COMPLEX NETWORKS

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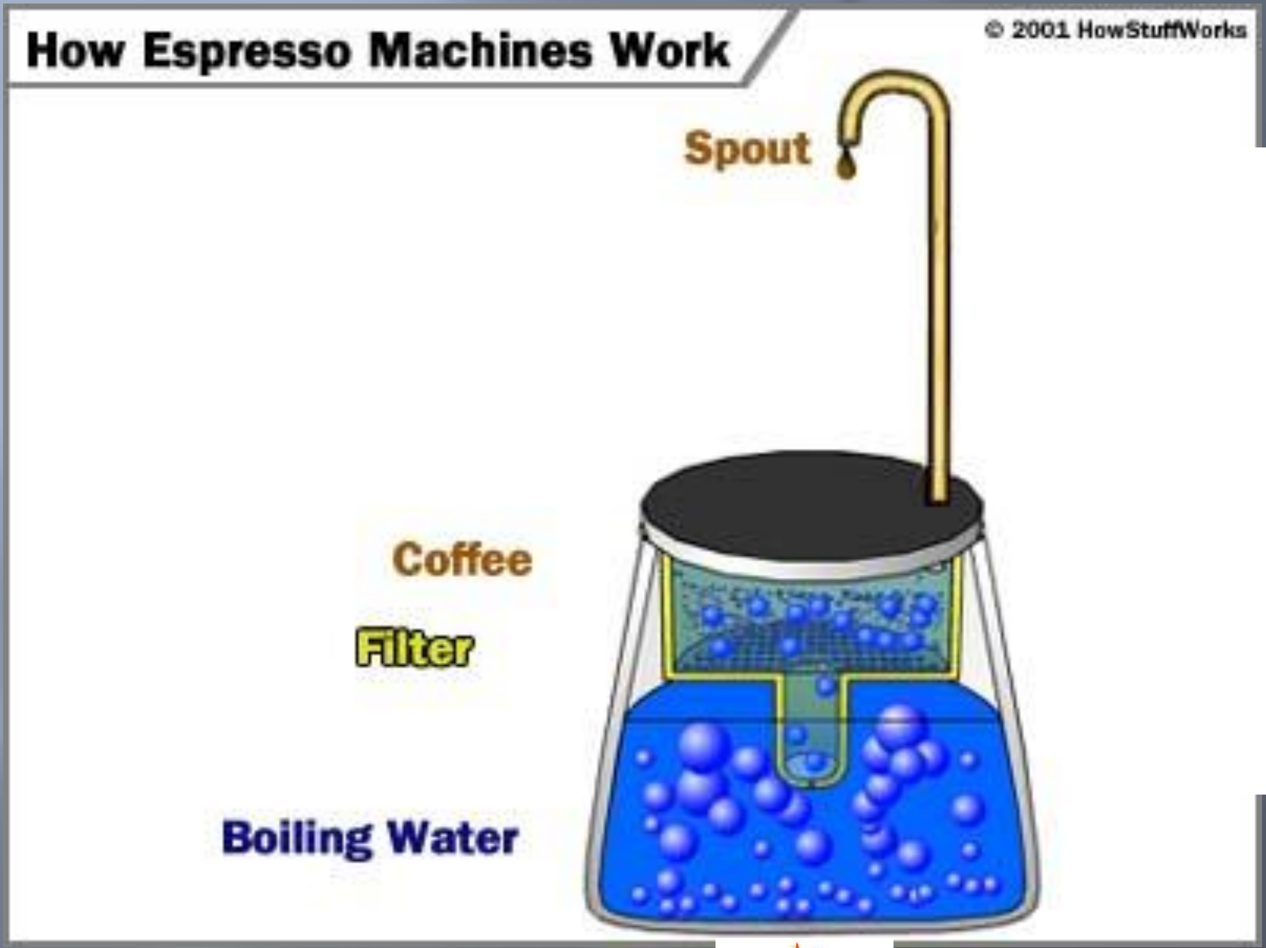
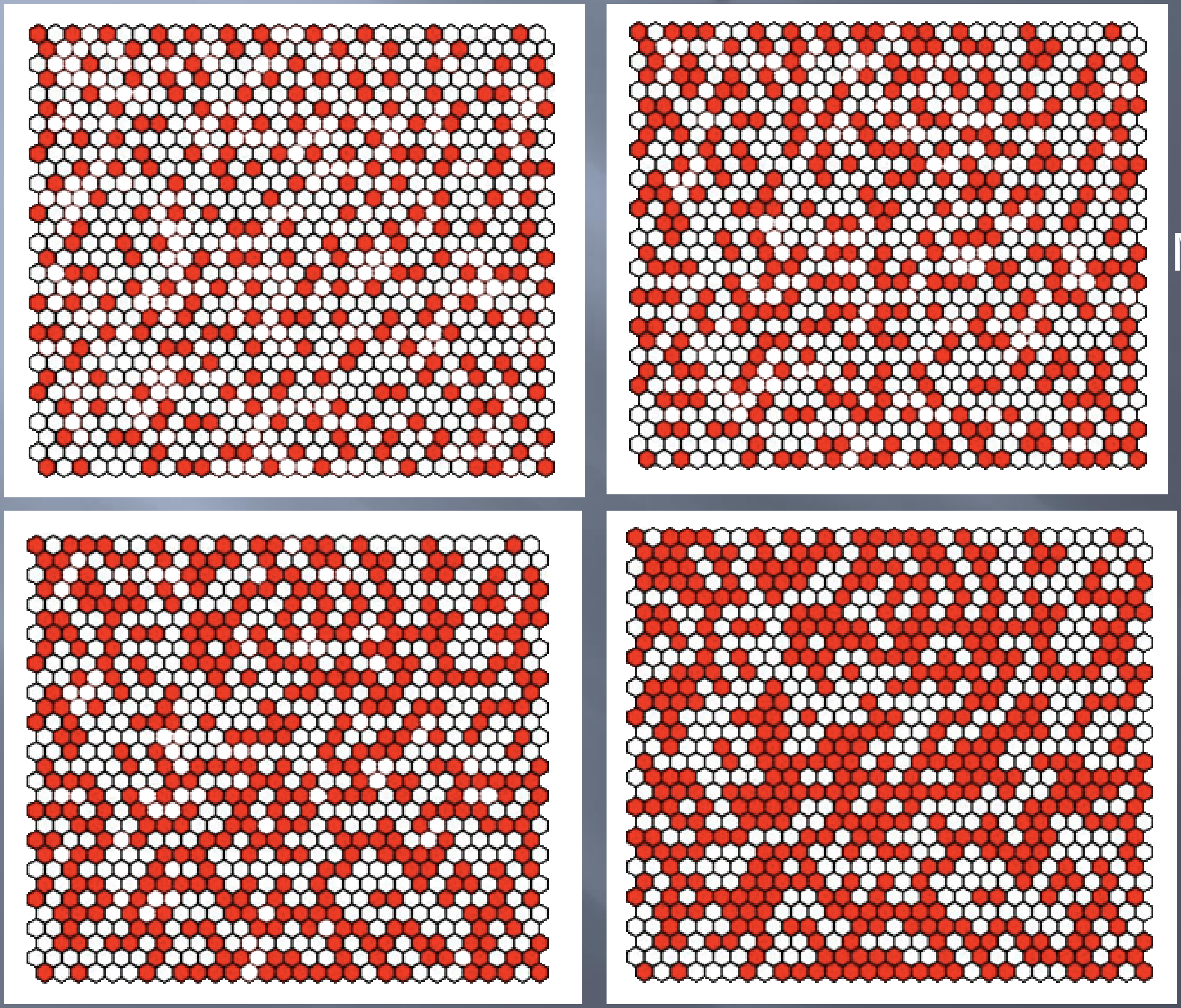
2. PERCOLATION THEORY AND SCALE FREENESS

How does a percolator work?



http://home.howstuffworks.com/espresso-machine2.htm

How does a percolator work?



Increasing density

↓

More ramified path

↓

Better coffee

CAUTION!
Do not pack
too densely!

http://www.tececo.com/technical.particle_packing.php

Percolation transition

There is an “optimal” density , where a path still exists but it is most ramified.

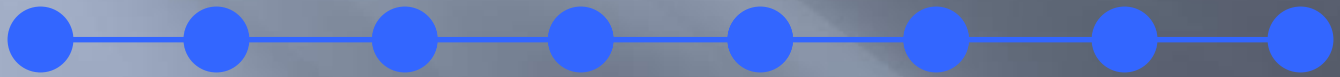
How to model this transition from the state
with a path
to a state
without a path through the sample

Although coffee grains do not sit at the sites of a regular lattice, we will first study percolation on regular lattices

Lattices are simple networks

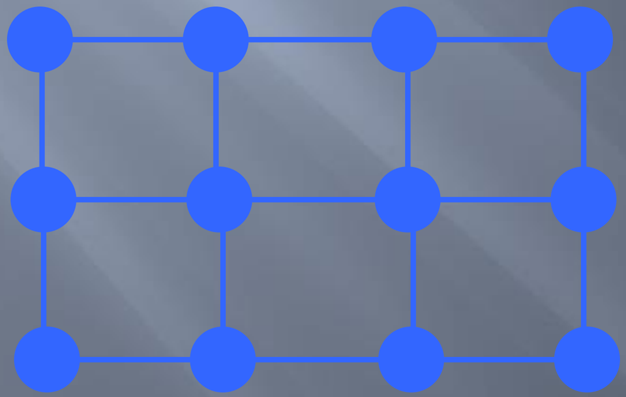
Periodic (crystal) structure:

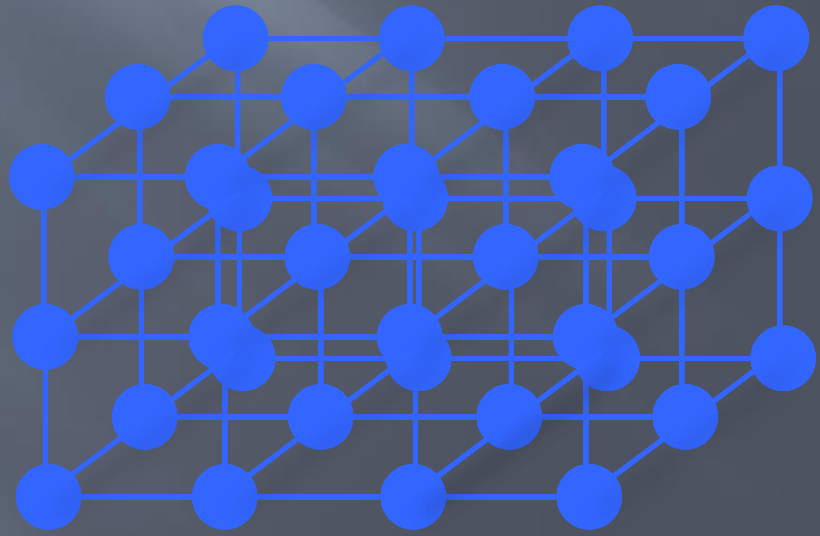
1 dimension:



3 dimensions:

2 dimensions:



