Exam of *Statistics for Data Science*

It is forbidden to consult any material during the test, with the exception of a scientific calculator. Duration of written exam is 1.5h.

Exercise 1 (6 points). Let X and Y be two independent random variables, where $X \sim N(2,5)$ and $Y \sim N(5,9)$. Define $Z = 3X - 2Y + 1$.

a Compute $E[Z]$ and $Var(Z)$.

b What is the distribution of Z?

c Compute $P(Z \leq 6)$.

Solution. See full solution of Ex. 11.4 (a,b,c) at page 456 of [1].

Exercise 2 (6 points). Let us generate a number x from a uniform distribution on the interval $[0, \theta]$. Let us now decide to test $H0 : \theta = 2$ against $H1 : \theta = 2.5$ by rejecting $H0$ if $x \leq 0.1$ or $x \geq 1.9$.

a Compute the probability of committing a type I error.

b Compute the probability of committing a type II error if the true value of θ is 2.5.

Solution. The text is from Ex. 26.3 of [1]. Here there are the solutions:

a A type I error is done when $H0$ is true but it is rejected. Hence it is the probability that $x \leq 0.1$ or $x \geq 1.9$. Given that the distribution is uniform this probability is $P(X \leq 0.1) + P(X \geq 1.9) = (0.1)/2 + (0.1)/2 = 0.1$ where $X \sim U(0,2)$.

b A type II error is done if the true value of θ is 2.5 but $H0$ is not rejected. Thus it is $P(0.1 \leq X \leq 1.9) = (1.9 - 0.1)/2.5 = 0.72$, where $X \sim U(0, 2.5)$.

Exercise 3 (6 points).

27.3 Table 27.5 lists the results of tensile adhesion tests on 22 U-700 alloy specimens. The data are loads at failure in MPa. The sample mean is 13.71 and the sample standard deviation is 3.55. You may assume that the data originated from a normal distribution with expectation μ. One is interested in whether the load at failure exceeds 10 MPa. We investigate this by means of a t-test for the null hypothesis $H_0 : \mu = 10$.

a. What do you choose as the alternative hypothesis?

b. Compute the value of the test statistic and report your conclusion, when performing the test at level 0.05.

Table 27.5. Loads at failure of U-700 specimens.

19.8	18.5	17.6	16.7	15.8
15.4	14.1	13.6	11.9	11.4
11.4	8.8	7.5	15.4	15.4
19.5	14.9	12.7	11.9	11.4
10.1	7.9			

Solution. (a) a one-tailed alternative hypothesis $H_1 : \mu > 10$ because we are interested in whether the load at failure exceeds 10 MPa. (b) Since we assume normality, but we do not know the variance, we use a one-tailed t-test with test statistics:

$$T = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim t(n-1) \text{ where } S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

With given data, the t-vaue is:

$$t = \frac{13.71 - 10}{3.55/\sqrt{22}} = 4.901815$$

The one-sided right critical value at $\alpha = 0.05$ for $m = n - 1 = 21$ degrees of freedom is $t_{m,\alpha} = 1.721$ (see Table B.2). Since $t > t_{m,\alpha}$, the null hypothesis H_0 can be rejected in favor of the alternative hypothesis H_1 at the significance level of α.

Exercise 4 (6 points). Discuss the multiple comparison problem for independent statistical tests, and possible correction methods.

Solution. Consider a statistical test with null hypothesis H_0 at a significance level α. The α is the Type I error, namely the probability of rejecting H_0 when it is actually true. By performing multiple tests, say m, the probability at least one Type I error is: $1 - (1 - \alpha)^m$ (one minus the probability that all of the null hypothesis are not rejected). Such a function grows exponentially with m, leading to the multiple comparison problem: falsely rejecting at least one null hypothesis in a series of tests. In such a formula, we are assuming independence of the test precisely in the calculation of $(1 - \alpha)^m$. The Family-wise Error Rate (FWER) is $\alpha_{FWER} = 1 - (1 - \alpha)^m$ and it should be controlled for. This can be done:

- by scaling the significance level at $\alpha' = 1 - (1 - \alpha)^{1/m}$ in such a way that $\alpha'_{FWER} = 1 - (1 - \alpha')^m = \alpha$, i.e., by setting α' the inverse of α_{FWER} in such a way that the resulting $FWER$ will be the desired α;
- or, by scaling the p-values: $p \leq \alpha'$ is equivalent to test $1 - (1 - p)^m \leq \alpha$

These correction strategies are known as the *Šidák correction*.

