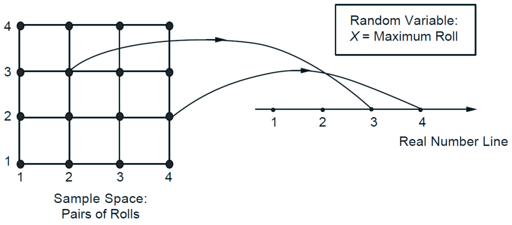
Department of Computer Science
University of Pisa, Italy
salvatore.ruggieri@unipi.it

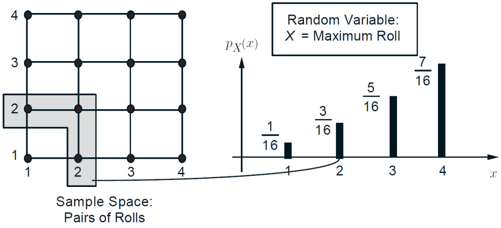
Experiments



- **Experiment**: roll two independent 4 sided die.
- We are interested in probability of the *maximum of the two rolls*.
- Modeling so far
  - $\Omega = \{1,2,3,4\} \times \{1,2,3,4\} = \{(1,1),(1,2),(1,3),(1,4),(2,1),\ldots,(4,4)\}$
  - $A = \{\text{maximum roll is } 2\} = \{(1,2),(2,1),(2,2)\}$
  - $P(A) = P(\{(1,2),(2,1),(2,2)\}) = {}^{3}/_{16}$

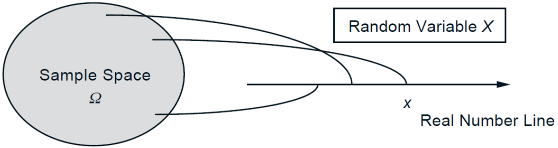
Random variables





- Modeling $X : \Omega \to \mathbb{R}$
  - $X((a, b)) = max(a, b)$
  - $A = \{\text{maximum roll is } 2\} = \{(a, b) \in \Omega \mid X((a, b)) = 2\} = X^{-1}(2)$
  - $P(A) = P(X^{-1}(2)) = 3/16$
  - We write $P_X(X = 2) \stackrel{\text{def}}{=} P(X^{-1}(2))$ *[Induced probabiity]*

(Discrete) Random variables



- A random variable is a function $X : \Omega \rightarrow \mathbb{R}$
  - it transforms Ω into a more tangible sample space $\mathbb{R}$
    - from (a, b) to $min(a, b)$
  - it decouples the details of a specific Ω from the probability of events of interest
    - from $\Omega = \{\text{H, T}\}$ or $\Omega = \{\text{good, bad}\}$ or $\Omega = \ldots$ to $\{0, 1\}$
  - it is not 'random' nor 'variable'

DEFINITION. Let Ω be a sample space. A *discrete random variable* is a function $X : \Omega \rightarrow \mathbb{R}$ that takes on a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$.

Probability Mass Function (PMF)

DEFINITION. The *probability mass function* p of a discrete random variable X is the function $p : \mathbb{R} \to [0, 1]$, defined by

$$p(a) = \mathrm{P}(X = a) \quad \text{for } -\infty < a < \infty.$$

- Support or domain of X is $dom(X) = \{a \in \mathbb{R} \mid P(X = a) > 0\} = \{a_1, a_2, \ldots a_i, \ldots\}$
  - $p(a_i) > 0$ for $i = 1, 2, \ldots$
  - $p(a_1) + p(a_2) + \ldots = 1$
  - $p(a) = 0$ if $a \notin dom(X)$