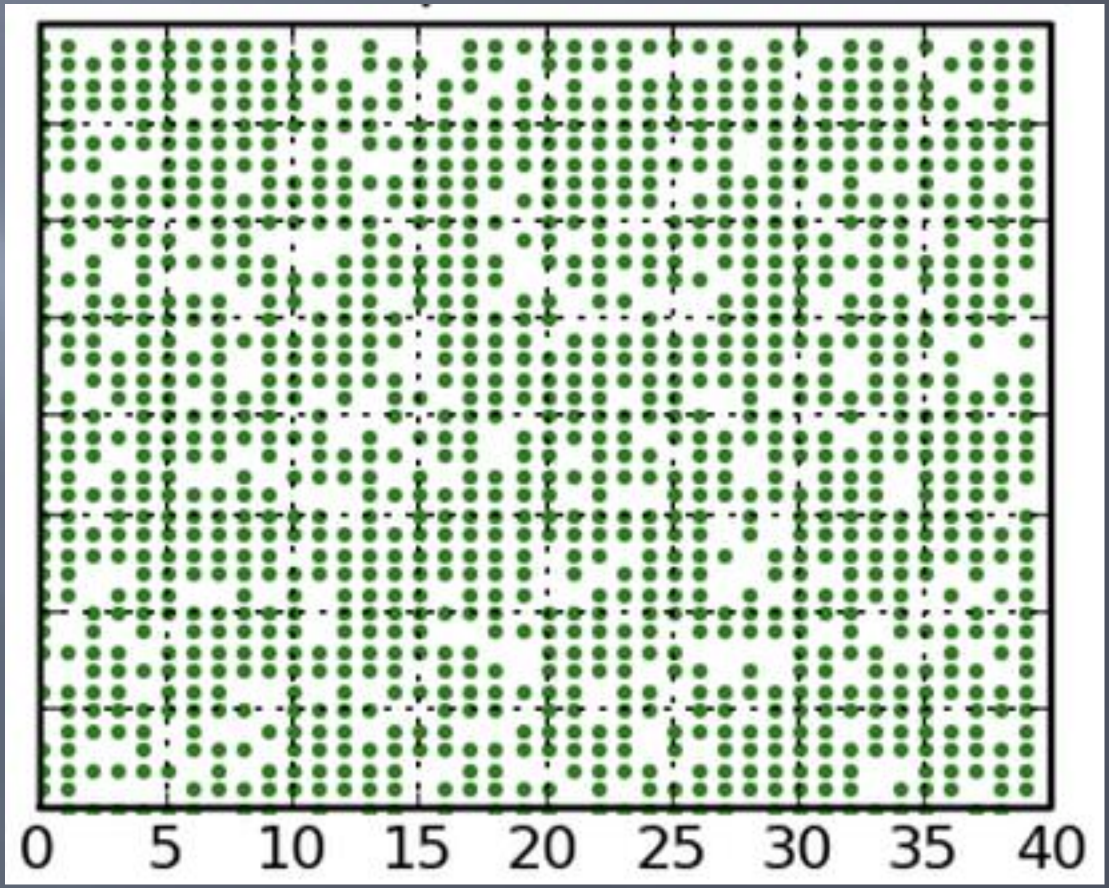


We assume they are very (infinitely) large

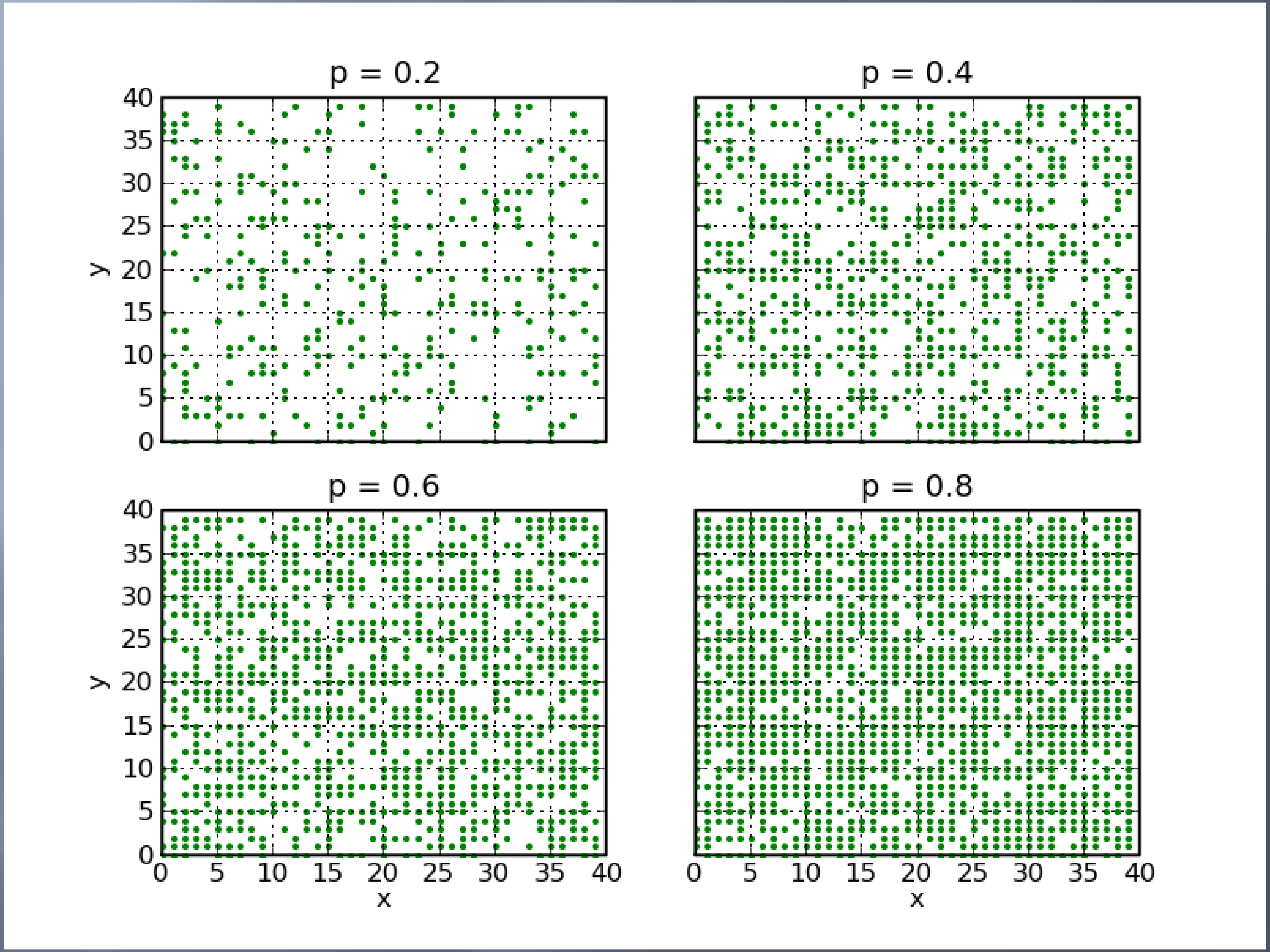
Disorder

40X40 square lattice,20% of the sites randomly removed.

Complementary view: 80 % of the sites present



Disorder



http://www.ippp.dur.ac.uk/compphys/Percolation/Lecture/pe3.html

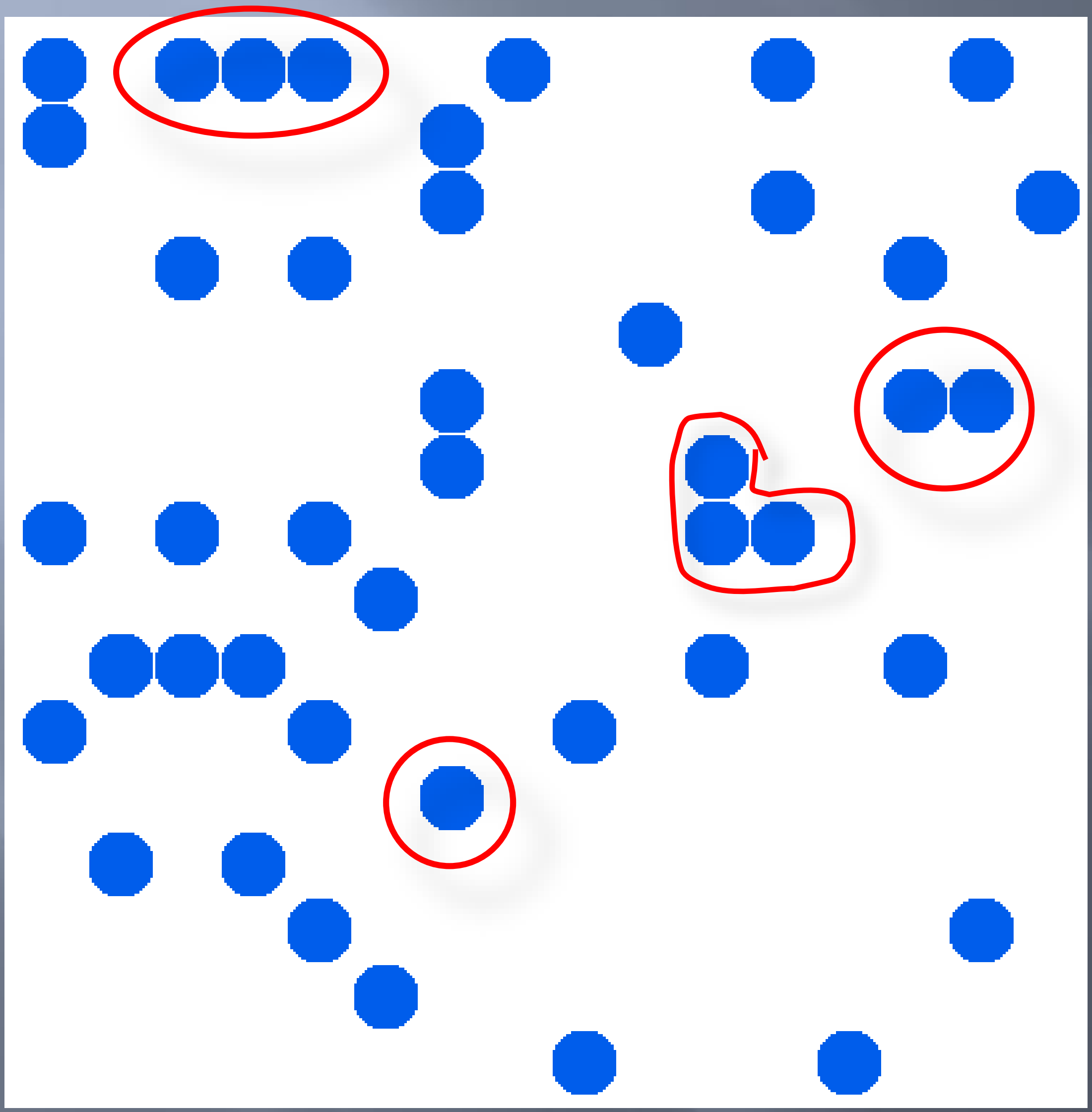
Percolation model

Nodes of an infinite (very large) network can be in two states: Occupied or empty. We occupy nodes with constant, independent probability called occupation probability *p.*

Set of nodes, which can be reached from each other by paths through occupied nodes are called clusters or components.

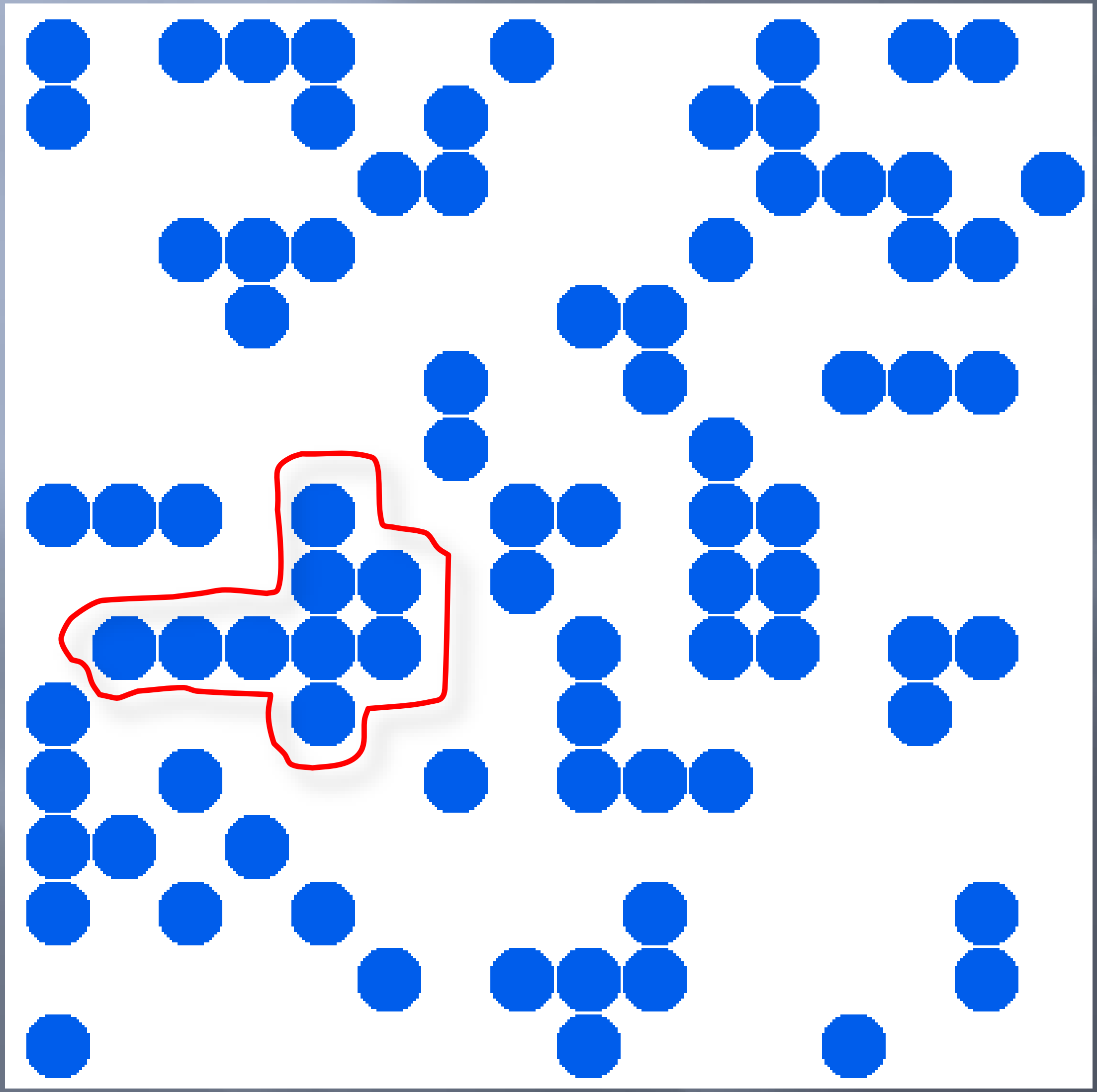
Below a threshold value p_c there is no infinite cluster (component) of occupied nodes, above it there is.