Exercise 5 (6 points). Code in R the calculation of bootstrapped confidence intervals for the test of mean using the library `boot`.

Solution.

```
data=rnorm(20) # example input
# actual answer
library(boot)
b <- boot(data, function(x,d) mean(x[d]), R = 1000)
quantile(b$t, c(0.025, 0.975))
# or
boot.ci(b, type = "norm")
```

References

[1] F.M. Dekking, C. Kraaikamp, H.P. Lopuhaä, and L.E. Meester. *A Modern Introduction to Probability and Statistics*. Springer, 2005.

Table B.1. Right tail probabilities $1 - \Phi(a) = \mathrm{P}(Z \geq a)$ for an $N(0, 1)$ distributed random variable Z.

a	0	1	2	3	4	5	6	7	8	9
0.0	5000	4960	4920	4880	4840	4801	4761	4721	4681	4641
0.1	4602	4562	4522	4483	4443	4404	4364	4325	4286	4247
0.2	4207	4168	4129	4090	4052	4013	3974	3936	3897	3859
0.3	3821	3783	3745	3707	3669	3632	3594	3557	3520	3483
0.4	3446	3409	3372	3336	3300	3264	3228	3192	3156	3121
0.5	3085	3050	3015	2981	2946	2912	2877	2843	2810	2776
0.6	2743	2709	2676	2643	2611	2578	2546	2514	2483	2451
0.7	2420	2389	2358	2327	2296	2266	2236	2206	2177	2148
0.8	2119	2090	2061	2033	2005	1977	1949	1922	1894	1867
0.9	1841	1814	1788	1762	1736	1711	1685	1660	1635	1611
1.0	1587	1562	1539	1515	1492	1469	1446	1423	1401	1379
1.1	1357	1335	1314	1292	1271	1251	1230	1210	1190	1170
1.2	1151	1131	1112	1093	1075	1056	1038	1020	1003	0985
1.3	0968	0951	0934	0918	0901	0885	0869	0853	0838	0823
1.4	0808	0793	0778	0764	0749	0735	0721	0708	0694	0681
1.5	0668	0655	0643	0630	0618	0606	0594	0582	0571	0559
1.6	0548	0537	0526	0516	0505	0495	0485	0475	0465	0455
1.7	0446	0436	0427	0418	0409	0401	0392	0384	0375	0367
1.8	0359	0351	0344	0336	0329	0322	0314	0307	0301	0294
1.9	0287	0281	0274	0268	0262	0256	0250	0244	0239	0233
2.0	0228	0222	0217	0212	0207	0202	0197	0192	0188	0183
2.1	0179	0174	0170	0166	0162	0158	0154	0150	0146	0143
2.2	0139	0136	0132	0129	0125	0122	0119	0116	0113	0110
2.3	0107	0104	0102	0099	0096	0094	0091	0089	0087	0084
2.4	0082	0080	0078	0075	0073	0071	0069	0068	0066	0064
2.5	0062	0060	0059	0057	0055	0054	0052	0051	0049	0048
2.6	0047	0045	0044	0043	0041	0040	0039	0038	0037	0036
2.7	0035	0034	0033	0032	0031	0030	0029	0028	0027	0026
2.8	0026	0025	0024	0023	0023	0022	0021	0021	0020	0019
2.9	0019	0018	0018	0017	0016	0016	0015	0015	0014	0014
3.0	0013	0013	0013	0012	0012	0011	0011	0011	0010	0010
3.1	0010	0009	0009	0009	0008	0008	0008	0008	0007	0007
3.2	0007	0007	0006	0006	0006	0006	0006	0005	0005	0005
3.3	0005	0005	0005	0004	0004	0004	0004	0004	0004	0003
3.4	0003	0003	0003	0003	0003	0003	0003	0003	0003	0002

Table B.2. Right critical values $t_{m,p}$ of the t-distribution with m degrees of freedom corresponding to right tail probability p: $\mathrm{P}(T_m \geq t_{m,p}) = p$. The last row in the table contains right critical values of the $N(0,1)$ distribution: $t_{\infty,p} = z_p$.

	Right tail probability p							
m	0.1	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	3.078	6.314	12.706	31.821	63.657	127.321	318.309	636.619
2	1.886	2.920	4.303	6.965	9.925	14.089	22.327	31.599
3	1.638	2.353	3.182	4.541	5.841	7.453	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
∞	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291