Cumulative Distribution Function (CDF) and CCDF

DEFINITION. The ***distribution function*** F of a random variable X is the function $F : \mathbb{R} \to [0,1]$, defined by

$$F(a) = \mathrm{P}(X \leq a) \quad \text{for } -\infty < a < \infty.$$

- $F(a) = P(X \in \{a_i \mid a_i \leq a\}) = P(X \leq a) = \sum_{a_i \leq a} p(a_i)$
- if $a \leq b$ then $F(a) \leq F(b)$ *[Non-decreasing]*
- $P(a < X \leq b) = F(b) - F(a) = \sum_{a < a_i \leq b} p(a_i)$

Complementary cumulative distribution function (CCDF)

$$\bar{F}(a) = P(X > a) = 1 - P(X \leq a) = 1 - F(a)$$

- $\bar{F}(a) = P(X \in \{a_i \mid a_i > a\}) = P(X > a) = \sum_{a_i > a} p(a_i)$

See R script

$X \sim U(m, M)$

Uniform discrete distribution

A discrete random variable X has the *uniform distribution* with parameters $m, M \in \mathbb{Z}$ such that $m \leq M$, if its pmf is given by

$$p(a) = \frac{1}{M - m + 1} \quad \text{for } a = m, m+1, \ldots, M$$

We denote this distribution by $U(m, M)$.

- **Intuition:** all integers in $[m, M]$ have equal chances of being observed.

$$F(a) = \frac{\lfloor a \rfloor - m + 1}{M - m + 1} \quad \text{for } m \leq a \leq M$$

- **Example:** classic 6-faces (fair) die ($m = 1,\ M = 6$)

See R script

$X \sim Ben$

Benford's law

A discrete random variable X has the *Benford's distribution*, if its pmf is given by

$$p(a) = \log_{10}\left(1 + \frac{1}{a}\right) \quad \text{for } a = 1, 2, \ldots, 9$$

We denote this distribution by *Ben*.

- X models the frequency distribution of leading digits in many real-life numerical datasets.
- **Example:** leading digits of 2^n
- See **Wikipedia** for its interesting history and applications!

See R script

$X \sim Ber(p)$

DEFINITION. A discrete random variable X has a *Bernoulli distribution* with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(1) = \mathrm{P}(X = 1) = p \quad \text{and} \quad p_X(0) = \mathrm{P}(X = 0) = 1 - p.$$

We denote this distribution by $Ber(p)$.

- X models success/failure
- **Example:** getting head (H,T) when tossing a coin, testing for a disease (infected, not infected), membership in a set (member, non-member), etc.
- p_X is the *pmf* (to distinguish from parameter p)
- Alternative definition: $p_X(a) = p^a \cdot (1-p)^{1-a}$ for $a \in \{0, 1\}$

See R script

i.d. random variables

Identically distributed random variables

Two random variables X and Y are said *identically distributed* (in symbols, $X \sim Y$), if $F_X = F_Y$, i.e.,

$$F_X(a) = F_Y(a) \quad \text{for } a \in \mathbb{R}$$

- Identically distributed does **not** mean equal
- Toss a fair coin
  - let X be 1 for H and 0 for T
  - let Y be $1 - X$
- $X \sim Ber(0.5)$ and $Y \sim Ber(0.5)$
- Thus, $X \sim Y$ but are clearly always different.

Joint p.m.f.

- For a same Ω, several random variables can be defined
  - Random variables related to the same experiment often influence one another
  - $\Omega = \{(i,j) \mid i,j \in 1,\ldots,6\}$ rolls of two dies
    - $X((i,j)) = i + j$ and $Y((i,j)) = max(i,j)$
    - $P(X = 4, Y = 3) = P(X^{-1}(4) \cap Y^{-1}(3)) = P(\{(3,1),(1,3)\}) = 2/36$
- In general:

$$P_{XY}(X = a, Y = b) = P(\{\omega \in \Omega \mid X(\omega) = a \text{ and } Y(\omega) = b\}) = P(X^{-1}(a) \cap Y^{-1}(b))$$

DEFINITION. The *joint probability mass function* p of two discrete random variables X and Y is the function $p : \mathbb{R}^2 \to [0,1]$, defined by

$$p(a,b) = \mathrm{P}(X = a, Y = b) \quad \text{for } -\infty < a, b < \infty.$$

Joint and marginal p.m.f.