Percolation clusters



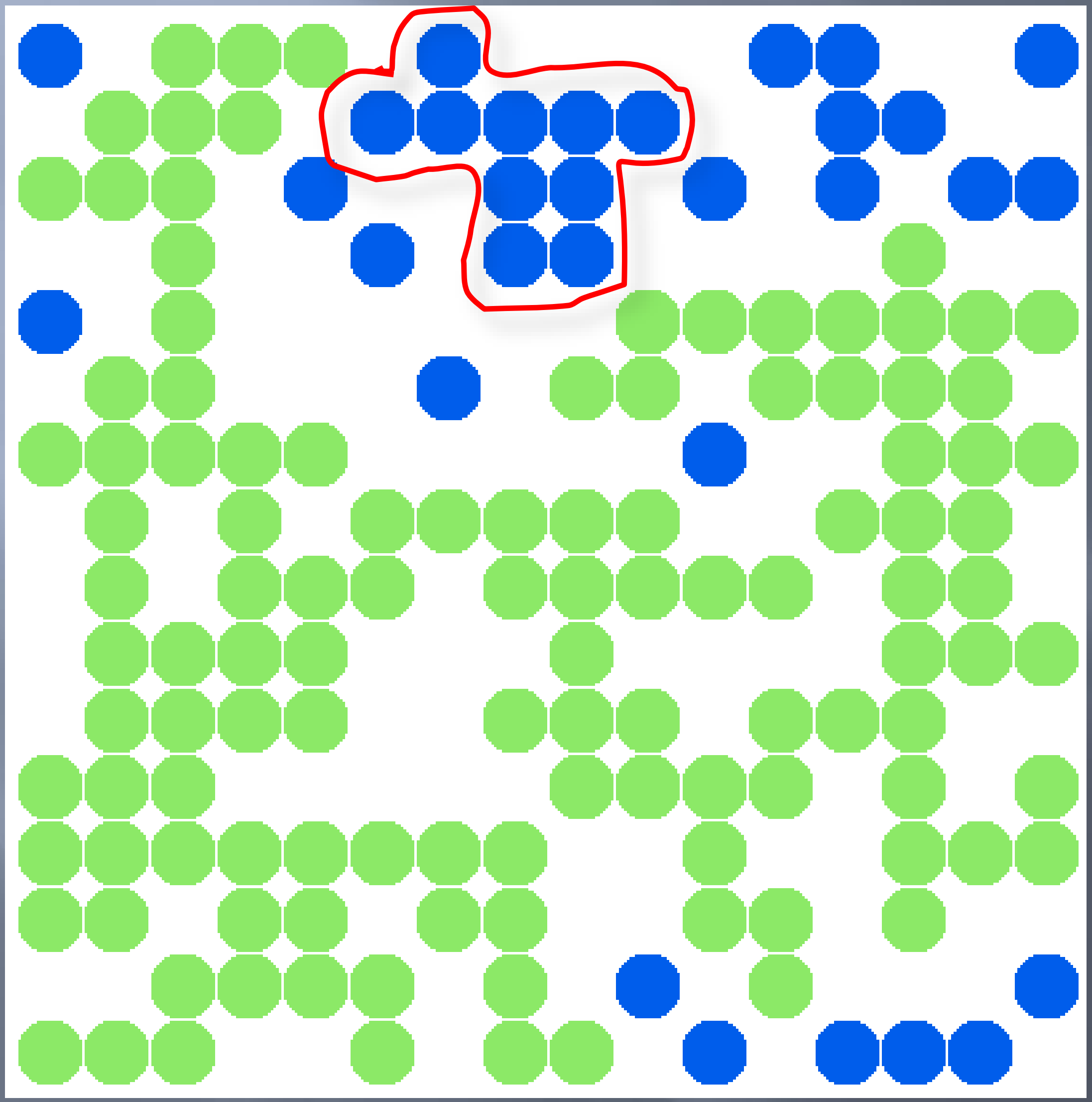
$p = 0.2 < p_c$

Percolation clusters



$p = 0.34 < p_c$

Percolation clusters

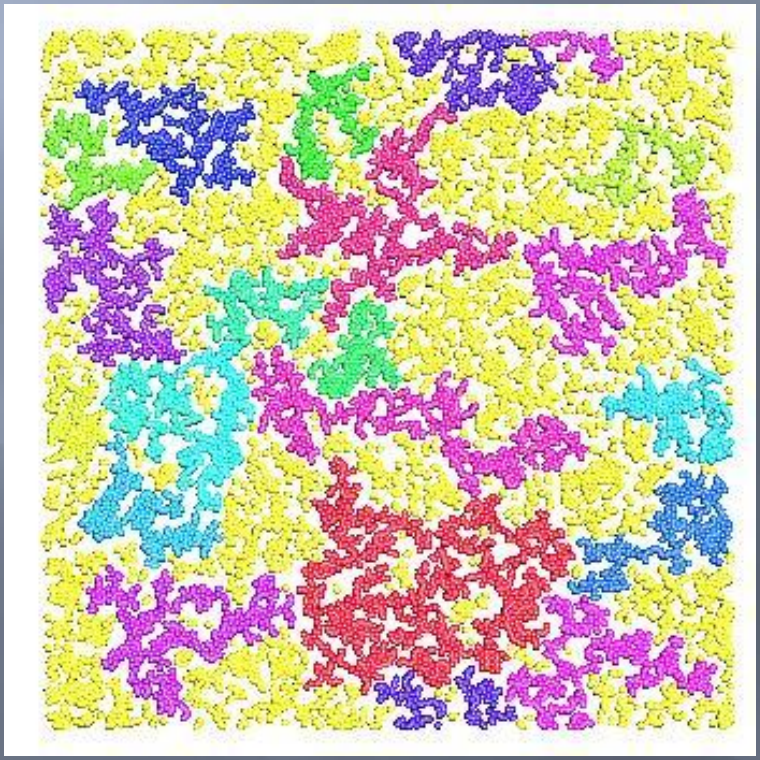


$p = 0.61 > p_c$

Percolation threshold

Below a threshold value p_c there is no infinite cluster (component) of occupied nodes, above it there is. The threshold is also called critical point.

http://simreplay.org/gallery.html



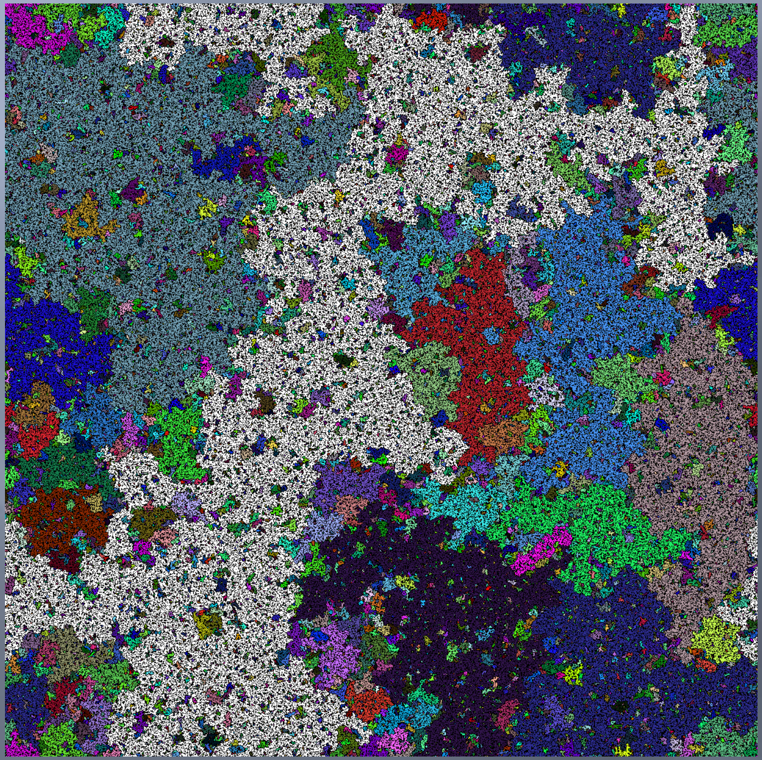
Twenty largest clusters shown in different colors (the smaller ones are all colored yellow)

$p < p_c$

We are close to the threshod, there are large clusters

Percolation threshold

Below a threshold value p_c there is no infinite cluster (component) of occupied nodes, above it there is.



http://www.opencourse.info/anderson/

Different clusters shown in different colors

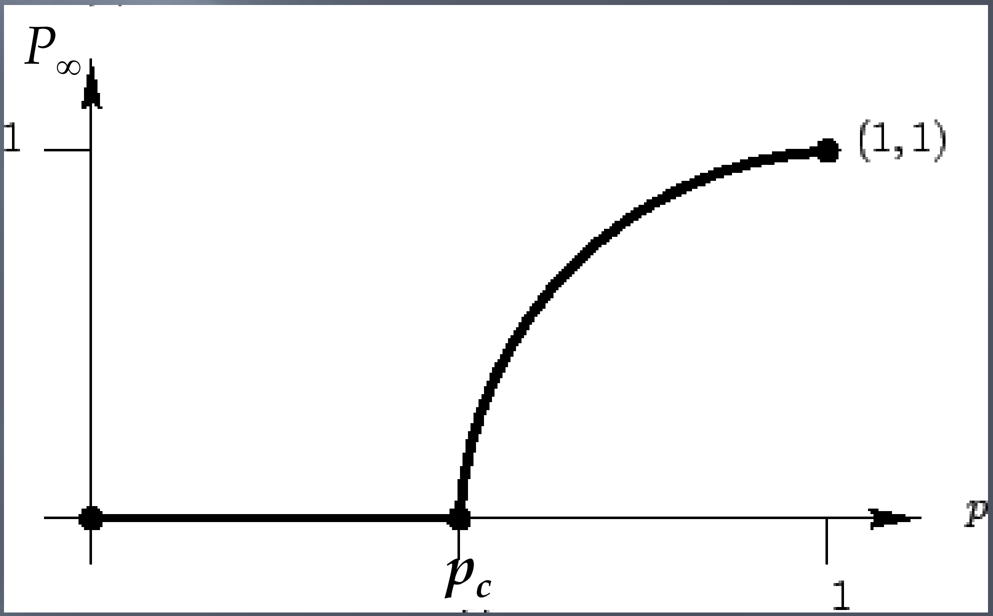
$p > p_c$

The “infinite” (spanning) cluster is the white one.

Percolation probability

Percolation probability P_∞ is the probability that a randomly chosen occupied node belongs to the infinite cluster. In other word: P_∞ is the relative weight or density of the infinite cluster.

Below p_c clearly $P_\infty = 0$. Above p_c it starts to grow and becomes 1 at $p = 1$.



Phase transition

Note non-linearity!

Percolation applet

http://www.physics.buffalo.edu/gonsalves/Java/Percolation.html

Note the finite size effects!

Bond percolation

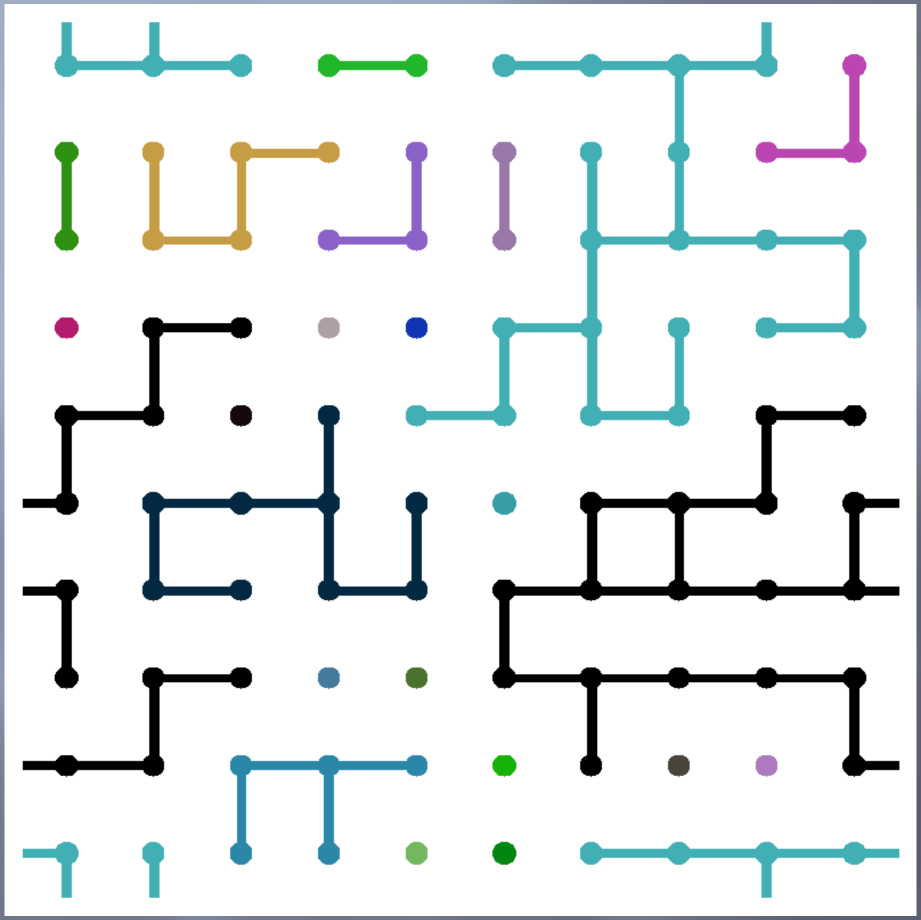
This was site (node) percolation

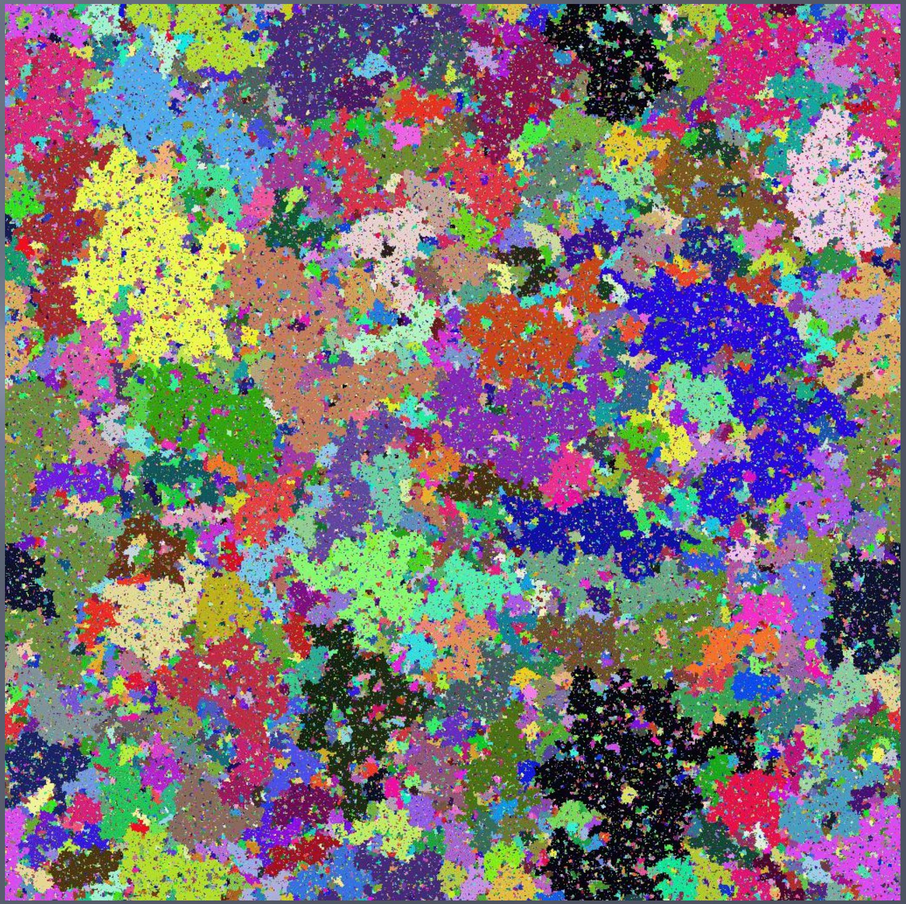
The orchard problem: What is the optimal distance between the trees?

Probability *p* of the transmission of disease decreases with distance

If the distance is too close ($p > p_c$) the disease spreads over the whole orchard!

Bond percolation clusters





http://pages.physics.cornell.edu/~myers/teaching/ComputationalMethods/ComputerExercises/Percolation/Percolation.html

Spreading

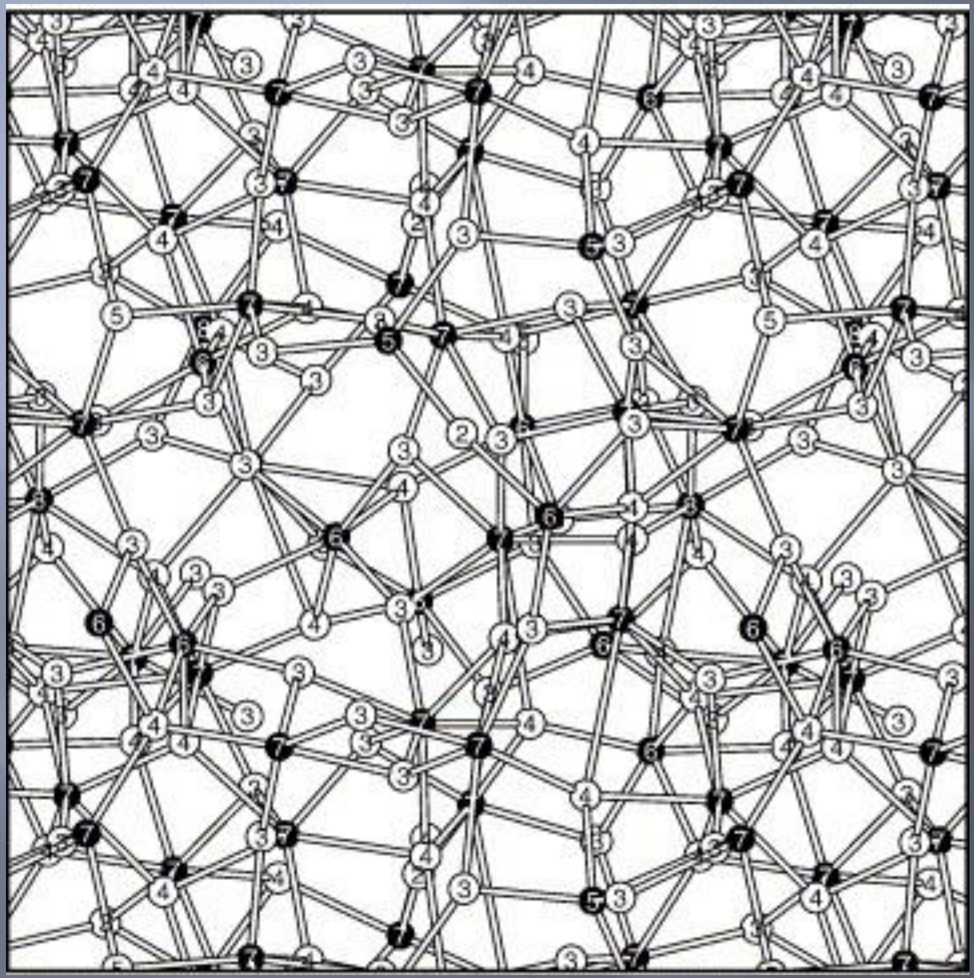
The orchard problem is an example for spreading.

Importance of spreading: Propagation of

- Disease (epidemics)
- Computer viruses
- Information, rumors
- Innovations

…

Random graph

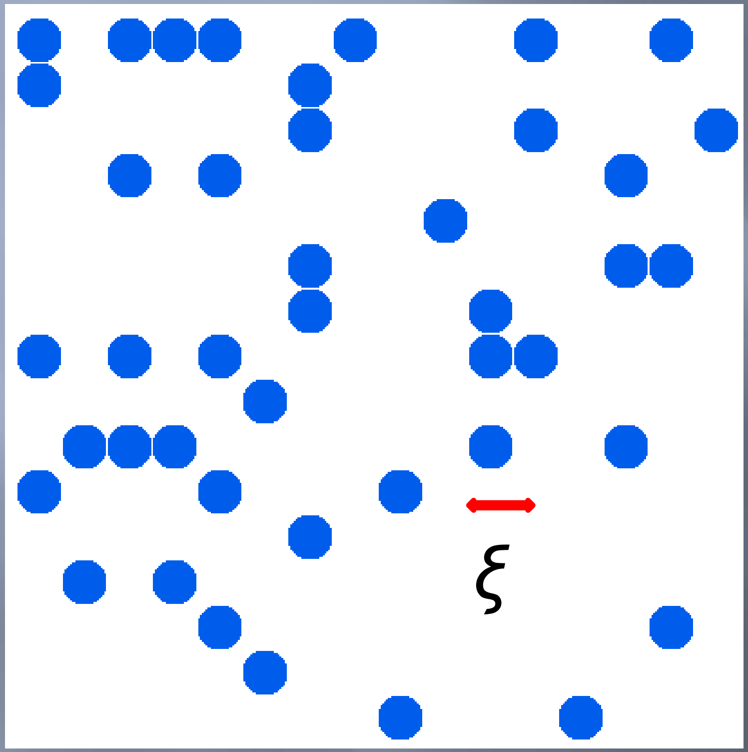


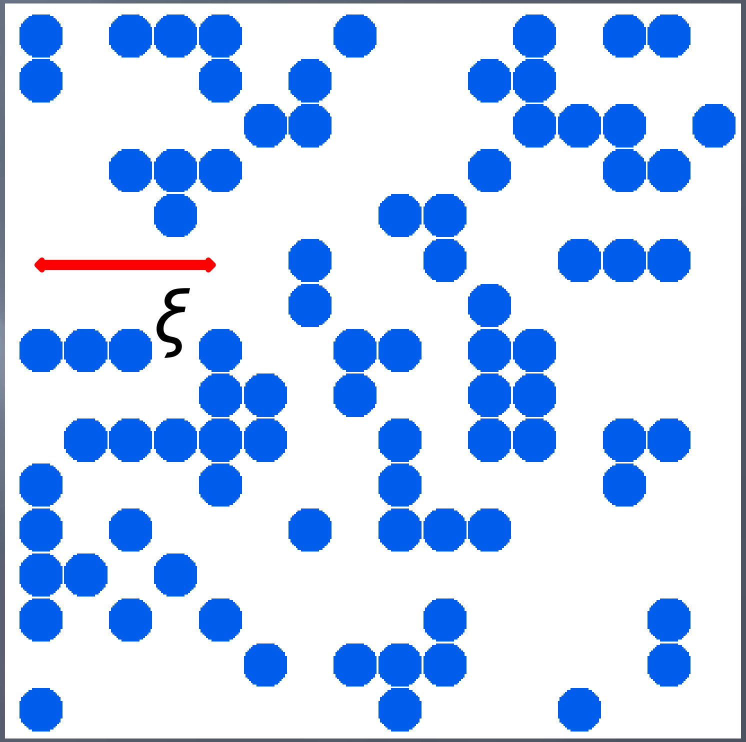
Percolation model can be defined on any (infinite) graph

D.Vanderbilt et al. : Thin Solid Films 486, 125-128 (2004)

Connectivity length

For $p < p_c$ we have first only small, isolated clusters.





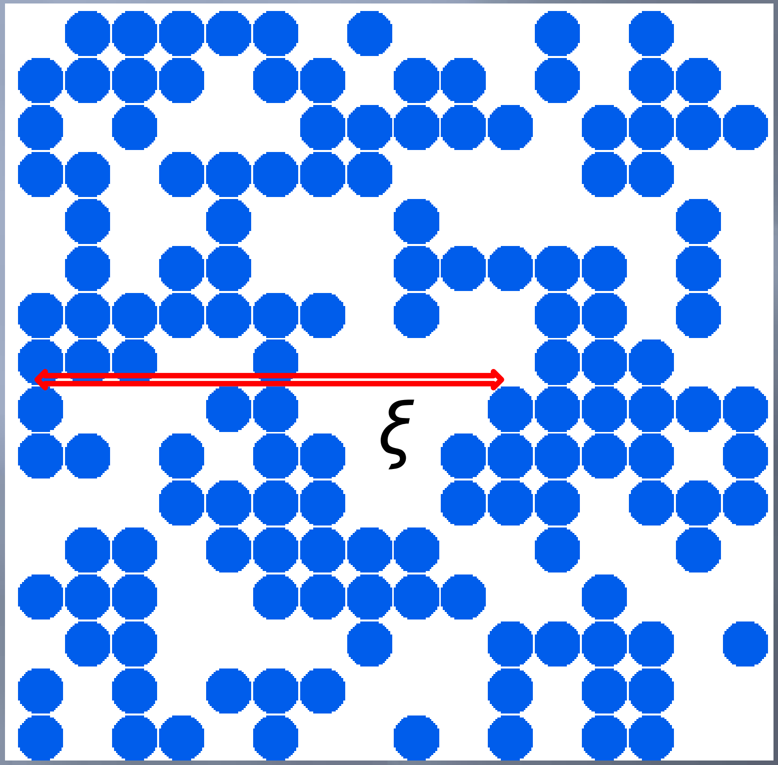
$p = 0.2 < p_c$

$p = 0.36 < p_c$

ξ is the characteristic size of the clusters.
It increases as p_c is approached

Connectivity length

Close to p_c



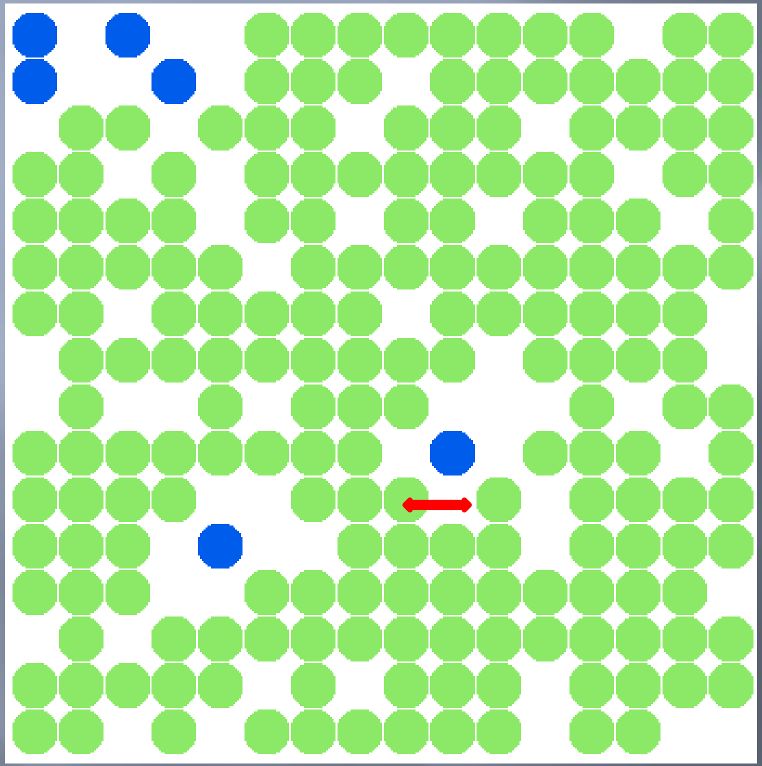
$p = 0.58 < p_c$

ξ increases as p_c is approached

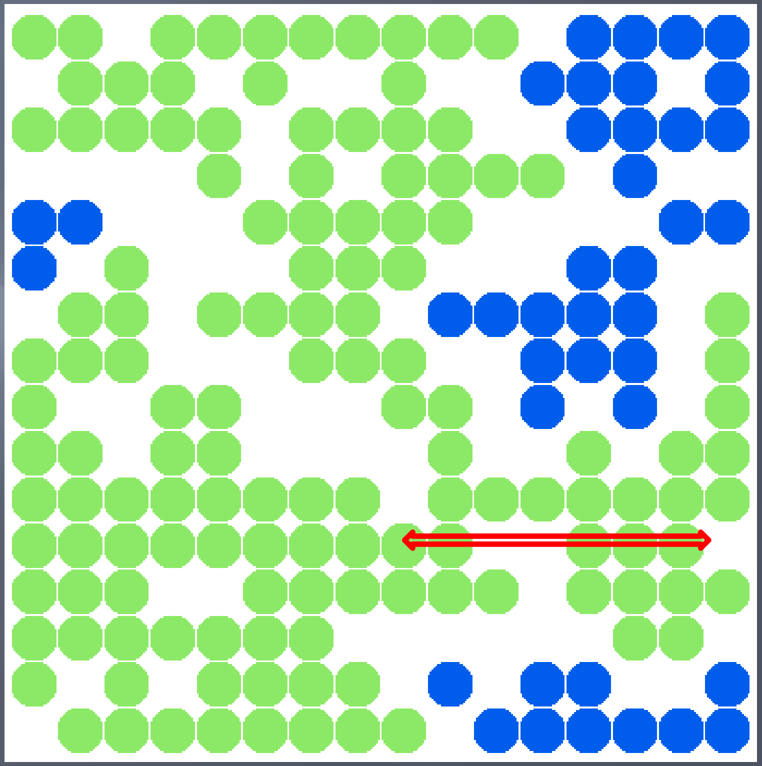
and it grows beyond any limit

Connectivity length

What if we start from the other limit?