- **Joint distribution function** $F : \mathbb{R} \times \mathbb{R} \to [0,1]$:

$$F_{XY}(a,b) = P(X \leq a, Y \leq b) = \sum_{a_i \leq a, b_i \leq b} p(a_i, b_i)$$

- By generalized additivity, the **marginal p.m.f.**'s can be derived: *[Tabular method]*

$$p_X(a) = P_X(X = a) = \sum_b P_{XY}(X = a, Y = b) \quad p_Y(b) = P_Y(Y = b) = \sum_a P_{XY}(X = a, Y = b)$$

and the marginal distribution function of X as:

$$F_X(a) = P_X(X \leq a) = \lim_{b \to \infty} F_{XY}(a,b) \qquad F_Y(b) = P_Y(Y \leq b) = \lim_{a \to \infty} F_{XY}(a,b)$$

- Deriving the joint p.m.f. from marginal p.m.f.'s is not always possible!
  - **Exercise at home.** Prove it (hint: find two joint p.m.f.'s with the same marginals)
- Deriving the joint p.m.f. from marginal p.m.f.'s is possible for independent events!
  - $\Omega = \{1,2,3,4\} \times \{1,2,3,4\}$, $X((a,b)) = a$, $Y((a,b)) = b$
  - $P(X = 1, Y = 2) = 1/16 = 1/4 \cdot 1/4 = P(X = 1) \cdot P(Y = 2)$

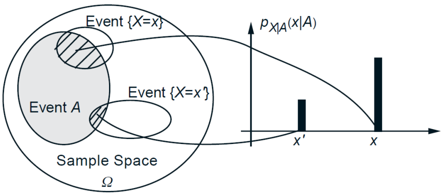
See R script

Conditional distribution

Conditional distribution

Consider the joint distribution P_{XY} of X and Y. The conditional distribution of X given $Y \in B$ with $P_Y(Y \in B) > 0$, is the function $F_{X|Y\in B} : \mathbb{R} \to [0, 1]$:

$$F_{X|Y\in B}(a) = P_{X|Y}(X \le a|Y \in B) = \frac{P_{XY}(X \le a, Y \in B)}{P_Y(Y \in B)} \quad \text{for} \quad -\infty < a < \infty$$



- Distribution of X after knowing $Y \in B$.
- Chain rule: $P_{XY}(X \le a, Y \in B) = P_{X|Y}(X \le a|Y \in B)P_Y(Y \in B)$
- What if the distribution does not change w.r.t. the prior P_X?

(Machine Learning) Binary Classifiers

- $\Omega = \{\text{f}, \text{m}\} \times \mathbb{N} \times \{+, -\}$
- Predictive Features and True-Class as Random Variables:
  - gender: $G((g, a, c)) = 1$ if g is f and 0 otherwise
  - age: $A((g, a, c)) = a$
  - has-covid: $Y((g, a, c)) = 1$ if $c = +$ and 0 otherwise
- Binary Classifier as a Random Variable:
  - $\hat{Y}((g, a, c)) = 1$ if $clf((g, a)) = +$ and 0 otherwise
    where $clf : \{\text{f}, \text{m}\} \times \mathbb{N} \rightarrow \{+, -\}$ is a function over predictive features
- $P(Y = \hat{Y})$, i.e., $P(\{\omega \in \Omega \mid Y(\omega) = \hat{Y}(\omega)\})$ *[True Accuracy]*
- $P(Y = 1 | \hat{Y} = 1)$ *[True Precision]*
- $P(\hat{Y} = 1 | Y = 1)$ *[True Recall]*
- Such probabilities are unknown! They can only be estimated on a sample (*test set*)