$p = 0.8 > p_c$



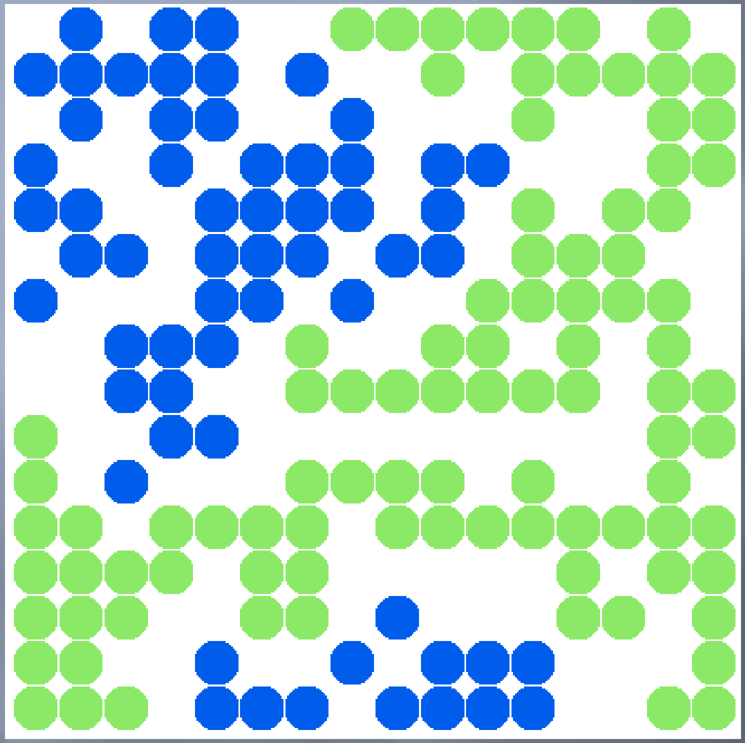
$p = 0.63 > p_c$

ξ is now the characteristic length of the finite clusters

Again, as we approach p_c it grows

At the critical point

The connectivity length ξ is infinity!



$p = 0.59 = p_c$

There is no characteristic length in the system; it is scale free!

The incipient infinite cluster is very ramified, with holes on every scale, where the finite clusters sit in.

Scale transformation

In a system with a characteristic length a scale transformation causes clear changes: The transformed object will be different from the original one.

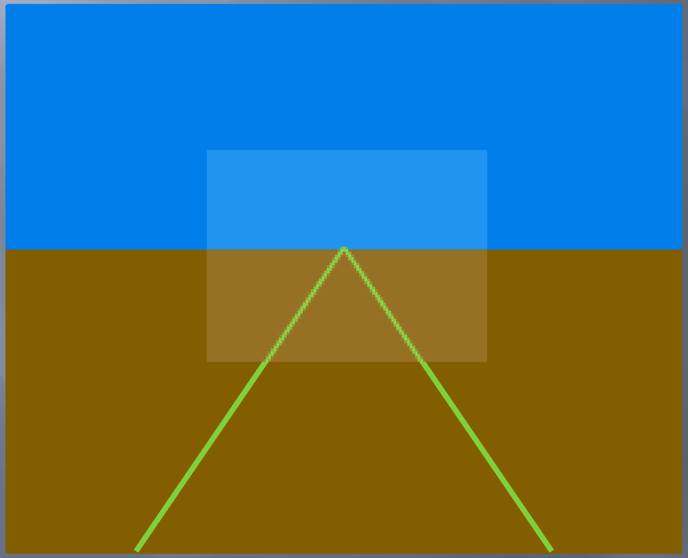
In the presence of a scale, we can tell "how far we are" from the object.

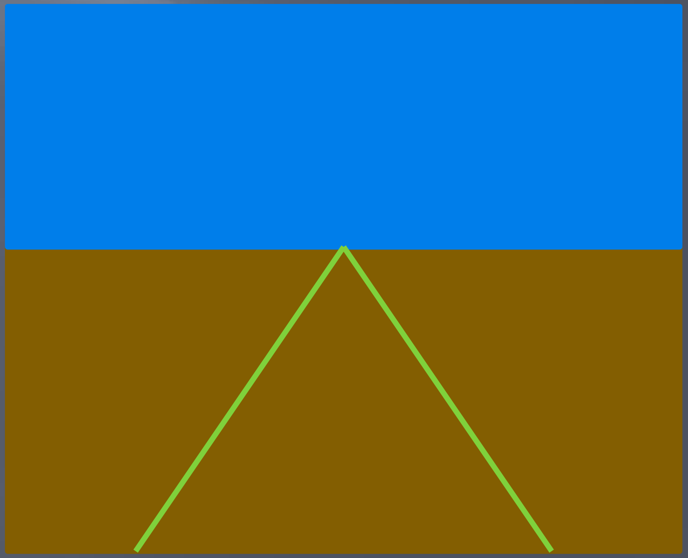
Photo by Matt Benton

Scale invariance

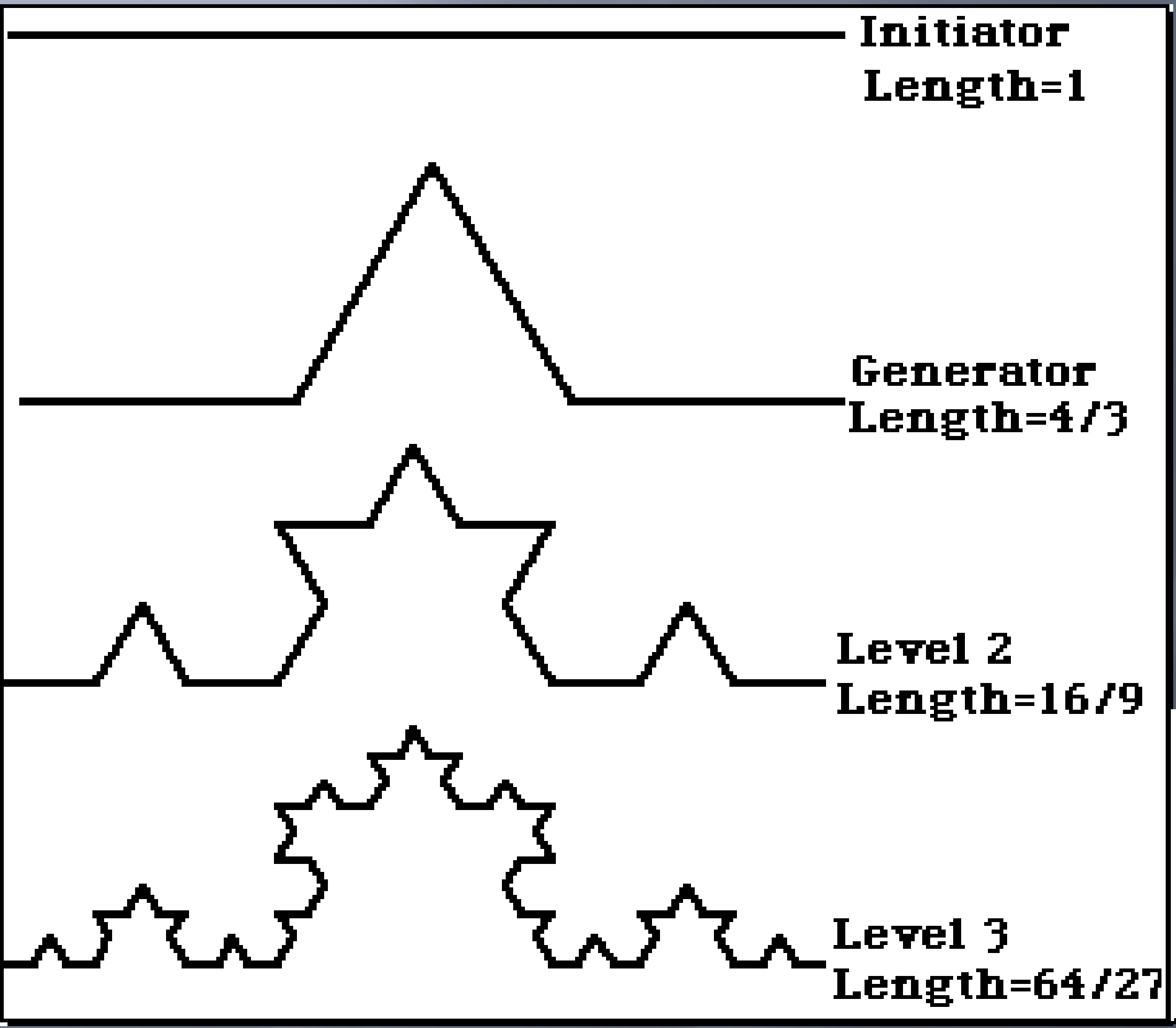
In a system without a characteristic length a scale transformation has no effect. The transformed object will be the same as the original one.

In the absence of a scale, we cannot tell “how far we are” from the object. Self-similarity

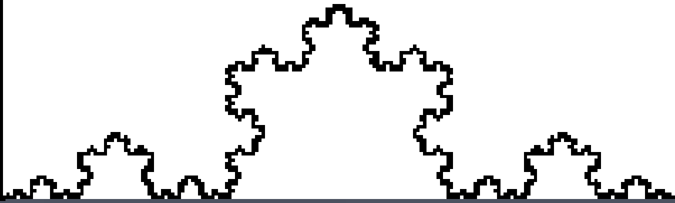




Self-similarity



In the asymptotic limit it is a strange object: No scale, self similar



Koch curve: extremely ramified object

Wikipedia: Koch curve


Euclid

Euclidean world

Human made world follows Euclid:

Simple laws: Characteristic length *a*

All other lengths = *const* * *a*

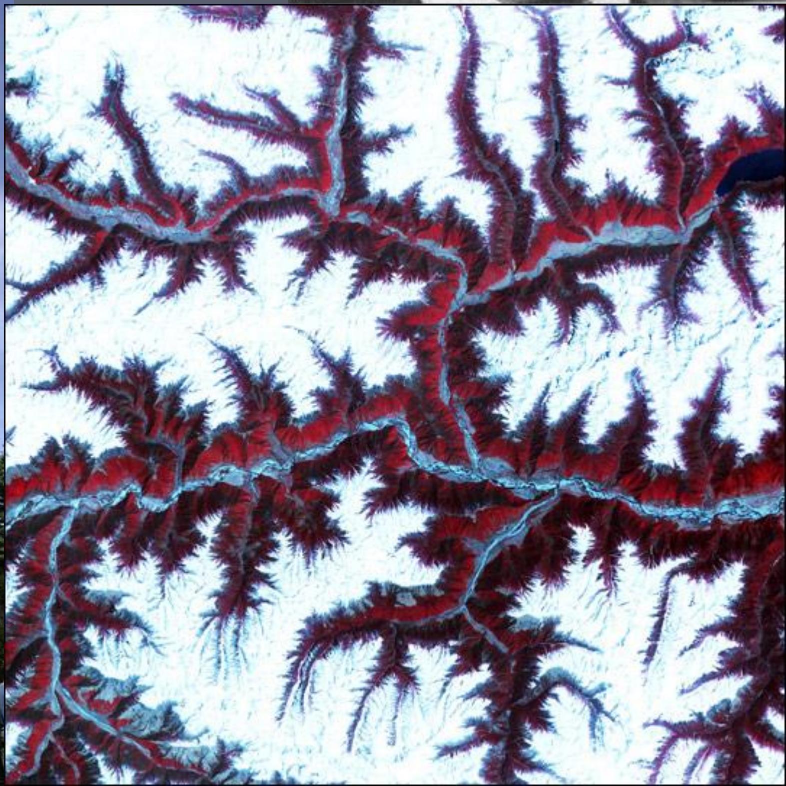
Area $\propto a^2$

Volume $\propto a^3$

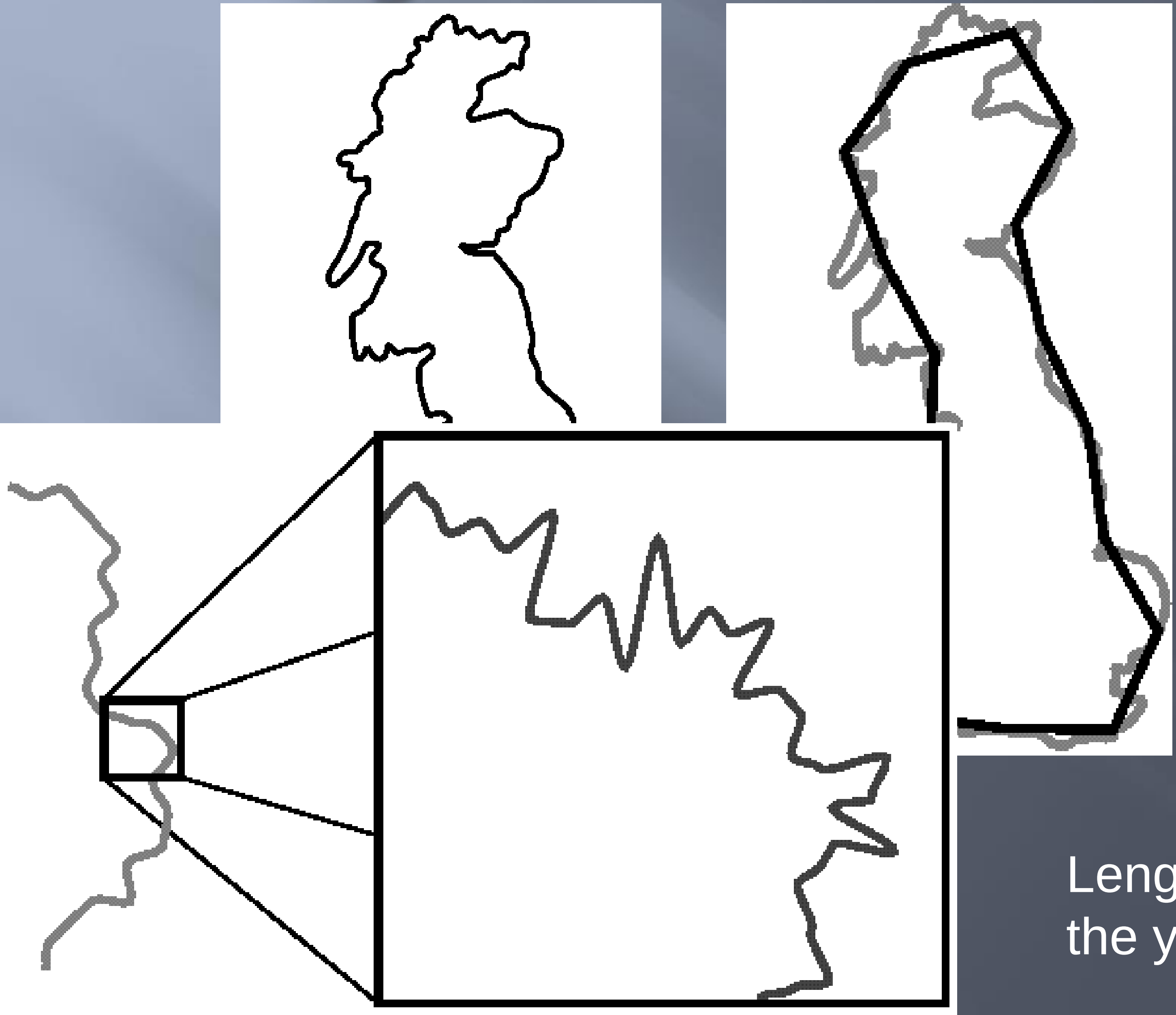
Nature's fractal geometry

„Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”

Mandelbrot



The length of a coastline



Length depends on the yardstick!

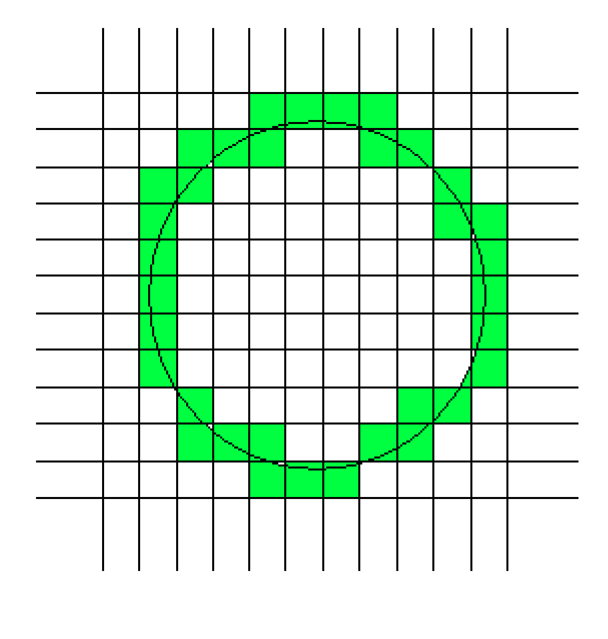
Measuring the length

Length (area, volume) **converges** when taking finer and finer measuring tools.

Put a grid of mesh size ℓ onto the object.

Count the number of boxes $N(\ell)$ covering the object.

$$\lim_{\ell \to 0} N(\ell)\ell^d = \begin{cases} L & \text{for } d = 1 \\ A & \text{for } d = 2 \\ V & \text{for } d = 3 \end{cases}$$



For the highly (infinitely) ramified objects the measure **diverges**.

Fractal dimension

Put a mesh onto the object.

Find *D* such that

$$\lim_{\ell \to 0} N(\ell)\ell^D = \text{finite}$$

from which

$$D = -\lim_{\ell \to 0} [\log N(\ell) / \log \ell]$$

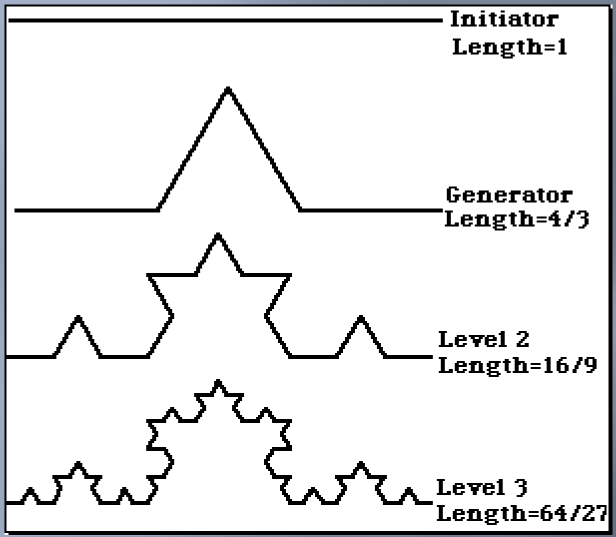
D: Hausdorff dimension: Non-integer. For Euclidean objects, D=d

Objects are embedded into an Euclidean space of dimension d_e and have a topological dimension d_t

$d_t \leq D \leq d_e$ If $d_t < D$ the object is a FRACTAL and *D* is the FRACTAL DIMENSION

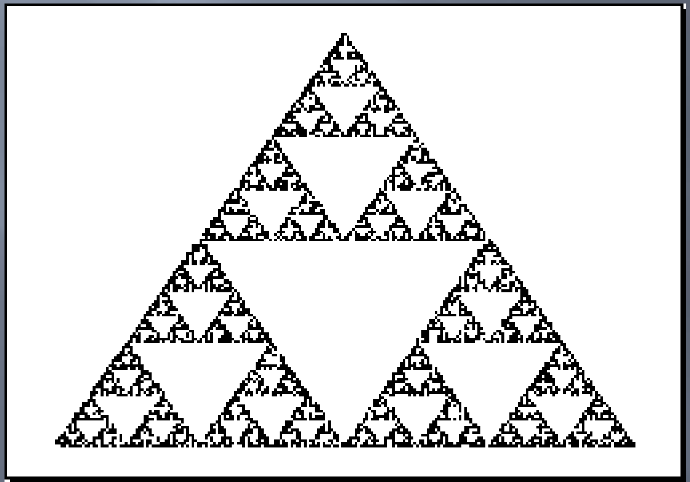
Fractal examples

Koch curve



i	ℓ	$N(\ell)$
0	1	1
1	1/3	4
2	$(1/3)^2$	4^2
3	$(1/3)^3$	4^3
4	$(1/3)^4$	4^4

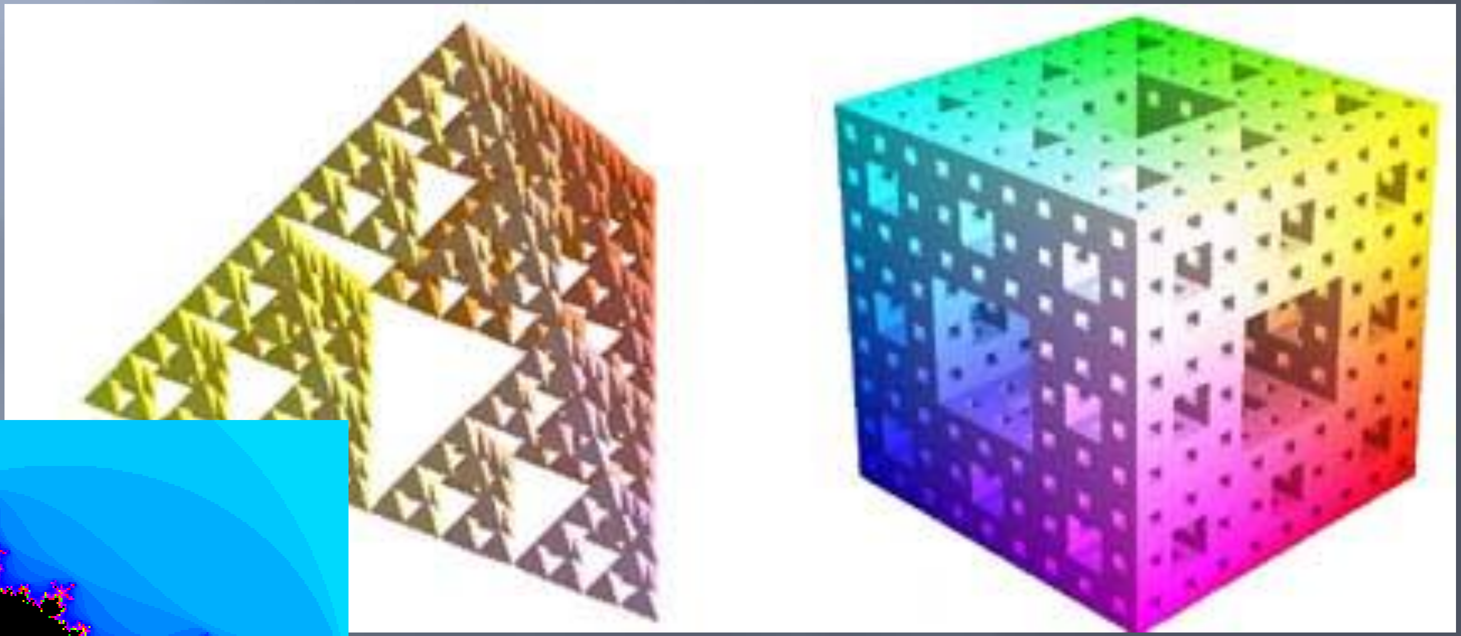
$$D = -\lim (\log N(\ell) / \log \ell) = \log 4 / \log 3$$

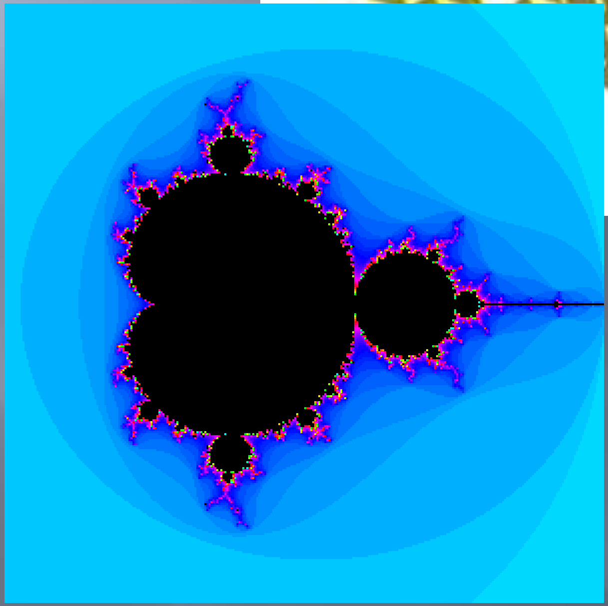


Sierpinski gasket

$D = ?$

Fractal examples





Beautiful video

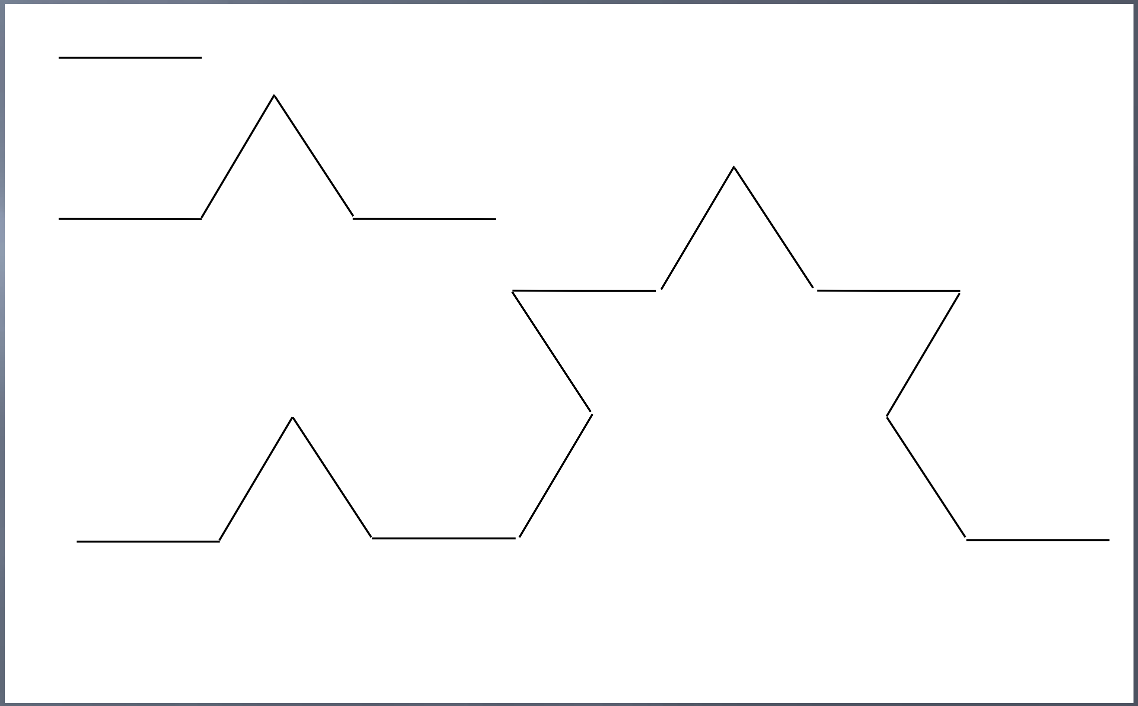
http://www.youtube.com/watch?v=VtsAduikmQU