Independence of two random variables

Independence $X \perp\!\!\!\perp Y$

A random variable X is independent from a random variable Y, if for all $P_Y(Y \leq b) > 0$:

$$P_{X|Y}(X \leq a | Y \leq b) = P_X(X \leq a) \quad \text{for } -\infty < a < \infty$$

- Properties
  - $X \perp\!\!\!\perp Y$ iff $P_{XY}(X \leq a, Y \leq b) = P_X(X \leq a) \cdot P_Y(Y \leq b) \quad \text{for } -\infty < a, b < \infty$
  - $X \perp\!\!\!\perp Y$ iff $Y \perp\!\!\!\perp X$ *[Symmetry]*
- For X, Y **discrete** random variables:
  - $X \perp\!\!\!\perp Y$ iff $P_{XY}(X = a, Y = b) = P_X(X = a) \cdot P_Y(Y = b) \quad \text{for } -\infty < a, b < \infty$
  - **Exercise at home.** Prove it!
  - $X \perp\!\!\!\perp Y$ iff $P_{XY}(X \in A, Y \in B) = P_X(X \in A) \cdot P_Y(Y \in B) \quad \text{for } A, B \subseteq \mathbb{R}$
  - **Exercise at home.** Prove it!

See R script

Sum of independent discrete random variables

ADDING TWO INDEPENDENT DISCRETE RANDOM VARIABLES. Let X and Y be two independent discrete random variables, with probability mass functions p_X and p_Y. Then the probability mass function p_Z of $Z = X + Y$ satisfies

$$p_Z(c) = \sum_j p_X(c - b_j) p_Y(b_j),$$

where the sum runs over all possible values b_j of Y.

- **Proof (sketch).**

$$\begin{aligned} P(Z = c) &= \sum_j P(Z = c | Y = b_j) \cdot P(Y = b_j) \\ &= \sum_j P(X = c - b_j | Y = b_j) \cdot P(Y = b_j) \\ &= \sum_j P(X = c - b_j) P(Y = b_j) \end{aligned}$$

Independence of multiple random variables

Independence (factorization formula)

Random variables $X_1, \ldots, X_n$ are independent, if:

$$P_{X_1,\ldots,X_n}(X_1 \leq a_1, \ldots, X_n \leq a_n) = \prod_{i=1}^{n} P_{X_i}(X_i \leq a_i) \quad \text{for } -\infty < a_1, \ldots, a_n < \infty$$

- $X_1, \ldots, X_n$ **discrete** random variables are independent iff:

$$P_{X_1,\ldots,X_n}(X_1 = a_1, \ldots, X_n = a_n) = \prod_{i=1}^{n} P_{X_i}(X_i = a_i) \quad \text{for } -\infty < a_1, \ldots, a_n < \infty$$

- **Definition:** $X_1, \ldots, X_n$ are **i.i.d.** (independent and identically distributed) if $X_1, \ldots, X_n$ are independent and $X_i \sim F$ for $i = 1, \ldots, n$ for some distribution F

$X \sim Bin(n, p)$

DEFINITION. A discrete random variable X has a *binomial distribution* with parameters n and p, where $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(k) = \mathrm{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

We denote this distribution by $Bin(n, p)$.

- X models the number of successes in n Bernoulli trials (How many H's when tossing n coins?)
- **Intuition**: for $X_1, X_2, \ldots, X_n$ such that $X_i \sim Ber(p)$ and independent (**i.i.d.**):

$$X = \sum_{i=1}^{n} X_i \sim Bin(n, p)$$

- $p^k \cdot (1-p)^{n-k}$ is the probability of observing first k H's and then $n-k$ T's
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ number of ways to choose the first k variables *[Binomial coefficient]*
- $p_X(k)$ computationally expensive to calculate (no closed formula, but approximation/bounds)
- **Exercise at home.** Prove $X_1 + X_2 \sim Bin(2, p)$ using the sum of independent random variables.

See R script

$X \sim Bin(n, p)$

DEFINITION. A discrete random variable X has a *binomial distribution* with parameters n and p, where $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(k) = \mathrm{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

We denote this distribution by *$Bin(n, p)$*.

- **Exercise**: there are c bikes shared among n persons. Assuming that each person needs a bike with probability p, what is the probability that all bikes will be in use?

$$P(X = c) = \binom{n}{c} p^c \cdot (1-p)^{n-c} = \texttt{dbinom(c-1, n, p)}$$

$X \sim Geo(p)$

DEFINITION. A discrete random variable X has a *geometric distribution* with parameter p, where $0 < p \leq 1$, if its probability mass function is given by

$$p_X(k) = \mathrm{P}(X = k) = (1-p)^{k-1} p \quad \text{for } k = 1, 2, \ldots .$$

We denote this distribution by $Geo(p)$.

- X models the number of Bernoulli trials before a success (how many tosses to have a H?)
- **Intuition**: for $X_1, X_2, \ldots$ such that $X_i \sim Ber(p)$ i.i.d.:

$$X = min_i\,(X_i = 1) \sim Geo(p)$$

- $\bar{F}(a) = P(X > a) = (1-p)^{\lfloor a \rfloor}$
- $F(a) = P(X \leq a) = 1 - \bar{F}(a) = 1 - (1-p)^{\lfloor a \rfloor}$

See R script

You cannot always lose

- H is 1, T is 0, $0 < p < 1$
- $B_n = \{$T in the first n-th coin tosses$\}$
- $P(\cap_{n \geq 1} B_i) = ?$
- $X \sim Geom(p)$
- $P(B_n) = P(X > n) = (1 - p)^n$
- $P(\cap_{n \geq 1} B_n) = lim_{n \to \infty} P(B_n) = lim_{n \to \infty} (1 - p)^n = 0$
- $P(\cap_{n \geq 1} B_n) = lim_{n \to \infty} P(B_n)$ for B_n non-increasing *[σ-additivity, see Lesson 01]*

But if you lost so far, you can lose again

Memoryless property

For $X \sim Geo(p)$, and $n, k = 0, 1, 2, \ldots$

$$P(X > n + k | X > k) = P(X > n)$$

Proof

$$\begin{aligned}
P(X > n + k | X > k) &= \frac{P(\{X > n + k\} \cap \{X > k\})}{P(\{X > k\})} \\
&= \frac{P(\{X > n + k\})}{P(\{X > k\})} \\
&= \frac{(1-p)^{n+k}}{(1-p)^k} \\
&= (1-p)^n = P(X > n)
\end{aligned}$$

Sum of independent random variables (repetita iuvant)

ADDING TWO INDEPENDENT DISCRETE RANDOM VARIABLES. Let X and Y be two independent discrete random variables, with probability mass functions p_X and p_Y. Then the probability mass function p_Z of $Z = X + Y$ satisfies

$$p_Z(c) = \sum_j p_X(c - b_j)p_Y(b_j),$$

where the sum runs over all possible values b_j of Y.

- Examples:
  - For $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$, $Z \sim Bin(n + m, p)$
  - For $X \sim Geo(p)$ (days radio 1 breaks) and $Y \sim Geo(p)$ (days radio 2 breaks):

$$p_Z(X + Y = k) = \sum_{l=1}^{k-1} p_X(l) \cdot p_Y(k - l) = (k - 1)p^2(1 - p)^{k-2}$$

$X \sim NBin(n, p)$

Negative binomial (or Pascal distribution)

A discrete random variable X has a negative binomial with parameters n and p, where $n = 0, 1, 2, \ldots$ and $0 < p \leq 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k+n-1}{k}(1-p)^k \cdot p^n \quad \text{for } k = 0, 1, 2, \ldots$$

- X models the number of failures before the n-th success in Bernoulli trials (how many T's to have n H's?)
- **Intuition**: for $X_1, X_2, \ldots, X_n$ such that $X_i \sim Geo(p)$ i.i.d.:

$$X = \sum_{i=1}^{n} X_i - n \sim NBin(n, p)$$

- $(1-p)^k \cdot p^n$ is the probability of observing first k T's and then n H's
- $\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$ number of ways to choose the first k variables among $k+n-1$ (the last one must be a success!)