Growing fractals

M : „mass” of the object

R : linear extent

$M \sim R^D$



$$D = \lim_{R \to \infty} [\log M(R) / \log R]$$

Mathematical vs physical fractals

Mathematically either way a limit is taken:

$\lim_{R \to \infty}$ or $\lim_{\ell \to 0}$ briefly $\lim_{R/\ell \to \infty}$

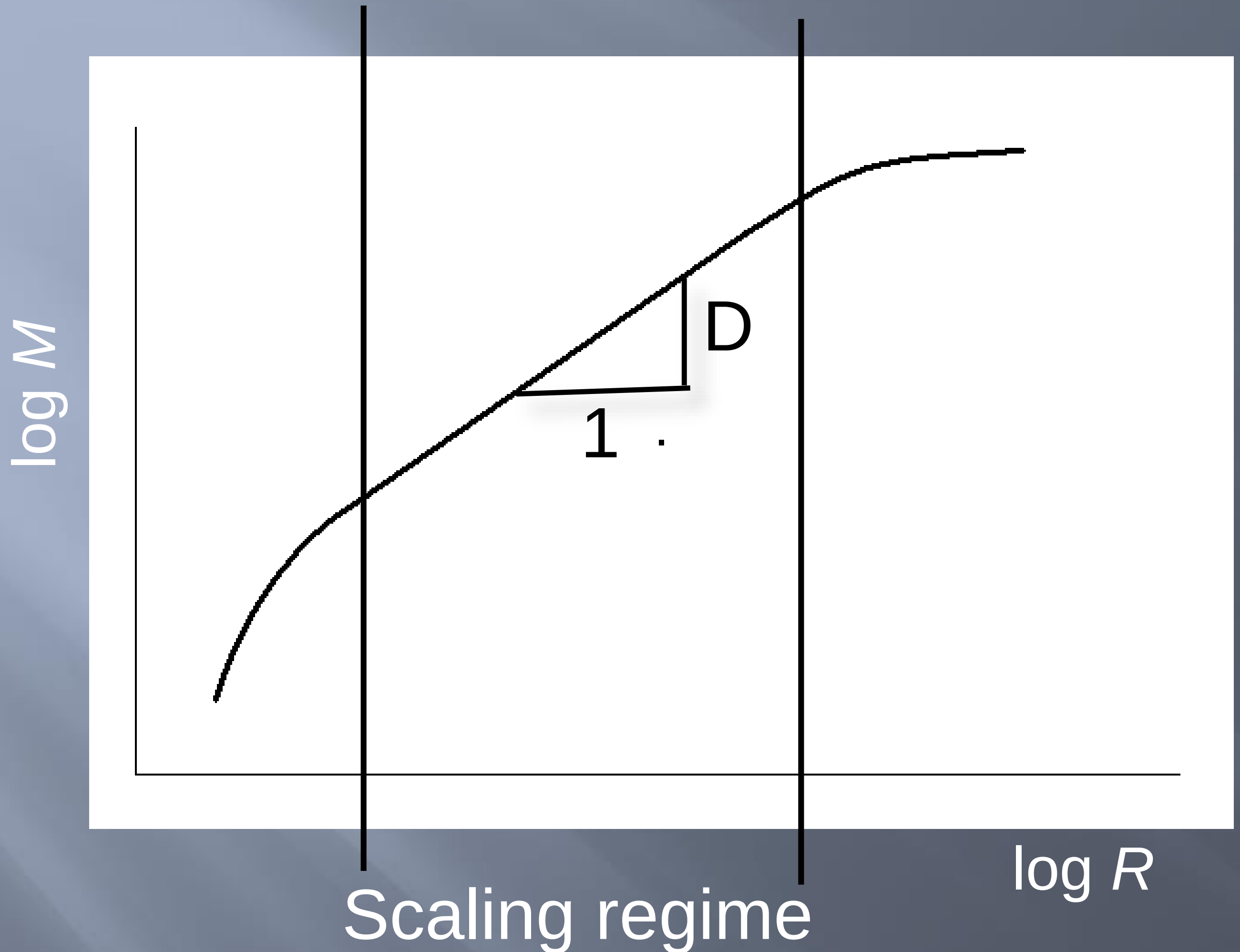
This is needed for perfect **self-similarity**

In physics two problems:

1. No perfect self-similarity because of randomness „Statistical self-similarity", *D* can be measured (percolation)
2. Neither limits can be carried out in practice

There is always a lower and an upper **cutoff**

Finite size effects



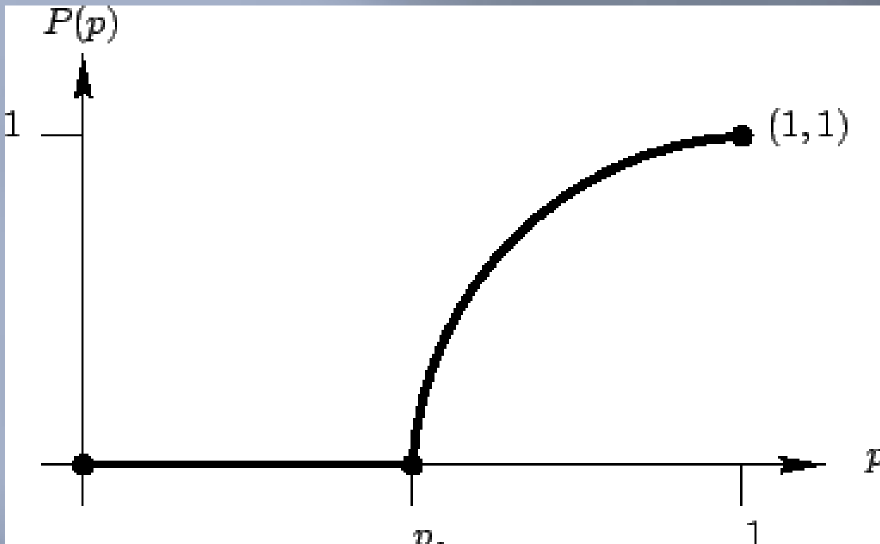
Typical plot

log-log scale

Finite size effect

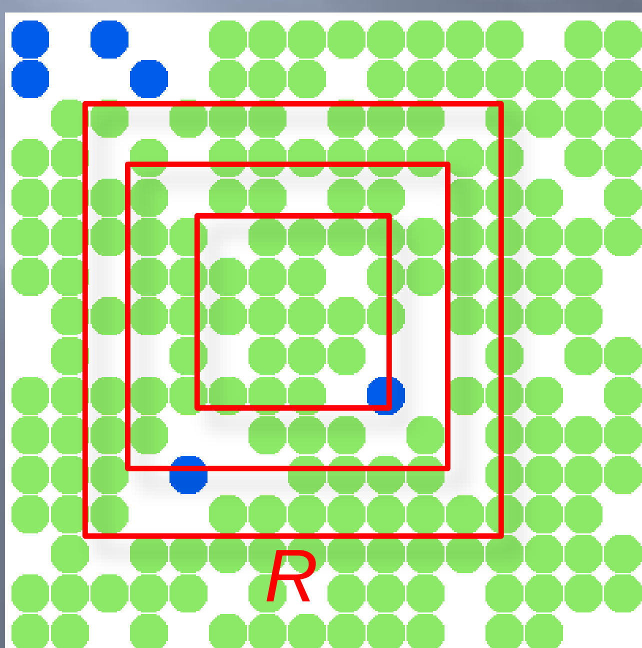
Rule of thumb: Scaling regime > 2 decades

Percolation cluster at threshold



At p_c its density is 0 but it exists!

How is this possible ?



$p > p_c$

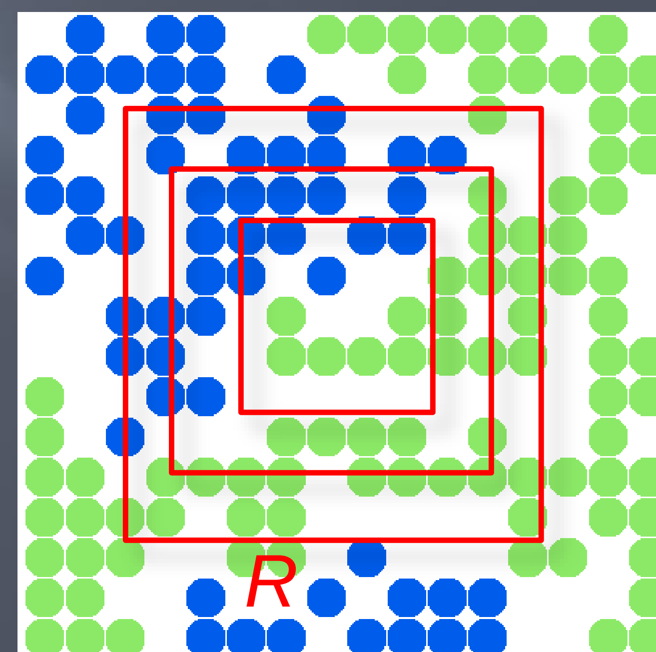
$M \sim R^d$

$M/R^d = P_\infty =$ cnst

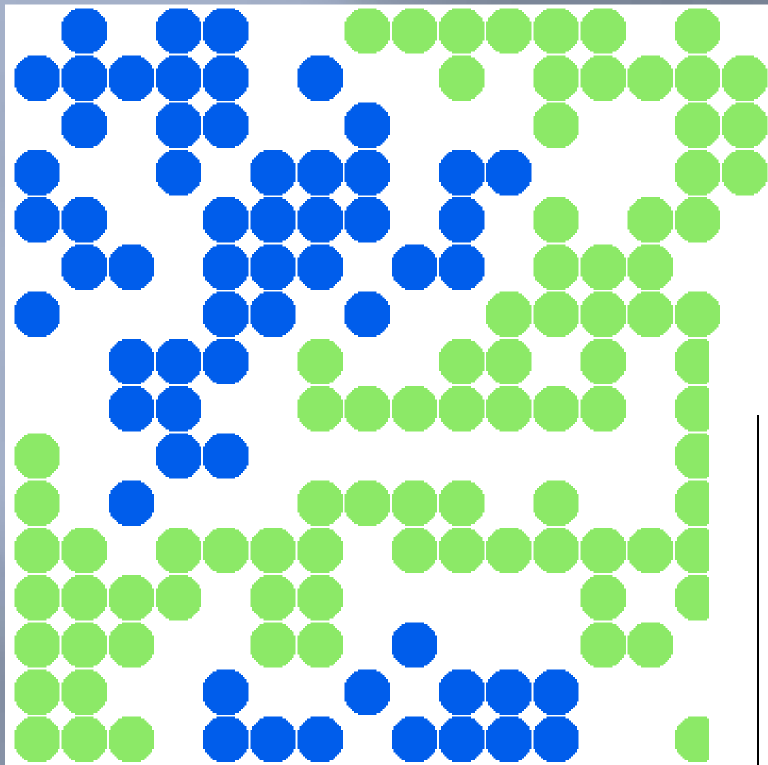
$P = p_c$

$M/R^d = P_\infty(R)$

$M \sim R^{D<d}$

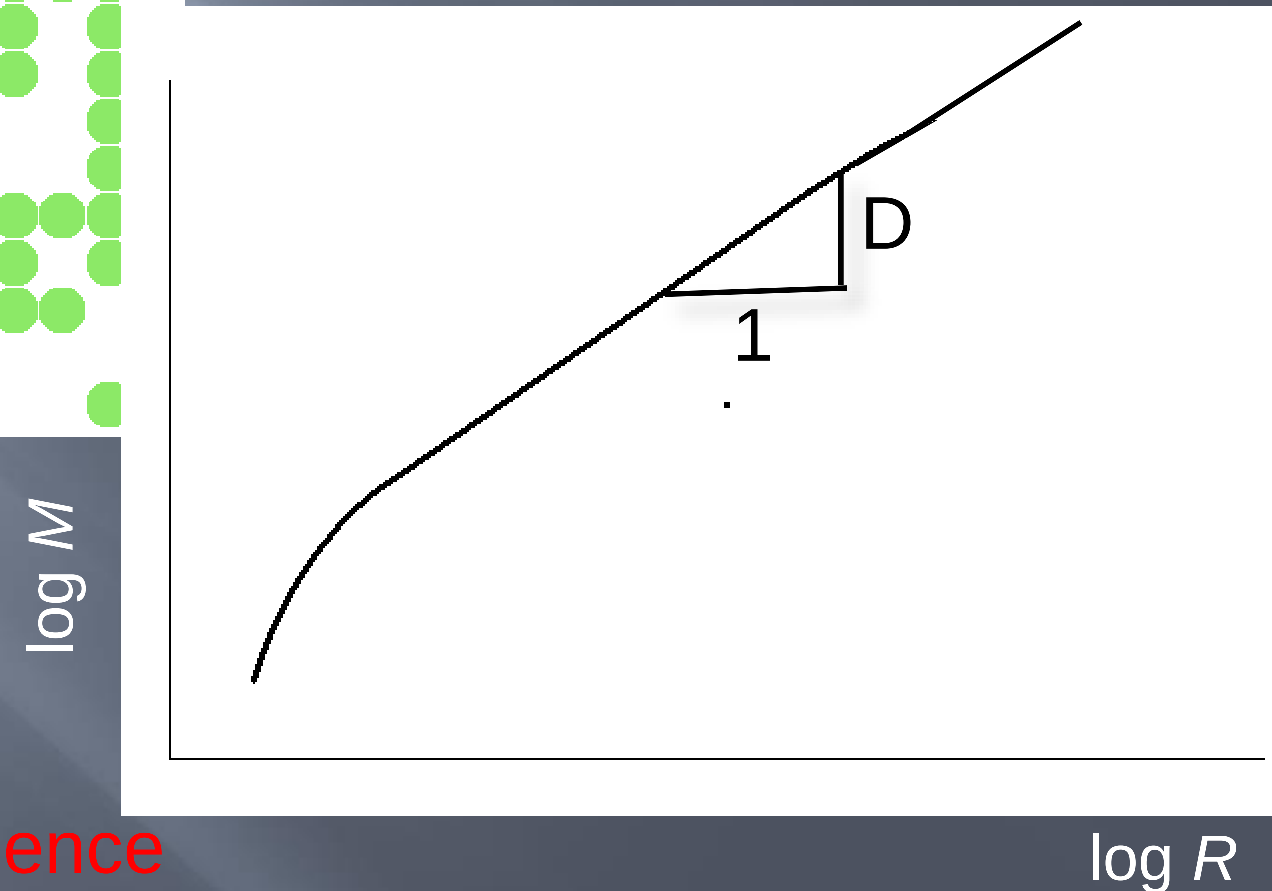


Incipient infinite cluster



Random fractal: $M \sim R^D$

Power law function: linear on log-log scale



Lack of scale
Power law dependence

Scale freeness and power laws

Some well known functions:

e^x

$\sin(x)$

e^{-x^2}

$\cos^2(x)$

...