See R script

$X \sim Poi(\mu)$

Definition. A discrete random variable X has a *Poisson distribution* with parameter μ, where $\mu > 0$ if its probability mass function p is given by

$$p(k) = \mathrm{P}(X = k) = \frac{\mu^k}{k!} \mathrm{e}^{-\mu} \quad \text{for } k = 0, 1, 2, \ldots.$$

We denote this distribution by $Pois(\mu)$.

- X models the number of events in a fixed interval if these events occur with a known constant mean rate μ and independently of the last event
  - telephone calls arriving in a system
  - number of patients arriving at an hospital
  - customers arriving at a counter
- μ denotes the mean number of events
- $Bin(n, \mu/n)$ is the number of successes in n trials, assuming $p = \mu/n$, i.e., $p \cdot n = \mu$
- When $n \to \infty$: $Bin(n, \mu/n) \to Poi(\mu)$ *[Law of rare events]*
  - Number of typos in a book, number of cars involved in accidents, etc.

See R script

The discrete Bayes' rule

BAYES' RULE. Suppose the events $C_1, C_2, \ldots, C_m$ are disjoint and $C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$. The conditional probability of C_i, given an arbitrary event A, can be expressed as:

$$\mathrm{P}(C_i \mid A) = \frac{\mathrm{P}(A \mid C_i) \cdot \mathrm{P}(C_i)}{\mathrm{P}(A \mid C_1)\mathrm{P}(C_1) + \mathrm{P}(A \mid C_2)\mathrm{P}(C_2) + \cdots + \mathrm{P}(A \mid C_m)\mathrm{P}(C_m)}.$$

Definition. Conditional p.m.f. of X given $Y = b$ with $P_Y(Y = b) > 0$

$$p_{X|Y}(a|b) = \frac{p_{XY}(a,b)}{p_Y(b)} \quad \text{i.e.,} \quad P_{X|Y}(X = a|Y = b) = \frac{P_{XY}(X = a, Y = b)}{P_Y(Y = b)}$$

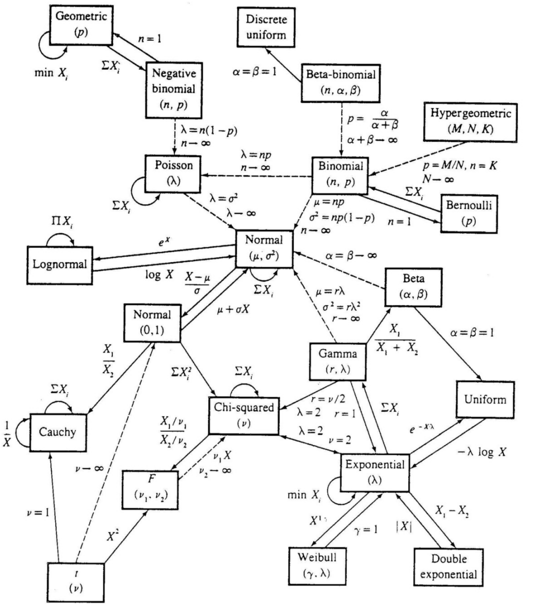
Discrete Bayes' rule:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_{a \in dom(X)} p_{Y|X}(y|a)p_X(a)}$$

Exercise at home. A machine fails after n days with a p.m.f. $X \sim Geo(p)$. p is known to be either $p = 0.1$ or 0.05 with equal probability. What can we say about the distribution of p given n? Code your solution in R.

Common distributions

- **Probability distributions at Wikipedia**
- **Probability distributions in R**
- C. Forbes, M. Evans,
  N. Hastings, B. Peacock (2010)
  Statistical Distributions, 4th Edition
  Wiley



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).