The arguments (*x*) must be dimensionless.

If a distance is involved, a characteristic size must be present:
$x = r / \xi$

If time is involved, a characteristic time is needed to make the argument dimensionless: $x = t / \tau$

There is one exception: power laws

Scale freeness and power laws

There is one exception: power laws

What does scale invariance mean mathematically?

$$f(ax) = a^k f(x)$$

For any (positive) *α*
(order *k* homogeneous function)

$$\frac{df(ax)}{da} = xf'(ax) = ka^{k-1} f(x)$$

$$xf'(x) = kf(x)$$

With the solution:

$$f(x) = Ax^k$$

Power law functions are characteristic for scale freeness

Power laws at criticality

A basic quantity in percolation is the number of s-size clusters per site: n_s

$$n_s = \frac{\#\,\text{of s-size clustes}}{N}$$

where $N = L^d$ is the total number of sites.

The probability that an occupied site belongs to a cluster of size s is $p_s = s n_s$. Conservation of prob.: $\sum_s p_s + P_\infty + (1-p) = 1$

The average size S of finite clusters is

$$S = \frac{\sum_s s^2 n_s}{\sum_s s n_s}$$

There is an intimate relationship between thermal critical phenomena and the percolation transition, which can be established using the theory of diluted magnets as well as that of the Potts magnetic models. P_∞ corresponds to the magnetization (order parameter), S to the susceptibility with p being the control parameter (~temperature). There is possibility to introduce the analogue of the magnetic field (ghost site).

The connectivity function $C(\mathbf{r})$ is the probability that two occupied sites belong to the same *finite* cluster.
Their characteristic size, the connectivity length ξ diverges at p_c as

$$\xi \sim |p - p_c|^{-\nu}$$

where the notation reminds to the thermal phase transitions (remark for physicists ☺).

ξ

$$P_\infty \sim (p - p_c)^\beta$$
$$S \sim |p - p_c|^{-\gamma}$$

(indicating that *S* plays the role of the susceptibility; no wonder, it contains the second moment of n_s).

The key task in simulating percolation systems is cluster counting, i.e., calculating n_s-s. There are efficient algorithms.

Connectivity function:
The probability that two sites at distance r belong to the same *finite* cluster. It is a homogeneous function of its variables:

$C(r, p-p_c) = b^{\kappa} C(r/b, (p-p_c) b^{y})$

Connectivity length: Characteristic size of fluctuations ≈ size of finite clusters.

$$\xi = |p - p_c|^{-\nu} \qquad y = 1/\nu$$

P_∞ (order parameter)
Let L be the linear dimension of the system. The critical point is $p=p_c$, $L \rightarrow \infty$. Due to scaling:

$$P_\infty(p - p_c, L) = b^{-\beta y} P_\infty((p - p_c) b^{y}, L/b) \rightarrow P_\infty(p=p_c, L) \sim L^{-\beta y}$$

Finite size scaling

$P_\infty(L) \sim L^{-\beta y}$

$P_\infty(p) \sim (p - p_c)^{\beta}$

$\xi \sim |p - p_c|^{-\nu}$

$M \sim L^{D}$

$M = P_\infty L^{d}$

$D = d - \beta/\nu$

Finite size scaling

If the characteristic length diverges, how can we get information about the infinite system from simulating finite samples?

Make virtue of necessity!

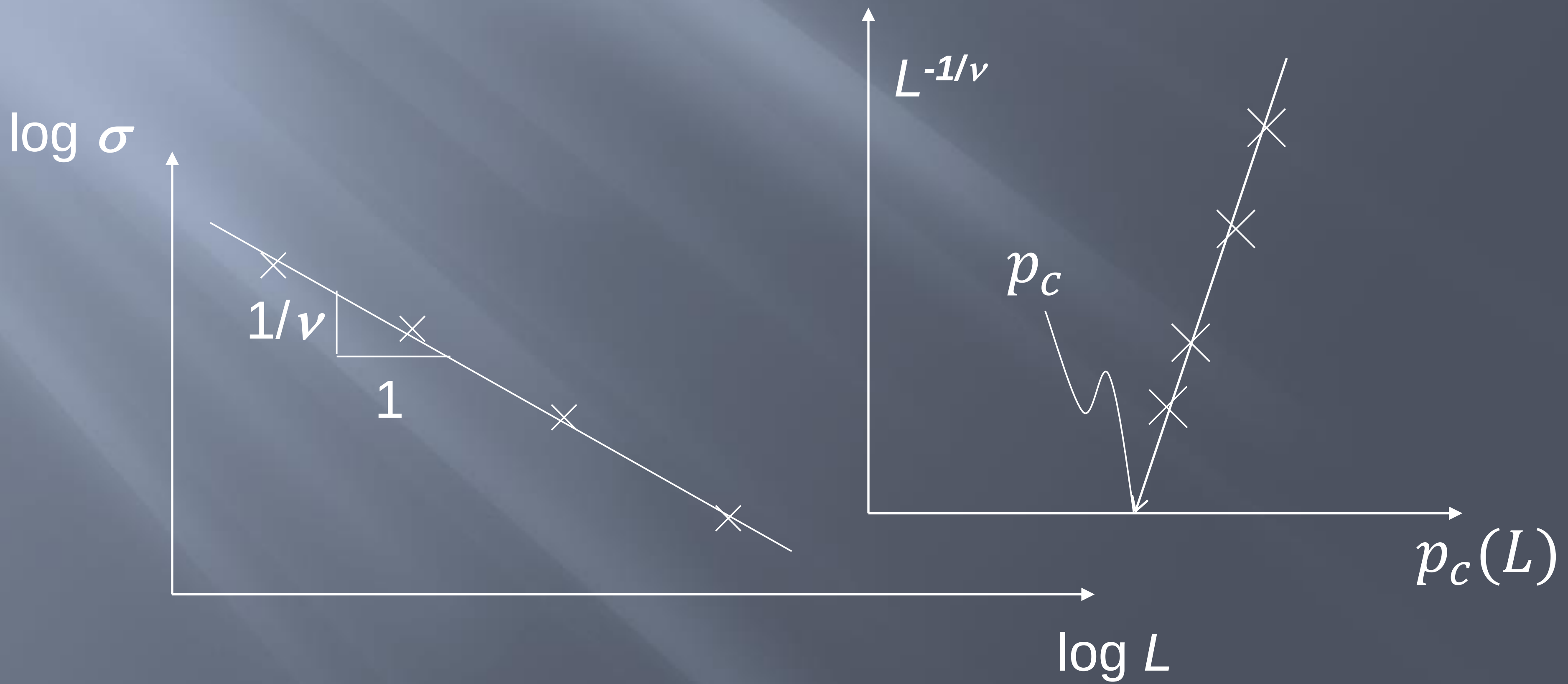
Observe, how the quantities *depend on L* ! (We already saw, $P_\infty(L) \sim L^{-\beta y}$.)

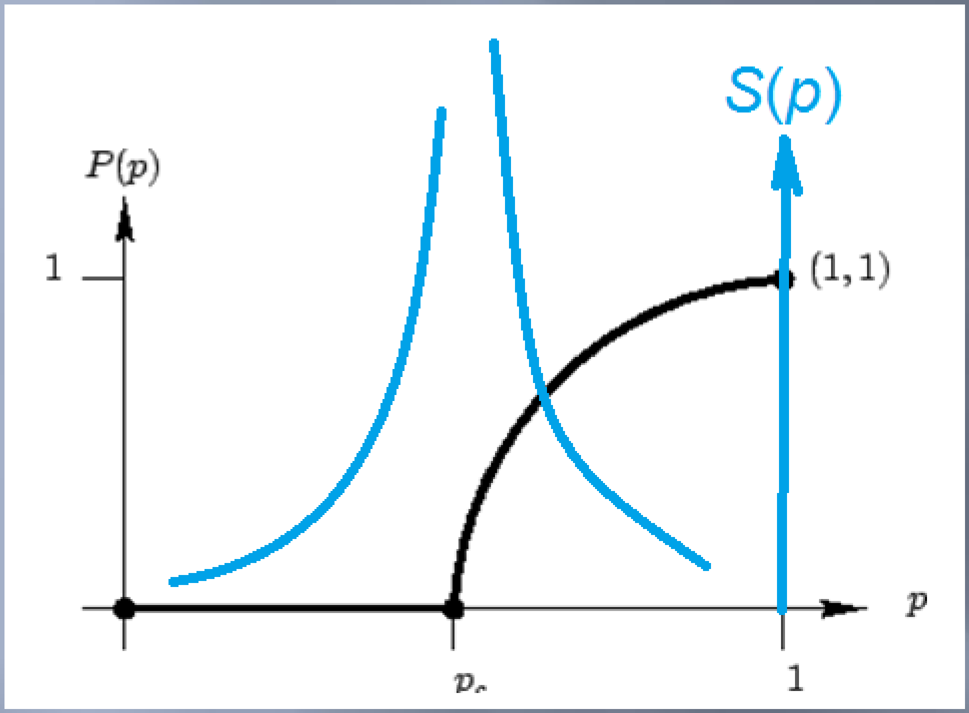
$\xi \sim |p - p_c|^{-\nu}$: In a finite size sample the linear size *L* will play the role of the correlation length, if ξ exceeds *L*. The characteristic $p(L)$, where this happens is given by $L \sim |p(L) - p_c|^{-\nu}$

If we measure, e.g., the sample to sample variation of the finite size analogue of the critical point $p_c(L)$, then this will have a scattering over $\sigma(L)$. Both are reflecting the deviation from the infinite size case, therefore we have

$|p_c(L) - p_c| \sim L^{-1/\nu}$ and $\sigma(L) \sim L^{-1/\nu}$

These two relationships are enough to determine p_c and ν.



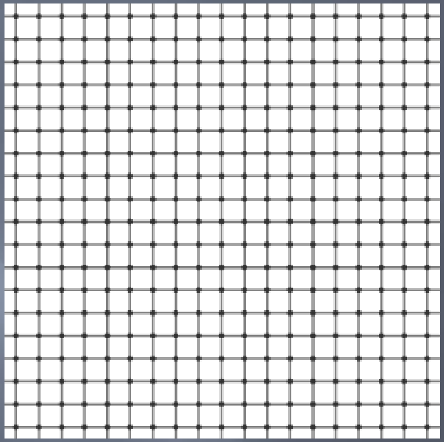


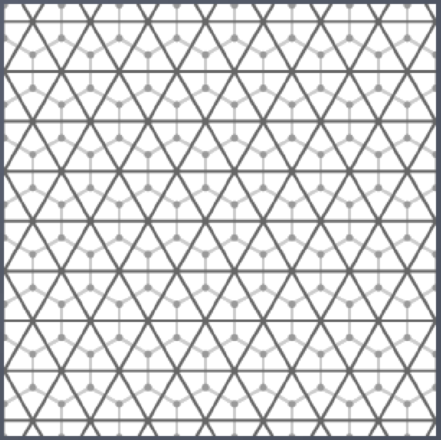
$P_\infty(R) \sim R^{D\text{-}d}$

$$P_\infty \sim (p - p_c)^\beta$$

$$S \sim |p - p_c|^{-\gamma}$$

$$\xi \sim |p - p_c|^{-\nu}$$





Exponents are universal:

There are classes of systems for which they are the same

They depend only on the dimension and the range of interaction but not on

- Type (site or bond)
- Lattice (triangular, square honeycomb etc.)

FSS for general graphs

Spanning from North to South has no meaning for a graph not embedded into a Euclidean space.
Below threshold: There are only components such that their relative size $\to 0$ with increasing system size.
Above threshold: there is one component, whose relative size converges to a constant > 0 with increasing system size. This is the "**giant component**".

The finite size scaling variable is not the linear extent L (no meaning) but the number of vertices N.

Summary

- Percolation is the paradigmatic model for randomness. The connectivity length is the typical size of finite clusters and it diverges when approaching the critical point. At the critical point there is no characteristic length in the system (scale freeness).

- Scale free geometric objects are self similar fractals. Their mass depends on the linear size of observation as $M \sim R^D$. The percolation incipient infinite cluster is a random fractal

- The mathematical description of self-similarity and scale freeness is given by power law functions. Exponents are universal within unv. classes.

Home work

Write a program, which checks, whether there is a "North-South" spanning cluster in a square lattice site percolation problem. Take most convenient boundary conditions (open, periodic, helical, mixed). Make statistics over sample dependent thresholds as a function of the system size L. Estimate the threshold value in the infinite size limit and the exponent ν.